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EXACT MULTIMAGNON STATES IN ONE-DIMENSIONAL FERROMAGNETIC SPIN CHAINS WITH A SHORT-RANGE INTERACTION

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After the celebrated Bethe's paper [1] on the solution of Heisenberg chain the next candidate for the role of integrable one-dimensional S=1/2 isotropic quantum spin model with possible applications in thermodynamics of magnetic systems was proposed almost sixty years ago by Haldane [2] and Shastry [3]. Contrary to [1], where the complete description of all possible states was given, in the long-range $1/r^2$ exchange case under periodic boundary conditions [2,3] only some sets of Jastrow-type wave functions were found. Recently, one of us has suggested the more complicated model [4] which for an infinite chain is determined by the one-parametric Hamiltonian

$$H=\frac{J_0}{2}\sum_{j,k=-\infty}^{\infty}\frac{\pi^2}{\kappa^2}\left[\sinh\frac{\pi}{\kappa}(j-k)\right]^{-2}\left(\frac{\vec{\sigma}_j\cdot\vec{\sigma}_k-1}{2}\right),\quad 0<\kappa<\infty. \qquad (1)$$

It contains both the Heisenberg and Haldane-Shastry models as limits at $\kappa\to 0,\infty$ under proper "renormalization" of the coupling J_0. A periodic version of the short-range interaction (I) can be obtained by replacing the trigonometric exchange integrals in (I) to the Weierstrass elliptic $\wp$ functions with the real period as a number of spins N [4]. The few integrals of motion and simplest two-magnon states were described but, similarly to [2,3], neither the origin of hidden symmetry nor the way of constructing all eigenvectors have been indicated.

In this Letter we report on an extension of the famous Bethe Ansatz which allows one to obtain the exact wave functions of M magnons with arbitrary quasimomenta in the ferromagnetic case $J_0<0$ of the model (I). The corresponding eigenvalue problem consists in finding complete symmetric tensors $\psi_{\{n_\alpha\}}$ $(1\le\alpha\le M,\ n_\alpha\in\mathbf{Z})$ obeying the equation

$$\mathfrak{L}\psi_{\{n_\alpha\}}=\sum_{\beta=1}^{M}\sum_{p\neq\{n_\alpha\}}h_{n_\beta p}\psi^{(p,n_\beta)}_{\{n_\alpha\}}+\psi_{\{n_\alpha\}}\left(\sum_{\beta\neq\gamma}^{M}h_{n_\beta n_\gamma}-J_0^{-1}\epsilon^{(M)}-M\epsilon_0\right)=0, \qquad (2)$$

where $h_{jk}=\left[\frac{\kappa}{\pi}\sinh\frac{\pi}{\kappa}(j-k)\right]^{-2}$, $\epsilon_0=\sum_{j\neq0}h_{j0}$ and $\psi^{(p,n_\beta)}_{\{n_\alpha\}}$

denotes the tensor obtained from $\psi_{\{n_\alpha\}}$ by replaicing the βth index n_β to p. As the eigenvectors of (I) have the form $|\psi_M\rangle = \sum_{\{n_\alpha\}} \psi_{\{n_\alpha\}} |\{n_\alpha\}\rangle$, where in the states $|\{n_\alpha\}\rangle$ spins at positions $\{n_\alpha\}$ are turned over ferromagnetic ground state, $\psi_{\{n_\alpha\}}$ should vanish for any pair of coinciding $\{n_\alpha\}$. In the case M = 2, the solution to (2) given in [4] can be written in the compact Bethe-like form

$$\psi_{n_1 n_2} = \sum_{P \in \pi_2} \exp\left(i \sum_{\alpha=1}^{2} k_{P_\alpha} n_\alpha\right) \frac{S_{12}^{(P)}}{s_{12}} . \tag{3}$$

Here $\{k_\alpha\}$ are magnon quasimomenta ($|\mathrm{Im}\, k_\alpha| < 2\pi/\kappa$), π_M is the group of all permutations $\{P\}$ of M indices and

$$S_{\alpha\beta}^{(P)} = \sinh\left[\frac{\pi}{\kappa}(n_\alpha - n_\beta) + \gamma(k_{P_\alpha}, k_{P_\beta})\right], \quad s_{\alpha\beta} = \sinh\frac{\pi}{\kappa}(n_\alpha - n_\beta) . \tag{4}$$

The two-magnon energy is $\epsilon^{(2)} = J_0\,[\epsilon(k_1) + \epsilon(k_2)]$ with

$$\epsilon(k) = -\Big\{ \wp(r_k) + 2\left[\zeta(r_k) - \frac{2r_k}{\omega}\zeta\left(\frac{\omega}{2}\right)\right] \times \\ \times \left[\zeta(2r_k) - \zeta(r_k) - \frac{2r_k}{\omega}\zeta\left(\frac{\omega}{2}\right)\right] + \frac{2}{\omega}\zeta\left(\frac{\omega}{2}\right)\Big\}, \tag{5}$$

where $\omega = i\kappa$, $r_k = -k\omega(4\pi)^{-1}$ and $\wp(x)$, $\zeta(x)$ are the usual Weierstrass elliptic functions with the periods 1 and ω. The phase shift in (4) depends on quasimomenta as

$$\coth\gamma(k_1, k_2) = \frac{\kappa}{2\pi}\left[f(k_1) - f(k_2)\right], \quad f(k) = \frac{k}{\pi}\zeta\left(\frac{\omega}{2}\right) - \zeta\left(\frac{k\omega}{2\pi}\right). \tag{6}$$

At $M \geq 2$ one can try to find the solution of (2) in the form of Bethe Ansatz, i.e. the symmetrized product of two-magnon amplitudes

$$\psi^{(0)}_{\{n_\alpha\}} = \sum_{P \in \pi_M} \Phi^{(0)}(P;\{n_\alpha\}), \quad \Phi^{(0)}(P;\{n_\alpha\}) = \exp\left(i\sum_{\alpha=1}^{M} k_{P\alpha} n_\alpha\right) \times \\ \times \prod_{\mu>\nu}^{M} \frac{S_{\mu\nu}^{(P)}}{s_{\mu\nu}} . \tag{7}$$

To calculate the sums in the left-hand side of (2), one needs the formula which can be obtained in the theory of elliptic functions,

$$\sum_{p\neq 0,\{-\ell_\alpha\}}^{\infty} \exp(ikp)\left[\frac{\kappa}{\pi}\sinh\frac{\pi}{\kappa}p\right]^{-2}\prod_{\alpha=1}^{K}\left[\sinh\frac{\pi}{\kappa}(p+\ell_\alpha)\right]^{-1}\times$$

$$\times\prod_{\beta=1}^{K-2J}\sinh\left[\frac{\pi}{\kappa}(p+\ell_\beta)+\Delta_\beta\right]=$$

$$=\prod_{\alpha=1}^{K}\left[\sinh\frac{\pi}{\kappa}\ell_\alpha\right]^{-1}\prod_{\beta=1}^{K-2J}\sinh\left(\frac{\pi}{\kappa}\ell_\beta+\Delta_\beta\right)\Big\{\epsilon(k)+\epsilon_0+\frac{\pi}{\kappa}f(k)\,\phi_{JK}- \tag{8}$$

$$-\frac{\pi^2}{2\kappa^2}\Big[\phi_{JK}^2+\sum_{\lambda=1}^{K}\left(\sinh\frac{\pi\ell_\lambda}{\kappa}\right)^{-2}-\sum_{\lambda=1}^{K-2J}\left[\sinh\left(\frac{\pi\ell_\lambda}{\kappa}+\Delta_\lambda\right)\right]^{-2}\Big]\Big\}+$$

$$+\frac{\pi}{\kappa}f(k)\left(\sum_{\lambda=1}^{K-2J}\eta_\lambda\sinh\Delta_\lambda+\sum_{\lambda=K-2J+1}^{K}\eta_\lambda\right)-$$

$$-\frac{\pi^2}{\kappa^2}\left(\sum_{\lambda=1}^{K-2J}\eta_\lambda\sinh\Delta_\lambda(\coth\Delta_\lambda+\xi_\lambda)+\sum_{\lambda=K-2J+1}^{K}\eta_\lambda\xi_\lambda\right).$$

Here K, J, $\{\ell_\alpha\}$ are $K+2$ integers such that $K-2J\geq 0$, $\prod_{\beta>\gamma}^{K}(\ell_\beta-\ell_\gamma)\neq 0$, $K-2J+1$ complex numbers $\{\Delta_\beta\}$, $\mathbf{k}$ are restricted to the strips $|\mathrm{Im}\,\Delta_\beta|<2\pi$, $|\mathrm{Im}\,k|<\frac{2\pi}{\kappa}(1+J)$, $\epsilon(k)$ and $f(k)$ are as in (5), (6) and

$$\phi_{JK}=\sum_{\lambda=1}^{K-2J}\coth\left(\frac{\pi}{\kappa}\ell_\lambda+\Delta_\lambda\right)-\sum_{\lambda=1}^{K}\coth\frac{\pi\ell_\lambda}{\kappa},$$

$$\eta_\lambda=\exp(-ik\ell_\lambda)\left[\sinh\frac{\pi\ell_\lambda}{\kappa}\right]^{-2}\prod_{\nu\neq\lambda}^{K-2J}\sinh\left[\frac{\pi}{\kappa}(\ell_\nu-\ell_\lambda)+\Delta_\nu\right]\times$$

$$\times\prod_{\rho\neq\lambda}^{K}\left[\sinh\frac{\pi}{\kappa}(\ell_\rho-\ell_\lambda)\right]^{-1},$$

$$\xi_\lambda=2\coth\frac{\pi\ell_\lambda}{\kappa}+\sum_{\nu\neq\lambda}^{K-2J}\coth\left[\frac{\pi}{\kappa}(\ell_\nu-\ell_\lambda)+\Delta_\nu\right]-\sum_{\nu\neq\lambda}^{K}\coth\frac{\pi}{\kappa}(\ell_\nu-\ell_\lambda).$$

After calculating $\mathfrak{L}\psi^{(0)}_{\{n_\alpha\}}$ by the use of (8) at J=0 we find that, contrary to the Heisenberg case, (7) is not a solution of (2). If the energy of M magnons is chosen as

$$\epsilon^{(M)} = J_0 \sum_{\alpha=1}^{M} \epsilon(\mathbf{k}_\alpha) ,$$

the left-hand side of (2) has the form

$$\mathfrak{L}\psi^{(0)}_{\{n_\alpha\}} = \frac{\pi^2}{\kappa^2} \sum_{P\in\pi_M} \Phi^{(0)}(P;\{n_\alpha\}) \sum_{\mu\neq\nu\neq\rho} \frac{C^{(P)}_{\mu\nu}\, C^{(P)}_{\mu\rho}\, C^{(P)}_{\nu\rho}}{S^{(P)}_{\mu\nu}\, S^{(P)}_{\mu\rho}\, S^{(P)}_{\nu\rho}} \Big\{ \frac{1}{6}(2 +$$

$$+ d_{\mu\nu}\mathfrak{T}^{(P)}_{\mu\nu} + d_{\mu\rho}\mathfrak{T}^{(P)}_{\mu\rho} + d_{\nu\rho}\mathfrak{T}^{(P)}_{\nu\rho}) - \frac{1}{2}\exp[\, ik_{P\mu}(n_\nu - n_\mu)]\, \mathfrak{T}^{(P)}_{\mu\nu} \times \tag{9}$$

$$\times (d_{\mu\nu} + d_{\nu\rho}) \prod_{\lambda\neq\mu,\nu,\rho}^{M} \sinh\Big[\frac{\pi}{\kappa}(n_\nu - n_\lambda) + \gamma(k_{P\mu}, k_{P\lambda})\, s_{\mu\lambda}\, (S^{(P)}_{\mu\lambda}\, s_{\nu\lambda})^{-1} \Big\},$$

where $s_{\mu\lambda}, S^{(P)}_{\mu\lambda}$ are given by (4) and $d_{\mu\nu} = \coth\frac{\pi}{\kappa}(n_\mu - n_\nu)$, $C^{(P)}_{\mu\nu} = \sinh\gamma(k_{P\mu}, k_{P\nu})$, $\mathfrak{T}^{(P)}_{\mu\nu} = \coth\gamma(k_{P\mu}, k_{P\nu})$. Due to the factors of the type $C^{(P)}_{\mu\nu}[S^{(P)}_{\mu\nu}]^{-1}$ (9) falls off exponentially with the increase of the distances $|n_\alpha - n_\beta|$ between all turned spins.

So (7) looks like a good approximation to the genuine solution. Motivated by the structure of (9) we proposed for the exact wave functions of M magnons with an arbitrary quasimomenta the extended Ansatz

$$\psi_{\{n_\alpha\}} = \sum_{P\in\pi_M} \Phi^{(0)}(P;\{n_\alpha\})\,[\,1 + \sum_{L=3}^{M} \Omega_L\,] , \tag{10}$$

where each term Ω_L vanishes if all $|n_\alpha - n_\beta|$ tend to infinity and can be represented as the sum Σ'_L over L indices varying from 1 to M so that their values do not coincide. The only requirement which determines the explicit form of Ω_L consists in the following: the contribution of Ω_L to the left-hand side of (2) must cancel in it all terms containing the sums of the type Σ'_L which arise from the previous $\{\Omega_\ell\}$ with $\ell < L$. For example, the contribution of $\psi^{(0)}_{\{n_\alpha\}}$ (9) which is of the type Σ'_3 is cancelled if one chooses Ω_3 as

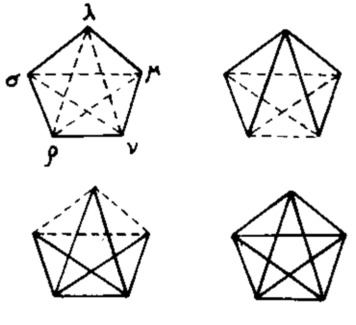
$$\Omega_3 = \frac{1}{12} \sum_{\mu\neq\nu\neq\rho}^{M} \frac{C^{(P)}_{\mu\nu}\, C^{(P)}_{\mu\rho}\, C^{(P)}_{\nu\rho}}{S^{(P)}_{\mu\nu}\, S^{(P)}_{\mu\rho}\, S^{(P)}_{\nu\rho}}, \tag{11}$$

but various sums with the structure $\Sigma_4', \Sigma_5', \Sigma_6'$ appear. By straightforward calculations with the use of (8) with J=1 we find also the next term,

$$\Omega_4 = \frac{1}{32} \sum_{\mu\neq\nu\neq\rho\neq\lambda}^{M} \frac{C^{(P)}_{\mu\nu}\, C^{(P)}_{\mu\lambda}\, C^{(P)}_{\rho\nu}\, C^{(P)}_{\rho\lambda}}{S^{(P)}_{\mu\nu}\, S^{(P)}_{\mu\lambda}\, S^{(P)}_{\rho\nu}\, S^{(P)}_{\rho\lambda}} \times \tag{12}$$

$$\times \{1 - \frac{\sinh[\frac{\pi}{\kappa}(n_\mu - n_\rho) + \gamma(k_{P_\nu}, k_{P_\lambda})] \sinh[\frac{\pi}{\kappa}(n_\nu - n_\lambda) - \gamma(k_{P_\mu}, k_{P_\rho})]}{S^{(P)}_{\mu\rho}\, S^{(P)}_{\nu\lambda}} .$$

The complexity of the Ω_L construction essentially grows with L. For $L \geq 5$ we can formulate at this stage only the following general procedure. Let us take on a plane L points supplied by summation indices $\{\mu\}$ with nonequal values. Connecting all these points by full lines so that an even number of them intersect at each point, we obtain a set of graphs $\{G\}$, as in the Figure for L = 5. Let us draw on each graph G also m(G) broken lines so that finally each pair of points must be connected by a line of any type, and denote as F_G the variety of full lines in G. To each line $f_{\alpha\beta} \subseteq F_G$ connecting two points with indices α and β take into corrrespondence the factor $Q_f = C^{(P)}_{\alpha\beta}(S^{(P)}_{\alpha\beta})^{-1}$ falling off exponentially with the growth of $|n_\alpha - n_\beta|$. As for broken lines, construct for each



Graphs representing the structure of Ω_L with L = 5 (see also text). Due to the evenness of the numbers of full lines intersecting at each point, the calculations of sums in (2) containing Ω_L can be performed by the use of (8).

pair (b_ℓ, b_n) of them connecting the points with indices (α,β) and (α',β'), $1 \le \ell, n \le m(G)$, the factor

$$\tilde{Q}_\delta(\ell,n) = \sinh\left[\frac{\pi}{\kappa}(n_\alpha - n_\beta) + \delta(\ell,n)\,\gamma(k_{P\alpha'}, k_{P\beta'})\right](S^{(P)}_{\alpha\beta})^{-1} ,$$

where $\delta(\ell,\ell)=1$, $\delta(\ell,n)$ may be equal to ± 1. The proper structure of Ω_L is now given by

$$\Omega_L = \sum_{\{\mu\}}{}'_L \sum_G \Big(\prod_{f\in F_G} Q_f\Big) \sum_{\tilde{P}\in\pi_{m(G)}} \sum_{\{\delta\}} A_L(G,\tilde{P},\{\delta\}) \prod_{\ell=1}^{m(G)} \tilde{Q}_\delta(\ell,\tilde{P}\ell) . \quad (13)$$

Here $A_L(G,\tilde{P},\{\delta\})$ are some numerical factors which do not depend on κ and magnon quasimomenta. They must be determined from the above definition of Ω_L. The explicit expressions (11) and (12) for $\Omega_{3,4}$ are in accordance with the rule (13) and have a relatively simple form because in both these cases there is only one graph on the type described above.

The number of graphs grows sharply with L ; it is equal to 4 for L = 5 (figure), but already at L = 6 there are 9 graphs with the numbers of full lines from 6 to 12. An effective algorithm for the calculation of $A_L(G,\tilde{P},\{\delta\})$ for $L \ge 5$ is not yet found.

At $M \le 4$ the relations (10)-(12) give the complete solution of the problem of interacting magnons in our model. The scattering matrix is determined only by the two-magnon phase shift (6), as it would be for intergable systems. In the limit $\kappa\to 0$ $\coth\gamma(k_1,k_2) \to \frac{i}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2})$, i.e. one obtains just the expression for the Bethe phase. All Ω_L vanish in that limit because $S^{(P)}_{\mu\nu}$ tend to infinity and $C^{(P)}_{\mu\nu}$ remain finite. So (19) reduces to the Bethe Ansatz. For the complex quasimomenta there are multimagnon bound states as in the infinite ferromagnetic Heisenberg chain. But contrary to it, the connections between the real total quasimomentum $K = \sum_{\alpha=1}^{M} k_\alpha$ and complex relative ones must be obtained through the solution of the system of highly transcendental equations which include elliptic ζ functions

$$\coth\gamma(k_\alpha, k_{\alpha+1}) = 1 , \qquad 1 \le \alpha \le M-1 . \quad (14)$$

The ground-state wave function for given K at the conditions (14) looks simpler than (10) and resembles the Jastrow structures as it can be seen from its explicit form in the simplest nontrivial case M=3,

$$\psi^{(g)}_{n_1 n_2 n_3} = (s_{32} s_{31} s_{21})^{-1} \sum_{P \in \pi_3} \exp[\, i(\sum_{\alpha=1}^{3} k_\alpha n_{P\alpha}) + \frac{\pi}{\kappa}(n_{P1} - n_{P3})](-1)^P , \quad (15)$$

where $(-1)^P$ is the parity of permutation P.

We conclude that, compared to the Heisenberg chain, the model (I) permits the exact investigation of multimagnon states in a more realistic case of non-nearest neighbour (but short-range) spin interaction. The price for that advantage is the large complexity of all calculations especially in the periodic version of the model where any analog of the simple wave functions (15) seems to be impossible. The question of rather a purely theoretical than applied nature concerning the extension of quantum inverse scattering method so as to include the treatment of (I) are, from our point of view, of most interest and still remain unsolved.

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