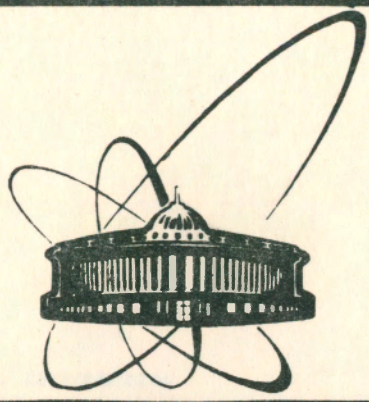


90-47

27



Объединенный  
Институт  
Ядерных  
Исследований  
Дубна

G 38

E5-90-47

V.P.Gerdt, A.Yu.Zharkov\*

COMPUTER GENERATION  
OF NECESSARY INTEGRABILITY CONDITIONS  
FOR POLYNOMIAL-NONLINEAR  
EVOLUTION SYSTEMS

Submitted to "International Symposium  
on Symbolic and Algebraic Computation,  
Tokyo, Japan, 20-24 August 1990

\*Saratov State University, Saratov, USSR

## 1. INTRODUCTION

Symmetry analysis is one of the domains in the theory of (in particular partial) differential equations to which computer algebra can be applied successfully [1]. The most general approach to solve these equations is based on the knowledge of their classic or Lie symmetries which are connected with the one-parametric group of transformation. Apart from very simple cases, these symmetries cannot be found in practice without a computer. In recent years, great attention has been paid also to the problem of finding the so-called generalized or higher infinitesimal (Lie-Bäcklund) symmetries [2] by computer [3,4]. These symmetries reveal important internal properties of the equations under consideration and are very useful for the construction of the exact solution and for their qualitative investigation.

Amongst the partial differential equations of physical interest, an important role is played by the class of integrable nonlinear evolution equations (NLEE). These equations possess a number of remarkable properties, for example, soliton solutions. Their integrability is closely connected with the existence of higher symmetries and conservation laws [5-7]. The symmetry approach to the investigation of integrability of NLEE is developed mainly [5] for equations in one-temporal and one-spatial dimensions. For instance, this approach can be applied effectively [6] to the systems of NLEE of the following form

$$\begin{aligned}u_t &= \Phi(x, u, \dots, u_N) = Au_N + F(x, u, \dots, u_{N-1}), \quad N \geq 2 \\u &= u(t, x), \quad u_x = \partial^k u / \partial x^k, \quad u = (u^1, \dots, u^M), \quad F = (F^1, \dots, F^M), \\A &= \text{diag}(\delta_1, \dots, \delta_N), \quad \delta_i \in \mathbb{C}, \quad \delta_i \neq 0, \quad \delta_i \neq \delta_j \quad (i \neq j).\end{aligned} \tag{1}$$

It should be noted that not only those NLEE which can be integrated by the inverse spectral transform but also equations which can be

linearized, like Burger's equation, satisfy the integrability conditions arising in the symmetry approach. These conditions are necessary for the existence of the higher symmetries and conservation laws. This fact plays an important role in the problem of classification of integrable NLEE, i.e., in obtaining a complete list of integrable equations of a given form, for example, some subset of (1), and in finding the invertible transforms connecting the equations in the obtained list [5].

In [6], algorithms are described for verifying the necessary integrability conditions for nonlinear evolution systems (1) and for finding higher symmetries and conservation laws. These algorithms have been implemented on a computer by using the algebra system FORMAC.

In the present paper we consider an important subclass of (1), namely the case when  $F$  is a polynomial. This subclass includes in particular the NLEE with uniform rank for which in the scalar case ( $M=1$  in (1)) packages in REDUCE for finding symmetries and conservation law densities have been developed [8]. The restriction for  $F$  being a polynomial has given us the possibility to establish an efficient program which uses REDUCE internal representation and built-in facilities for polynomial manipulations [9]. This program makes it possible to verify the necessary integrability conditions for (1), in the case where the r.h.s. in (1) has arbitrary parameters as coefficients, and to generate an equivalent system of algebraic equations for these parameters.

## 2. BASIC DEFINITIONS AND FORMULAS

We recall some definitions of the symmetry approach and give an essential set of the necessary formulas (for more details see [5,6]).

A vector function  $f=(f^1, \dots, f^M)$  of a finite number of dynamical variables taken from an infinite set  $x, u, u_1, \dots$  is a (generator of infinitesimal) symmetry of the system (1) if it leaves (1) invariant under the transformation  $t'=t, x'=x, u'=u+\tau f(x, u, u_1, \dots)$  within order  $\tau$ . This means that  $f$  satisfies

$$\frac{df}{dt} = \Phi_*(f), \quad (2)$$

where  $\Phi_*$  is a matrix differential operator

$$\Phi_* = \Phi_u + \Phi_{u_1} D + \dots + \Phi_{u_N} D^N, \quad [\Phi_{u_i}]_{k,j} = \frac{\partial \Phi^k}{\partial u_i^j}, \quad (3)$$

and where  $d/dt$  and  $D=d/dx$  are the total differentiation operators with respect to  $t$  and  $x$ , respectively,

$$\frac{d}{dt} = \sum_{i=1}^N \sum_{j=0}^{\infty} D^j (\Phi^i) \frac{\partial}{\partial u_i^j}, \quad D = \frac{\partial}{\partial x} + \sum_{i=1}^N \sum_{j=0}^{\infty} u_i^{j+1} \frac{\partial}{\partial u_i^j}. \quad (4)$$

Equation (2) is equivalent to the operator relation

$$L_t - [\Phi_*, L] = (\Phi_*)_{\tau}, \quad (5)$$

where

$$L = f_* = f_u + f_{u_1} D + \dots + \sum_{k=0}^n f_{u_k} D^k, \quad \deg(L) = n, \quad (6)$$

is the operator series (compare with (3)) of degree  $n$  and  $d/dt$  is defined by (4) with  $f$  instead of  $\Phi$ , i.e.  $d/dt = d/dt|_{\Phi \rightarrow f}$ .

Theorem 1 ([5,6]). The existence of an  $n_1$ -order symmetry with  $n_1 > N = \deg(\Phi)$  implies the existence of an  $n$ -order formal symmetry with  $n_1 \geq n > N$ . This is a formal series of degree  $n$  with matrix coefficients  $A_k$  depending on a finite set of dynamical variables  $x, u_i$ :

$$L = \sum_{k=m}^n A_k D^k = A_m D^m + \dots + A_0 + A_{-1} D^{-1} + \dots, \text{res}(L) = A_{-1} \quad (7)$$

with

$$\deg(L_t - [\Phi_*, L]) \leq m + N - n. \quad \square \quad (8)$$

From (1), (7), (8) it follows that

$$A_m = \text{diag}(\mu_1, \mu_2, \dots, \mu_N), \quad \mu_i \in \mathbb{C}. \quad (9)$$

A formal symmetry is called non-degenerated if  $\det A_m = \prod_k \mu_k \neq 0$  and degenerated otherwise. It is easy to see that formal symmetry is defined up to the addition of an arbitrary diagonal matrix (this can be eliminated by the normalization  $\text{diag} A_0 = 0$ ).

From (5)-(8) it is evident that (3) is not only the  $N$ -order

symmetry but also the N-order formal symmetry. The conditions for the existence of formal symmetries of higher order are given by a theorem which is of importance in our computations:

**Theorem 2** ([5,6]). Suppose that a non-generated formal symmetry of order  $n=N+i$ ,  $i \geq 0$  exists. Then a formal symmetry of order  $n=N+i+1$  exists iff

$$\frac{d}{dt}(R(i,j)) \in \text{Im}D, \quad j=1,2,\dots,M, \quad (10)$$

where

$$R(i,j) = \begin{cases} \frac{\partial F}{\partial u_{N-1}^j}, & i=0 \\ \frac{\partial}{\partial \mu_j} \text{trace}(\text{res}(L)), & i>0. \end{cases} \quad (11)$$

$L$  is a formal symmetry of order  $i+2$  and degree  $i$  with the leading coefficient (7) depending on  $M$  arbitrary parameters  $\mu_1, \mu_2, \dots, \mu_M$ , and  $\text{res } L \equiv A_{-1}$ .

Condition (10) means that the l.h.s. is a total derivative with respect to  $x$  of some function in a finite number of dynamical variables. In other words the expressions  $R(i,j)$  in (11) are conservation law densities of the system (1). For integrable NLEEs, formula (11) determines the algorithm for constructing the infinite series of the conservation law densities, which is called the canonical series. The coefficients  $A_k$  of (7) are calculated from the following recurrence relations which can be obtained [6] from (1), (3), (7), (8):

$$\begin{aligned} A_k &= 0, \quad k > m, \\ [A_m]_{1j} &= \begin{cases} 0, & i \neq j \\ \mu_1, & i = j \end{cases} \\ [A_k]_{1j} &= \begin{cases} \frac{1}{\lambda_1 - \lambda_j} \left[ \frac{d}{dt}([A_{N+k}]_{1j}) - [C_{N+k}]_{1j} \Big|_{A_k=0} \right], & i \neq j \\ \frac{1}{N\lambda_1} D^{-1} \left[ \frac{d}{dt}([A_{N+k-1}]_{11}) - [C_{N+k-1}]_{11} \Big|_{\text{diag}(A_k)=0} \right] + \nu_1^k, & i = j, \end{cases} \end{aligned} \quad (12)$$

where  $\mu_1, \nu_1^k \in \mathbb{C}$  are arbitrary constants, and where  $C_k$  are the coefficients of the commutator  $[\Phi_x, L] = \sum C_k D^k$ .

### 3. ALGORITHMS AND IMPLEMENTATION

According to Theorems 1,2 of Sect.2 the algorithm for verifying necessary integrability conditions, i.e., the existence of higher order symmetries, is reduced to the successive computation (for  $i=0,1,2,\dots$ ) of the expressions  $R(i,j)$  from (11) and to the test whether they fulfill the conditions (10). One starts by testing the  $M$  conditions (10) for  $i=0$  (first integrability conditions). These conditions are equivalent (Theorem 2) to the existence of a  $N+1$ -order non-degenerated formal symmetry. Then one constructs the formal symmetry of order 1 and degree 3 and tests the  $M$  conditions (10) for  $i=1$  (second integrability conditions). If these conditions are satisfied then the  $N+2$ -order formal symmetry also exists and so on. The elements of the matrix coefficients  $A_k$  for the  $m$ -th degree formal symmetry (7) are found from (12).

The above algorithm for the computation of the canonical law densities and for verifying the integrability conditions has been implemented using the symbolic mode of REDUCE (version 3.2). The implementation is based on the built-in recursive representation for polynomials in "standard form" (s.f.) and effectively uses the following ordering rule for variables ("kernels")

$$v < u(i,j) < u(i,j+1) < u(i+1,k), \quad i,j,k=0,1,2,\dots, \quad (13)$$

where the kernel  $u(i,j)$  represents the dynamical variable  $u_i^j$  and  $v$  can be any other variable. The statement

ALGEBRAIC KORDER u\$

defines the ordering for polynomials in accordance with (13).

Algebraic operations over polynomials and their derivatives are realized in symbolic mode by calls to the corresponding built-in procedures acting at s.f. and "standard quotients" (s.q.).

The most laborious step in the above computational scheme is the application of the operator  $D^{-1}$ , i.e., the integration with respect to the spatial variable  $x$  in (12). The problem is to find  $Q$  as a solution

of the equation

$$D(Q) = S, \quad (14)$$

where  $S$ , in general, is a polynomial in the variables  $u(i,j)$  with a finite number of arbitrary parameters. From (14) it follows [6] that

$$\frac{\partial S}{\partial u^i} = 0, k=0; \quad \frac{\partial^2 S}{\partial u_k^i \partial u_k^j} = 0, \quad \frac{\partial^2 S}{\partial u_k^i \partial u_{k-1}^j} - \frac{\partial^2 S}{\partial u_k^j \partial u_{k-1}^i} = 0, \quad k>0 \quad (i,j=1,2,\dots,M).$$

If these conditions are satisfied, then  $S$  can be rewritten as

$$S = Dq + \bar{S}, \quad \text{ord}(\bar{S}) < \text{ord}(S). \quad (15)$$

After having determined  $q$  we can represent the solution of (14) as  $Q=q+\bar{Q}$ , where  $\bar{Q}$  satisfies the equation  $D(\bar{Q}) = \bar{S}$ . Thus the condition  $S \in \text{Im}D$  is reduced to the condition  $\bar{S} \in \text{Im}D$  with the order of  $\bar{S}$  lower than the order of  $S$ .

The transformation (15) is the basis of our algorithm INTX which performs the operation  $D^{-1}$ . As a result we obtain either the explicit form of the integral in  $x$ , possibly together with an additional set of equations in arbitrary parameters which represent conditions for the existence of the integral, or the message "non-integrable expression" which means that the input expression cannot be represented as a total derivative with respect to  $x$  of another expression. As an example for the first case, the expression

$$\alpha * u(0,1) * u(2,2) + \beta * u(1,1) * u(2,1)$$

can be represented as a total derivative  $D(\alpha * u(0,1) * u(1,2))$  only if  $\alpha = \beta$ .

When describing the algorithm INTX, we use the following internal REDUCE notations [9]:

- mvar (main variable),
- ldeg (leading degree),
- lt (leading term),
- lc (leading coefficient),
- red (reductum).

Indices  $i,j$  of  $u(i,j)$  can be arbitrary non-negative integers.

Algorithm: INTX

Input :S % polynomial in u(i,j) and possibly in some parameters

Output:either  $D^{-1}(S)$  and equations in the parameters,

or the message 'non-integrable expression' ( $S \notin \text{Im}D$ )

r:=0; % r accumulates the value of  $D^{-1}(S)$

while S $\neq$ 0 do

begin

while (mvar S=u(i,j) and ldeg S>1) or mvar S=u(0,j) do

begin

split lt S;

S:=red S;

end;

if mvar S $\neq$ u(i,j) then return S;

% below mvar S=u(i,j), i>0

C:=lc S;

S:=red S;

while mvar C>u(i-1,j) do

begin

split lt C;

c:=red C;

end;

q:=int(C,u(i-1,j));

r:=r+q;

S:=S+C\*u(i-1,j)-D(q); % lowering of order of S

end;

Algorithm INTX requires only derivation and elementary operations on polynomial. It calls two procedures:

(i) int performs the polynomial integration  $\int (\partial S / \partial u_k^1) du_{k-1}^1$  which is needed for finding the function q in (15). We note that int integrates in a variable which has at least the order of the main variable of the integrand. It requires the list processing only at the highest list level and does not use the REDUCE integrator.

(ii) split (<expression = polynomial in u(i,j)>) picks out the coefficients of the different combinations of u(i,j) and sets them equal to zero. If a numerical coefficient different from zero remains, then the message "non-integrable expression" appears.



#### 4. SOME APPLICATIONS. CONCLUSION

We have used our program to compute several first conservation law densities (11) and to generate, from (10), the necessary integrability conditions in the form of algebraic equations for the following NLEEs [10] (all  $\lambda_i \in \mathbb{C}$  are parameters):

1. The seventh order scalar NLEE of KdV type ( $M=1, N=7$  in (1))

$$u_t = u_7 + \lambda_1 u u_5 + \lambda_2 u_1 u_4 + \lambda_3 u_2 u_3 + \lambda_4 u^2 u_3 + \lambda_5 u u_1 u_2 + \lambda_6 u_1^3 + \lambda_7 u^3 u_1. \quad (16)$$

For (16) we have computed the densities (11) for  $i=0, 2, 4, 6$  (odd  $i$  do not lead to any restrictions on  $\lambda_i$ ) and have obtained a system of thirteen algebraic equations in seven variables  $\lambda_i$ .

2. The seventh order scalar NLEE of a modified KdV (MKdV) type ( $M=1, N=7$  in (1))

$$u_t = u_7 + \lambda_1 (u_1 u_5 + 3u_2 u_4 + 2u_3^2) + \lambda_2 (u^2 u_5 + 6u u_1 u_4 + 10u u_2 u_3 + 6u_1^2 u_3 + 7u_1 u_2^2) + \lambda_3 (2u_1 u_2^2 + u_1^2 u_3) + \lambda_4 (3u^2 u_1 u_3 + 12u u_1^2 u_2 + 3u^2 u_2^2 + 2u_1^4) + \lambda_5 (u^4 u_3 + 8u^3 u_1 u_2 + 6u^2 u_1^3) + \lambda_6 u^6 u_1. \quad (5)$$

The densities for  $i=0, 2, 4$  (odd  $i$  give no restrictions) lead to a system of nine algebraic equations in six variables.

3. The system of two coupled nonlinear equations of KdV type ( $M=2, N=3$  in (1)) which satisfies to conditions (10) for  $i=0, 1, 2, 3$  [10]

$$\begin{aligned} u_t &= \lambda_1 u_3 + \lambda_5 u u_1, \\ v_t &= (\lambda_1 - 1)v_3 + \lambda_2 u u_1 + \lambda_3 v u_1 + \lambda_4 u v_1. \end{aligned} \quad (18)$$

The conditions (10) for  $i=4, 5, 6, 7$ ;  $j=1, 2$  generate a system of four algebraic equations in five variables.

The comparison with our previous (more general-purpose) program [6], written in FORMAC, which has been used for the classification of eqs. (16)-(18) in previous work (see [6],[10] and refs. therein), shows that our present implementation in REDUCE is almost ten times faster, which is in contrast to the sometimes reported opinion that FORMAC is much more effective than REDUCE (see, for example, [3]). Moreover, all our (rather cumbersome) computations have been carried out on an IBM PC AT-like computer.

The algebraic equations which are generated as integrability conditions appear to be complicated and have in general infinitely many solutions [10,11]. However, for the equations (16)-(18) and in other cases they can be solved [11] by using the technique of Groebner basis [12]. Hence, besides their importance for applications, they are very useful for testing other computer algebra methods.

#### REFERENCES

1. Schwarz F. Symmetries of differential equations: from Sophus Lie to computer algebra, *SIAM Rev.* 30,3,450-481,1988.
2. Ibragimov N.H. Transformation groups applied to mathematical physics, Reidel, Boston, 1985.
3. Fushchich W.I., Korniyak V.V. Computer algebra application for determining Lie and Lie-Bäcklund symmetries of differential equations, *J. Symb. Comp.* 7, 611-619, 1989.
4. Bocharov A.V., Bronstein M.L. Efficiently implementing two methods of the geometrical theory of differential equations: an experience in algorithm and software design, *Acta Appl. Math.* 16, 143-166, 1989.
5. Mikhailov A.V., Shabat A.B., Yamilov R.I. Symmetry approach to classification of nonlinear equations. The complete list of integrable systems. *Usp. Mat. Nauk.* 42,3-53,1987 (in Russian).
6. Gerdt V.P., Shabat A.B., Svinolupov S.I., Zharkov A.Yu. Computer algebra application for investigating integrability of nonlinear evolution systems. In: "EUROCAL'87", Lecture Notes in Computer Science 378, 81-92, Springer-Verlag, Berlin, 1989.
7. Fokas A.S. Symmetries and integrability. *Stud. Appl. Math.*, 77: 253-299, 1987.
8. Ito M., Kako F. A REDUCE program for finding conserved densities of partial differential equations with uniform rank, *Comp. Phys. Comm.*, 38, 415-419, 1985.  
Ito M. A REDUCE program for finding symmetries of nonlinear evolution equations with uniform rank, *Comp. Phys. Comm.* 42, 351-375, 1986.
9. Norman A.R. Symbolic and algebraic modes in REDUCE, *REDUCE Newsletter*, 3, 5-9, Univ. of Utah, 1976.  
Edneral V.F., Kryukov A.P., Rodionov A.Ya. The language REDUCE for Analytic Calculations, Chap. 4, Moscow State University, Moscow, 1989.
10. Gerdt V.P., Kostov N.A., Zharkov A.Yu. Nonlinear evolution equations and solving of algebraic systems: the importance of computer algebra, Preprint JINR, E5-89-624, Dubna, 1989.
11. Gerdt V.P., Zharkov A.Yu., Khutornoy N.V. Solving algebraic systems arising as necessary integrability conditions for polynomial-nonlinear evolution equations, Preprint JINR E5-90-48, Dubna, 1990. Submitted to ISSAC'90, Tokyo, 1990.
12. Buchberger B. Groebner basis: a method in symbolic mathematics. In: (Bose N.K., ed.) *Multidimensional Systems Theory*, 184-232, Dordrecht: Reidel, 1985.

Received by Publishing Department  
on January 25, 1990.