ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА



E5-90-48

V.P.Gerdt, N.V.Khutornoy, A.Yu.Zharkov*

# SOLVING ALGEBRAIC SYSTEMS WHICH ARISE AS NECESSARY INTEGRABILITY CONDITIONS FOR POLYNOMIAL-NONLINEAR EVOLUTION EQUATIONS

Submitted to "International Symposium on Symbolic and Algebraic Computation, Tokyo, Japan, 20-24 August 1990

*Saratov State University, Saratov, USSR

1990

## 1. INTRODUCTION

The investigation of the problem of integrability of nonlinear evolution equations (NLEEs) can often be reduced to the problem of finding the exact solution of a complicated systems of nonlinear algebraic equations. Such systems arise in particular in the problem of classification [1] of integrable NLEEs of the form

$$u_t=\Delta u_N+F(u,u_1,\ldots u_{N-1};\lambda_1,\ldots\lambda_K),\ u=u(x,t)=(u^1,\ldots u^M),\ u_i=\mathbb{D}^i(u),\ \mathbb{D}=d/dx$$
$$F=(F^1,\ldots F^M),\ \Delta=\mathrm{diag}(\delta_1,\delta_2,\ldots\delta_M),\ \lambda_i,\delta_j\in\mathbb{C},\ \delta_i\neq 0,\ \delta_i\neq\delta_j(i\neq j) \tag{1}$$

where F is a polynomial in its arguments. In order to be integrable, (1) must satisfy the necessary integrability conditions which follow from the existence of higher infinitesimal symmetries and which have the form of conservation laws

$$\frac{d}{dt}(R(i,j))\in\mathrm{Im}\mathbb{D},\ i\in\mathbb{N},j=1,2,\ldots M. \tag{2}$$

The densities R(i,j) depend on a finite number of dynamical variables taken from an infinite set $\{u,u_1,u_2,\ldots\}$. Any of the conditions (2) means that the l.h.s. is a total derivative of some other function of a finite number of dynamical variables with respect to x. If arbitrary numerical parameters ($\lambda_i$ and/or $\delta_j$) are present in (1) the conditions (2) lead to a system of algebraic equations in these parameters.

The densities R(i,j) can be calculated, and, in the absence of arbitrary parameters, the conditions (2) can be verified completely automatically by using special computer algebra packages [2,3]. In the presence of arbitrary parameters, in order to select from (1) all integrable NLEEs, we have to find all the solutions of the algebraic system which is equivalent to (2).

Finding the solution of a system of algebraic equations is one of the most interesting area of application of computer algebra. The main results in this field have been achieved by using the fundamental results obtained by Buchberger on the Groebner basis method [4] and in a number of other papers, for example, [5-9].

Here we apply this method to obtain all (infinitely many) solutions of the systems of algebraic equations which are obtained by the above procedure for the classification of the three different multiparametric families of NLEEs: the seventh order scalar KdV-like equations, the seventh order MKdV-like equations, and the third order coupled KdV-like systems. These equations have certain homogeneity properties with respect to the scale transformation $x \Rightarrow \alpha x$, $u \Rightarrow \beta u$ and form an important subclass of (1). They are called NLEEs with uniform rank [10].

We would like to emphasize that all our computations have been carried out by using the computer algebra system REDUCE (version 3.2) on an IBM PC AT-like computer.

## 2. SYSTEMS OF ALGEBRAIC EQUATIONS FROM CLASSIFICATION PROBLEMS

In this section we give an explicit form of three different systems of algebraic equations which arise in classification problems [11]. These systems and others which follow from the necessary integrability conditions (2) are generated completely by computer [3]. In general, each condition (2) generates several algebraic equations, some of them may be identical with others which follow from a different condition (2). In order to simplify the generated system, it is sufficient to reduce the input polynomials according to the prescription of the Buchberger algorithm [4].

I. The seventh order scalar NLEEs of KdV type (M=1, N=7 in (1))

$$u_t=u_7+\lambda_1 uu_5+\lambda_2 u_1u_4+\lambda_3 u_2u_3+\lambda_4 u^2u_3+\lambda_5 uu_1u_2+\lambda_6 u_1^3+\lambda_7 u^3u_1. \qquad (3)$$

The first seven conditions (2) ($0\le i\le 6$) lead to a system of thirteen algebraic equations in seven variables

$$\begin{aligned}
&\lambda_1(\lambda_4-\lambda_5/2+\lambda_6)=(2/7\ \lambda_1^2-\lambda_4)(-10\lambda_1+5\lambda_2-\lambda_3)=(2/7\ \lambda_1^2-\lambda_4)(3\lambda_4-\lambda_5+\lambda_6)=0,\\
&a_1(-3\lambda_1+2\lambda_2)+21a_2=a_1(2\lambda_4-2\lambda_5)+a_2(-45\lambda_1+15\lambda_2-3\lambda_3)=0,\\
&2a_1\lambda_7+a_2(12\lambda_4-3\lambda_5+2\lambda_6)=b_1(2\lambda_2-\lambda_1)+7b_2=b_1\lambda_3+7b_2=0,\\
&b_1(-2\lambda_4-2\lambda_5)+b_2(2\lambda_2-8\lambda_1)+84b_3=0,\\
&b_1(8/3\ \lambda_5+6\lambda_6)+b_2(11\lambda_1-17/3\ \lambda_2+5/3\ \lambda_3)-168b_3=0,\\
&15b_1\lambda_7+b_2(5\lambda_4-2\lambda_5)+b_3(-120\lambda_1+30\lambda_2-6\lambda_3)=0,\\
&-3b_1\lambda_7+b_2(-\lambda_4/2+\lambda_5/4-\lambda_6/2)+b_3(24\lambda_1-6\lambda_2)=3b_2\lambda_7+b_3(40\lambda_4-8\lambda_5+4\lambda_6)=0,
\end{aligned} \qquad (4)$$

where

$$a_1=-2\lambda_1^2+\lambda_1\lambda_2+2\lambda_1\lambda_3-\lambda_2^2-7\lambda_5+21\lambda_6,\quad a_2=7\lambda_7-2\lambda_1\lambda_4+3/7\ \lambda_1^3,$$
$$b_1=\lambda_1(5\lambda_1-3\lambda_2+\lambda_3),\quad b_2=\lambda_1(2\lambda_6-4\lambda_4),\quad b_3=\lambda_1\lambda_7/2,$$

define the structure of conservation law densities in (2) (with $R_{i+1}=R(i,1), i=0,1,..$).

$$R_1=\lambda_1 u,\quad R_3=(2/7\ \lambda_1^2-\lambda_4)u^2,\quad R_5=a_1u_1^2+a_2u^3,\quad R_7=b_1u_2^2+b_2uu_1^2+b_3u^4.$$

Densities $R_i$ (i=2,4,..) do not produce any restrictions on $\lambda_j$.

II. The seventh order scalar NLEEs of a modified KdV (MKdV) type (M=1, N=7 in (1))

$$u_t=u_7+\lambda_1(u_1u_5+3u_2u_4+2u_3^2)+\lambda_2(u^2u_5+6uu_1u_4+10uu_2u_3+6u_1^2u_3+7u_1u_2^2)+\lambda_3(2u_1u_2^2+ u_1^2u_3)+\lambda_4(3u^2u_1u_3+12uu_1^2u_2+3u^2u_2^2+2u_1^4)+\lambda_5(u^4u_3+8u^3u_1u_2+6u^2u_1^3)+\lambda_6u^6u_1. \quad (5)$$

The computation of the densities $R_i$ (i=1,3,5) and the verification of (2) gives a system of nine algebraic equations in six variables

$$a_1\lambda_1=a_1\lambda_2+14a_2=a_1\lambda_4=a_1(6\lambda_2+2\lambda_3+3\lambda_4)+168a_2=a_1\lambda_5+5a_2\lambda_2=0,$$
$$5b_1\lambda_1+21b_2=10b_1\lambda_2+14b_3=105b_4-5b_1\lambda_5-b_3\lambda_2=5b_1\lambda_4+2b_2\lambda_2=0, \quad (6)$$

where

$$a_1=-4\lambda_2+\lambda_3-2/7\ \lambda_1^2,\quad a_2=\lambda_5-2/7\ \lambda_2^2,\quad b_1=7\lambda_2,\quad b_2=6\lambda_1\lambda_2-2\lambda_1\lambda_3-7\lambda_4+3/7\ \lambda_1^3,$$
$$b_3=-42\lambda_5-6\lambda_1\lambda_4-2\lambda_2\lambda_3+9/7\ \lambda_1^2\lambda_2+16\lambda_2^2,\quad b_4=-2\lambda_2\lambda_5+7\lambda_6+3/7\ \lambda_2^3.$$

As in the previous case, we have from the structure of the conservation law densities

$$R_1=\lambda_2u^2,\qquad R_3=a_1u_1^2+a_2u^4,\qquad R_5=b_1u_2^2+b_2u_1^3+b_3u^2u_1^2+b_4u^6.$$

III. The system of two coupled nonlinear equations of KdV type (M=2, N=3 in (1)) which satisfies the conditions (10) for i=0,1,2,3 [11]

$$u_t=\lambda_1u_3+\lambda_5uu_1,$$
$$v_t=(\lambda_1-1)v_3+\lambda_2uu_1+\lambda_3vu_1+\lambda_4uv_1. \quad (7)$$

The conditions (2) for i=4,5,6; j=1,2 do not depend on $\lambda_2$ and generate a system of four algebraic equations in four variables

$$
\begin{aligned}
&-2\lambda_4^3\lambda_1+(3\lambda_4^2\lambda_1-2\lambda_4^2-6\lambda_4\lambda_3\lambda_1+6\lambda_4\lambda_3+6\lambda_3^2\lambda_1-6\lambda_3^2)\lambda_5-\lambda_4\lambda_1\lambda_5^2 = 0,\\
&18\lambda_4^3\lambda_1^2-9\lambda_4^3\lambda_1-18\lambda_4^2\lambda_3\lambda_1^2+18\lambda_4^2\lambda_3\lambda_1+18\lambda_4\lambda_3^2\lambda_1^2-18\lambda_4\lambda_3^2\lambda_1+(-27\lambda_4^2\lambda_1^2+24\lambda_4^2\lambda_1-5\lambda_4^2+\\
&63\lambda_4\lambda_3\lambda_1^2-78\lambda_4\lambda_3\lambda_1+15\lambda_4\lambda_3-63\lambda_3^2\lambda_1^2+78\lambda_3^2\lambda_1-15\lambda_3^2)\lambda_5+9\lambda_4\lambda_1^2\lambda_5^2 = 0,\\
&-8\lambda_4^4\lambda_1+(6\lambda_4^3\lambda_1-6\lambda_4^3-12\lambda_4^2\lambda_3\lambda_1+12\lambda_4^2\lambda_3+12\lambda_4\lambda_3^2\lambda_1-12\lambda_4\lambda_3^2)\lambda_5+\\
&(5\lambda_4^2\lambda_1-4\lambda_4^2-18\lambda_4\lambda_3\lambda_1+18\lambda_4\lambda_3+18\lambda_3^2\lambda_1-18\lambda_3^2)\lambda_5^2-3\lambda_4\lambda_1\lambda_5^3 = 0,\\
&(3\lambda_1-5)\lambda_4^2\lambda_3-15(\lambda_1-1)\lambda_4\lambda_3^2+10(\lambda_1-1)\lambda_3^3+(\lambda_4\lambda_3+3\lambda_3^2\lambda_1-3\lambda_3^2)\lambda_5-\lambda_3\lambda_1\lambda_5^2 = 0.
\end{aligned}
\tag{8}
$$

## 3. THE COMPLETE SET OF SOLUTIONS

The systems (4),(6),(8), as well as higher necessary integrability conditions (2) for NLEEs (3),(5),(7), always have, evidently, the trivial solution $\lambda_1=0$ which corresponds to (integrable) linear evolution equations. If there is a nontrivial solution, then it follows from the structure of the multiparametric families (3),(5),(7) that there are infinitely many solutions. Indeed, these families are invariant under the scale transformation

$$x\Rightarrow\alpha x,\ t\Rightarrow\alpha^N t,\ u\Rightarrow\rho u,\ v\Rightarrow\sigma v,\quad \alpha,\rho\in\mathbb{C},\ \alpha\neq0,\rho\neq0,\sigma\neq0, \tag{9}$$

where N is the order of NLEEs (see (1)). It is obvious that the transformation (9) does not change the property of integrability. Thus, the set of solutions is invariant under the corresponding scale transformation of $\lambda_1$.

Because of the fact that the computer algebra system REDUCE 3.2 (in particular on IBM PC, and unlike REDUCE 3.3), has no built-in package for computation of Groebner basis, we have written our own program in Rlisp in order to solve systems of algebraic equations using Buchberger's algorithm [4]. To make the program efficient we have used the distributive form for the internal representation of polynomials together with multivariate factorization.

In order to obtain the (infinitely many) solutions, we construct a lexicographic Groebner basis, then we compute, according to [12], the dimension and independent sets of variables for the ideal which is generated by the input system. Thereafter, we consider each set of variables as free parameters and compute a Groebner basis leaving the order of the others unchanged.As a result we obtain a set of Groebner bases with a simple structure, and the solution can be found in an easy way. From all the sets of solutions found by this procedure, we choose a complete independent subset. This is not difficult, because we have obtained previously all solutions in explicit form. In the worst case the problem is reduced to quadratic equations.

To find a good ordering we have started the process by using an heuristic procedure given in [5]. It turned out that for the systems (6),(8) this is the best choice. For the system (4), however, we did not succeed in constructing a Groebner basis in this way on an IBM PC because of lack of memory . Using a different ordering (see below), we were able to overcome this problem.

It should be noted that our system (4) had been obtained already in [13] and was included in a list of examples for Groebner basis computations [5]. It had been analyzed also in [9,12]. It seems, however, that no solution has been published as yet. The system (6) has been obtained in [14].

We now give a complete list of solutions for the problems of Sect.2., as obtained by the above procedure. The total computing time is given for an IBM PC AT-like computer (10 Mhz) and for the non-compiled program

I. Ordering: $\lambda_7>\lambda_6>\lambda_5>\lambda_4>\lambda_3>\lambda_2>\lambda_1$, time (min): 48.

Solution 1: free variable $\lambda_1$

$\lambda_2=7/2\ \lambda_1, \lambda_3=6\lambda_1, \lambda_4=2/7\ \lambda_1^2, \lambda_5=9/7\ \lambda_1^2, \lambda_6=5/14\ \lambda_1^2, \lambda_7=4/147\ \lambda_1^3.$

Solution 2: free variable $\lambda_1$

$\lambda_2=3\lambda_1, \lambda_3=5\lambda_1, \lambda_4=5/14\ \lambda_1^2, \lambda_5=10/7\ \lambda_1^2, \lambda_6=5/14\ \lambda_1^2, \lambda_7=5/98\ \lambda_1^3.$

Solution 3: free variable $\lambda_1$

$\lambda_2=2\lambda_1, \lambda_3=3\lambda_1, \lambda_4=2/7\ \lambda_1^2, \lambda_5=6/7\ \lambda_1^2, \lambda_6=1/7\ \lambda_1^2, \lambda_7=4/147\ \lambda_1^3.$

Solution 4: free variables $\lambda_2, \lambda_4$

$\lambda_1=0, \lambda_3=5\lambda_2, \lambda_5=1/14\ \lambda_2^2+9/2\ \lambda_4, \lambda_6=1/14\ \lambda_2^2+3/2\ \lambda_4, \lambda_7=0.$

Solution 5: free variables $\lambda_3, \lambda_6$

$\lambda_1=0, \lambda_2=r_1, \lambda_4=0, \lambda_5=-2/161\ r_1\lambda_3+40/23\ \lambda_6, \lambda_7=10/25921\ r_1\lambda_3^2-74/483\ r_1\lambda_6-16/3703\ \lambda_3\lambda_6$, where $r_1=(5\lambda_3+\sqrt{25\lambda_3^2-5152\lambda_6})/46$.

Solution 6: free variables $\lambda_3, \lambda_6$

$\lambda_1=0, \lambda_2=r_2, \lambda_4=0, \lambda_5=-2/161\ r_2\lambda_3+40/23\ \lambda_6, \lambda_7=10/25921\ r_2\lambda_3^2-74/483\ r_2\lambda_6-16/3703\ \lambda_3\lambda_6$, where $r_2=(5\lambda_3-\sqrt{25\lambda_3^2-5152\lambda_6})/46$.

Solution 7: free variables $\lambda_2, \lambda_3, \lambda_5$

$\lambda_1=0, \lambda_4=0, \lambda_6=1/21\ \lambda_2^2+1/3\ \lambda_5, \lambda_7=0$.

II. Ordering: $\lambda_6>\lambda_5>\lambda_4>\lambda_3>\lambda_2>\lambda_1$, time (s): 24.

Solution 1: free variable $\lambda_1$

$\lambda_2=-1/7\ \lambda_1^2, \lambda_3=-2/7\ \lambda_1^2, \lambda_4=-2/147\ \lambda_1^3, \lambda_5=2/343\ \lambda_1^4, \lambda_6=-4/50421\ \lambda_1^6$.

Solution 2: free variable $\lambda_2$

$\lambda_1=0, \lambda_3=3\lambda_2, \lambda_4=0, \lambda_5=5/14\ \lambda_2^2, \lambda_6=5/98\ \lambda_2^3$.

III. Ordering: $\lambda_3>\lambda_4>\lambda_1>\lambda_5$, time (min): 11.

Solution 1: free variable $\lambda_3$

$\lambda_1=1, \lambda_4=0, \lambda_5=0$.

Solution 2: free variable $\lambda_4$

$\lambda_1=0, \lambda_3=\lambda_4, \lambda_5=0$.

Solution 3: free variable $\lambda_4$

$\lambda_1=0, \lambda_3=\lambda_4/2, \lambda_5=0$.

Solution 4: free variable $\lambda_4$

$\lambda_1=1/3, \lambda_3=\lambda_4, \lambda_5=-\lambda_4$.

Solution 5: free variable $\lambda_4$

$\lambda_1=-1/3, \lambda_3=\lambda_4/2, \lambda_5=\lambda_4$.

Solution 6: free variables $\lambda_3, \lambda_6$

$\lambda_3=0, \lambda_4=0$.

## 4. CONCLUSION

Our analysis shows that the Groebner basis method allows us to obtain the complete set of exact solutions for systems of nonlinear algebraic equations which are the necessary integrability conditions for NLEEs and therefore to select all integrable evolution equations. It turns out from practice that if the first integrability conditions (2) are fulfilled, then often all the others are fulfilled as well. In other words, the fact that some necessary conditions are fulfilled is often sufficient for integrability. In particular, this is the case for the problems I and II. All the solutions correspond to integrable NLEEs from the families (3),(5) which are the higher infinitesimal symmetries of well-known integrable equations of lower order [13-14].

In the case of the problem III we have a more complicated situation. The first three solutions do not satisfy the restrictions on the eigenvalues of the matrix $\Delta$ in (1) ($\delta_1=\lambda_1\neq0$, $\delta_2=1-\lambda_1\neq0$) and should therefore be omitted in view of the integrability of (1). The solutions 4-6 in the above list lead to integrable NLEEs of the form (7) only in the case $\lambda_2=0$ [15]. Note that $\lambda_2$ is not present in the system (8), and the same is probably true also for higher integrability conditions. On the other hand, higher infinitesimal symmetries (which means integrability [1]) exist only for $\lambda_2=0$.

It is clear that the solvability of the above systems (with infinitely many solutions) and of even more complicated ones (see, for example [15]) is closely connected with the property of integrability. In addition to their importance in the theory and application of NLEEs, such systems are very useful for testing different computer algebra algorithms.

## REFERENCES


1. Mikhailov A.V.,Shabat A.B.,Yamilov R.I. Symmetry approach to classification of nonlinear equations.The complete list of integrable systems.*Usp.Mat.Nauk*.42,3-53,1987 (in Russian).
2. Gerdt V.P.,Shabat A.B.,Svinolupov S.I.,Zharkov A.Yu. Computer algebra application for investigating integrability of nonlinear evolution systems. In: ."EUROCAL'87",Lecture Notes in Computer Science 378,81-92,Springer-Verlag,Berlin,1989.

3. Gerdt V.P.,Zharkov A.Yu. Computer generation of necessary integrability conditions for polynomial-nonlinear evolution equations,Preprint JINR E5-90-47, Dubna,1990.Submitted to ISSAC'90, Tokyo, Japan.
4. Buchberger B. Groebner basis: a method in symbolic mathematics. In: (Bose N.K.,ed.) Multidimensional Systems Theory, 184-232, Dordrecht: Reidel,1985.
5. Boege W.,Gebauer R.,Kredel H. Some examples for solving systems of algebraic equations by calculating Groebner basis,*J. Symb. Comp.* 2, 83-98,1986.
6. Kobayashi H.,Fujise T.,Furukawa A. Solving systems of algebraic equations by a general elimination method, *J. Symb. Comp.* 5,303-320, 1988.
7. Kobayashi H.,Moritzugu S.,Hogan R. Solving systems of algebraic equations,in "ISSAC'88",Lecture Notes in Computer Science 358, 139-149,Springer-Verlag,Berlin,1989.
8. Czapor S.R. Solving algebraic equations: combining Buchberger's algorithm with multivariate fectorization,*J. Symb. Comp.* 7,49-53,1989.
9. Melenk H.,Möller H.M.,Neun W. On Groebner bases computation on a supercomputer using REDUCE,Konrad-Zuse-Zentrum für Informations-technik Berlin,Preprint SC 88-2,1988.
10. Ito M.,Kako F. A REDUCE program for finding conserved densities of partial differential equations with uniform rank,*Comp. Phys. Comm.*, 38,415-419,1985.
Ito M. A REDUCE program for finding symmetries of nonlinear evolution equations with uniform rank,*Comp. Phys. Comm.* 42,351-375, 1986.
11. Gerdt V.P.,Kostov N.A.,Zharkov A.Yu. Nonlinear evolution equations and solving of algebraic systems: the importance of computer algebra,Preprint JINR,E5-89-624,Dubna,1989.
12. Kredel H.,Weispfenning V. Computing dimension and independent sets for polynomial ideals,*J. Symb. Comp.* 6,231-247,1988.
13. Gerdt V.P.,Shvachka A.B.,Zharkov A.Yu. Classification of integrable high-order KdV-like equations, Preprint JINR R5-84-489,Dubna,1984.
14. Gerdt V.P.,Zharkov A.Yu. Computer classification of integrable seventh-order MKdV-like equations.In: "EUROCAL'87",Lecture Notes in Computer Science 378,93-94,Springer-Verlag,Berlin,1989.
15. Gerdt V.P.,Zharkov A.Yu. Computer classification of integrable coupled KdV-like systems, Preprint JINR E5-89-232,Dubna,1989. To be published in *J. Symb. Comp.*

Received by Publishing Department
on January 25, 1990.