ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E5-90-370

H.Neidhardt*

MOVING POTENTIALS AND COMPLETENESS OF WAVE OPERATORS

Existence and Completeness

Submitted to "Annales de L'I.H.P., Physique Theorique"

*On leave of absence from Karl-Weierstrass-Institut für Mathematik, AdW der DDR, 1086 Berlin, Mohrenstr. 39, DDR.1

1990

1. INTRODUCTION

In this note we put away the investigations of our generalized charge transfer model studied in [1,2] which is defined as follows. In $\mathfrak{H} = L^2(\mathbb{R}^n)$, $n \geq 1$, we consider the Schrödinger equation

$$i\frac{\partial u}{\partial t} = H(t)u \equiv (H_0 + V(t))u, \qquad u|_{t=s} = u_0, \tag{1.1}$$

where H_0 is the free Hamiltonian given as usual, i.e. $H_0 = -\frac{1}{2}\Delta$, and $\{V(t)\}_{t\in\mathbb{R}^1}$ is a time-dependent perturbation of the form

$$V(t) = \sum_{j=1}^{N} V_j(t), \tag{1.2}$$

where the time-dependent perturbations $\{V_j(t)\}_{j=1}^{N}$, $t \in \mathbb{R}^1$, arise from time-dependent potentials q_j as follows:

$$(V_j(t)f)(x) = q_j(t, x - x_j(t)), \quad f \in \mathfrak{H},\ t \in \mathbb{R}^1, \tag{1.3}$$

$x_j(.)\colon \mathbb{R}^1 \longrightarrow \mathbb{R}^n$, $j = 1,2,\ldots,N$.

In the following, by $C^1_{loc}(\mathbb{R}^m)$ and $C^1_{loc}(\mathbb{R}^m,\mathbb{R}^k)$, $m,k \geq 1$, we denote the sets of all functions defined on $\mathbb{R}^m$ with values in $\mathbb{R}^1$ and $\mathbb{R}^k$, respectively, whose first derivatives exist and are continuous.

ASSUMPTION P. – The potentials q_j, $j = 1,2,\ldots,N$, belong to $C^1_{loc}(\mathbb{R}^{n+1})$ and satisfy the properties

$$|q_j(t,x)| \leq M_j(1+|x|)^{-1-\varepsilon}, \quad (t,x) \in \mathbb{R}^{n+1},\ \varepsilon > 0, \tag{1.4}$$

$$|x||\nabla q_j| \in L^\infty(\mathbb{R}^{n+1}), \quad \lim_{|x|\to\infty} x\nabla q_j(t,x) = 0, \quad t \in \mathbb{R}^1, \tag{1.5}$$

$$\dot q_j \in L^\infty(\mathbb{R}^{n+1}), \quad \sup_{x\in\mathbb{R}^n} |\dot q_j(t,x)| \in L^1(\mathbb{R}^1), \tag{1.6}$$

$j = 1,2,\ldots,N$, where we have used the notation $\dot q_j = \frac{\partial}{\partial t} q_j$.

In the sequel, we are interested in the scattering theory and, therefore, in the behavior of the potentials at infinity. Consequently, we have omitted local singularities of the potentials. But it seems to us quite possible to include local singularities.

The function $x_j(.): \mathbb{R}^1 \longrightarrow \mathbb{R}^n$ can be regarded as a trajectory along which the potentials q_j move. Concerning the trajectories we assume the following.

ASSUMPTION T. – The trajectories $x_j(.)$, $j = 1,2,\ldots,N$, belong to $C^1_{loc}(\mathbb{R}^1,\mathbb{R}^n)$ such that

$$\lim_{t\to\pm\infty} \frac{1}{t} x_j(t) = v_j^\pm, \quad j = 1,2,\ldots,N, \tag{1.7}$$

exist and, moreover,

$$\sup_{\pm t\geq 0} |x_j(t) - v_j^\pm t| < +\infty, \quad j = 1,2,\ldots,N, \tag{1.8}$$

$$\sup_{\pm t>0} |t\dot x_j(t) - v_j^\pm t| < +\infty, \quad j = 1,2,\ldots,N. \tag{1.9}$$

If $q_j \in C^1_{loc}(\mathbb{R}^{n+1})$, q_j, $|\nabla q_j|$ and $\dot q_j \in L^\infty(\mathbb{R}^{n+1})$ as well as $x_j(.) \in C^1_{loc}(\mathbb{R}^1,\mathbb{R}^n)$, $j = 1,2,\ldots,N$, by Proposition 2.2 and Remark 2.1 of [1] with Eq. (1.1) we can associate a unique propagator $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ consisting of unitary operators and obeying the properties of Proposition 2.2 of [1]. Using

this propagator the scattering states are defined as follows.

DEFINITION 1.1. – The state f belongs to the scattering subspace $\mathfrak{h}_\pm^{sc}(s)$, $s \in \mathbb{R}^1$, if for every $R > 0$ we have

$$\lim_{T\to\pm\infty} \frac{1}{T} \int_s^{s+T} dt \,\|F(|X| < R)e^{ix_j(t)P}\, U(t,s)f\|^2 = 0, \tag{1.10}$$

$j = 1,2,\ldots,N$, and if for every $\varepsilon_j^\pm > 0$ there exist $\eta_j^\pm > 0$ and $\tau_j^\pm > 0$ such that

$$\sup_{\pm t>\tau_j^\pm} \|F(|P - v_j^\pm| < \eta_j^\pm)U(t,s)f\| < \varepsilon_j^\pm, \tag{1.11}$$

$j = 1,2,\ldots,N$.

REMARK 1.2. – We note that for the Cesaro mean it is unessential whether the function under the integral is taken by power two or one provided the function is bounded. Thus, it is possible to replace $\|\ldots\|^2$ in (1.10) by $\|\ldots\|$.

In accordance with Enss [3] by $F(.)$ we denote the spectral projection of the self-adjoint operator to the part of the spectrum as indicated in the parenthesis. By X and P we denote the commuting n-tuples $X = \{X_1,X_2, \ldots ,X_n\}$ and $P = \{-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, \ldots, -i\frac{\partial}{\partial x_n}\} = -i\nabla$ of position and impulse operators, respectively.

REMARK 1.3. – (i) If the potentials q_j, $j = 1,2,\ldots, N$, are nonmoving, i.e. $x_j(t) \equiv 0$, $j = 1,2,\ldots,N$, and time-independent, i.e $q_j(t,x) = q_j(x)$, $j = 1,2,\ldots,N$, then condition (1.10) coincides with those of Ruelle [4] and Amrein-Georgescu [5]. Moreover, condition (1.11) is a consequence of (1.10) and Assumption P, as can be seen from [3].

(ii) If the potentials q_j, $j = 1,2,\ldots,N$, are nonmoving but time-dependent, our definition of the scattering subspace coincides with Definition 5.1 of Kitada and Yajima [6]. See also [7,8,9]. As it has been pointed out by Kitada and Yajima the condition (1.11) is essential by a counter example given by Yafaev [10,11]. The same takes place in our case despite the fact that we have a slightly stronger condition (1.6) than Kitada and Yajima.

Therefore, it seems to us that Definition 1.1 is a natural generalization of the definition of the scattering subspace to moving time-dependent potentials.

The goal of the paper is to show the existence of the wave operators $W_\pm(s)$,

$$W_\pm(s) = \underset{t\to\pm\infty}{s\text{-}\lim}\, U(t,s)^* e^{-i(t-s)H_o}, \tag{1.12}$$

and to establish the completeness of them, i.e.

$$\mathfrak{R}(W_\pm(s)) = \mathfrak{H}_\pm^{sc}(s). \tag{1.13}$$

REMARK 1.4. – (i) If the potentials q_j, $j = 1,2,\ldots,N$, are nonmoving and time-independent on account of Remark 1.3 (i), the problem coincides with the existence and completeness problem for short range potentials which is solved.

(ii) If the potentials q_j, $j = 1,2,\ldots,N$, are nonmoving but time-dependent, the problem was solved by Kitada and Yajima [6,12] even for long range potentials.

(iii) If the potentials are moving but time-independent a stronger asymptotic completeness result than (1.13) was proved by Yajima [13], Graf [14], Hagedorn [15] and Wüller [16,17]. It can be shown that the relation (1.13) follows for time-independent short range potentials from [13] or [16,17] but under stronger assumptions concerning the trajectories $x_j(.)$, $j = 1,2,\ldots,N$, and the behavior of the potentials q_j, $j = 1,2,\ldots,N$, at infinity.

The proof of (1.13) relies on a phase space analysis, in particular, on the famous paper of Enss [18] on the propagating properties of quantum observables. We consider only the short-range case. The long-range case will be the contents of a forthcoming paper.

In the following we need the notation $C^\infty(\mathbb{R}^n)$, $n \geq 1$, denoting the set of bounded functions on $\mathbb{R}^n$ which are infinitely often differentiable. By $C_o^\infty(\mathbb{R}^n)$ we denote the subset of functions with compact supports of $C^\infty(\mathbb{R}^n)$. If $\mathcal{M}$ is a closed subset of $\mathbb{R}^n$ we set $C^\infty(\mathbb{R}^n\setminus\mathcal{M}) = \{f \in C^\infty(\mathbb{R}^n) : f|\mathcal{M} = 0\}$ and, similarly, $C_o^\infty(\mathbb{R}^n\setminus\mathcal{M}) = \{f \in C_o^\infty(\mathbb{R}^n) : f|\mathcal{M} = 0\}$.

2. TECHNICAL PRELIMINARIES

For simplicity and since it will be unessential in the following that the trajectories $x_j(.)$ have different asymptotics for past and future we assume throughout this section that $v_j^+ = v_j^- = v_j$, $j = 1,2,\ldots,N$. This agreement has the advantage that instead of (1.8) we have now

$$\sup_{t\in\mathbb{R}^1} |x_j(t) - v_j t| < +\infty, \quad j = 1,2,\ldots,N. \tag{2.1}$$

Basic in the sequel will be the following proposition of Enss.

PROPOSITION 2.1 [18]. – *Let* $g \in C^\infty(\mathbb{R}^1)$ *such that* $g' \in C_o^\infty(\mathbb{R}^1)$. *If* $\operatorname{supp} g \subset (v_o, +\infty]$, *then for any* $k \in \mathbb{N}$ *there is a constant* C_k *such that*

$$\|F(X_1<R+v_o t)e^{-itH_o}\, g(P_1)F(X_1>R)\| \le C_k(1+t)^{-k}, \tag{2.2}$$

$t \ge 0$. *If* supp $g \subset [-\infty, v_o)$, *then*

$$\|F(X_1>R+v_o t)e^{-itH_o}\, g(P_1)F(X_1<R)\| \le C_k(1+t)^{-k}, \tag{2.3}$$

$t \ge 0$. *The constants* C_k *depend on the shape of* g *and on* dist$(v_o,$supp $g)$, *but are independent of* v_o *and* $R \in \mathbb{R}^1$.

Furthermore, in the following we assume that the velocities $v_j = \{v_{1j}, v_{2j}, \ldots, v_{nj}\}$, $j = 1,2,\ldots,N$, are ordered by

$$v_{11} \le v_{12} \le \ldots \le v_{1N}. \tag{2.4}$$

Proposition 2.1 allows one to establish the following

LEMMA 2.2. – *If the conditions* (1.4) *and* (1.7) *are satisfied and if* $g \in C^\infty(\mathbb{R}^1 \setminus \bigcup_{j=1}^{N}\{v_{1j}\})$, $g' \in C_o^\infty(\mathbb{R}^1)$, *then for every* $l = 1,2,\ldots,N$ *and every* $a,b \in \mathbb{R}^1$ *we have*

$$\|V_l(t)e^{-itH_o}\, g(P_1)F(a < X_1 < b)\| \in L^1(\mathbb{R}^1_+, dt). \tag{2.5}$$

Proof. – Fixing l and introducing $\delta_l = \frac{1}{2}$dist$(v_{1l},$supp $g)$ we have to distinguish the following two cases:

(i) supp $g \subset (v_{1l} + \delta_l, +\infty]$

(ii) supp $g \subset [-\infty, v_{1l} - \delta_l)$.

Assuming (i) and applying (2.2) we get

$$\|F(X_1<a+(v_{1l}+\delta_l)t)e^{-itH_o}\, g(P_1)F(a<X_1<b)\| \tag{2.6}$$

$$\in L^1(\mathbb{R}^1, dt).$$

Taking into account the estimate

$$\|V_l(t)e^{-itH_o}\, g(P_1)F(a < X_1 < b)\| \le$$

$$\|V_l(t)F(X_1 > a + (v_{1l} + \delta_l)t)\|\ \|g(P_1)\| + \tag{2.7}$$

$$\|V_l(t)\|\ \|F(X_1<a+(v_{1l}+\delta_l)t)e^{-itH_o}\, g(P_1)F(a< X_1 <b)\|$$

and $\sup_{t\in\mathbb{R}^1}\|V_l(t)\| \le M_l < +\infty$ (see (1.4)) the relation (2.5) follows if we show that

$$\|V_l(t)F(X_1 > a + (v_{1l} + \delta_l)t)\| \in L^1(\mathbb{R}^1_+, dt). \tag{2.8}$$

By (1.4) we get

$$\|V_l(t)F(X_1 > a + (v_{1l} + \delta_l)t)\| \le \tag{2.9}$$

$$M_l \sup_{x_1 > a+(v_{1l}+\delta_l)t} (1 + |x_1 - x_{1l}(t)|)^{-1-\varepsilon}.$$

Since $\lim_{t\to+\infty} \frac{x_{1l}(t)}{t} = v_{1l}$ we find a $t_o > 0$ such that $|x_{1l}(t) - v_{1l}t| < \frac{\delta}{2}t$. Therefore, we get

$$|x_1+a+(v_{1l}+\delta_l)t - x_{1l}(t)| \ge x_1 + a + \tfrac{1}{2}\delta_l t, \tag{2.10}$$

$t > t_o$, $x_1 \ge 0$, which immediately yields the estimate

$$\|V_1(t)F(X_1>a+(v_{11}+\delta_1)t)\| \le M_1(1+a+\tfrac{1}{2}\delta_1 t)^{-1-\varepsilon}, \qquad (2.11)$$

$t > \max\{t_o, -\frac{2a}{\delta_1}\}$. But (2.11) proves (2.8).

The proof for the case (ii) can be done in the same manner using instead of (2.2) the estimate (2.3).■

Lemma 2.2 allows a further refinement. To this end we introduce the intervals $\Delta_o = [-\infty, v_{11})$, $\Delta_j = (v_{1j}, v_{1(j+1)})$, $j = 1,2,\dots,N-1$, and $\Delta_N = (v_{1N}, +\infty]$.

LEMMA 2.3. - *If the conditions* (1.4), (1.7) *and* (1.8) *are satisfied and if* $g \in C^{\infty}(\mathbb{R}^1)$, $g' \in C_o^{\infty}(\mathbb{R}^1)$, $\operatorname{supp} g \subset \Delta_j$, $j = 0,1,2,\dots,N$, *then for every* $l = 1,2,\dots,N$ *we have*

$$\sup_{t>0} \|V_l(t+s)e^{-isH_o} g(P_1)F(\tfrac{X_1}{t} \in \Delta_j)\| \in L^1(\mathbb{R}^1_+, ds). \qquad (2.12)$$

Proof. - Let us introduce the multiplication operator $\tilde{V}_l(t)$ defined by

$$(\tilde{V}_l(t)f)(x) = q_l(t, x + v_l t - x_l(t))f(x), \quad x \in \mathbb{R}^n, \qquad (2.13)$$

$f \in \mathfrak{h}$. Since the formula

$$e^{i(t+s)v_l P} e^{-isH_o} g(P_1)F(\tfrac{X_1}{t} \in \Delta_j) = e^{is\frac{1}{2}v_l^2} \times \qquad (2.14)$$

$$\times\, e^{iv_l x_l} e^{-isH_o} g(P_1 + v_{1l})F(\tfrac{X_1}{t} + v_{1l} \in \Delta_j)e^{-iv_l x} e^{itv_l P}$$

holds, we find

$$\|V_l(t+s)e^{-isH_o} g(P_1)F(\tfrac{X_1}{t} \in \Delta_j)\| = \qquad (2.15)$$

$$\|\tilde{V}_l(t+s)e^{-isH_o} g(P_1 + v_{1l})F(\tfrac{X_1}{t} + v_{1l} \in \Delta_j)\|.$$

If $j \ge l$, then the problem (2.13) will be solved if we show that for $\operatorname{supp} \tilde{g} \subset (0, +\infty]$ we have

$$\sup_{t>0} \|\tilde{V}_l(t+s)e^{-isH_o} \tilde{g}(P_1)F(X_1 \ge 0)\| \in L^1(\mathbb{R}^1_+, ds). \qquad (2.16)$$

If $j < l$, then we have to establish that for $\operatorname{supp} \tilde{g} \subset [-\infty, 0)$ the relation

$$\sup_{t>0} \|\tilde{V}_l(t+s)e^{-isH_o} \tilde{g}(P_1)F(X_1 \le 0)\| \in L^1(\mathbb{R}^1_+, ds) \qquad (2.17)$$

holds.

To prove (2.16) we set $\delta = \frac{1}{2}\operatorname{dist}(0, \operatorname{supp} \tilde{g})$ and $\varrho_l = \sup_{t\in\mathbb{R}^1} |x_l(t) - v_l t|$ which is finite by (2.1). Using Proposition 2.1 we find

$$\|F(X_1 < \delta s)e^{-isH_o} \tilde{g}(P_1)F(X_1 > 0)\| \in L^1(\mathbb{R}^1_+, ds). \qquad (2.18)$$

Hence, on account of the estimate

$$\|\tilde{V}_l(t+s)e^{-isH_o} \tilde{g}(P_1)F(X_1 \ge 0)\| \le$$

$$\|\tilde{V}_l(t+s)F(X_1 > \delta s)\|\|\tilde{g}(P_1)\| + \qquad (2.19)$$

$$\|\tilde{V}_l(t+s)\|\|F(X_1 < \delta s)e^{-isH_o} \tilde{g}(P_1)F(X_1 > 0)\|$$

and $\sup_{t\in\mathbb{R}^1} \|\tilde{V}_l(t)\| \le M_l < +\infty$ the relation (2.16) follows if we show that

$$\sup_{t>0} \|\tilde{V}_1(t+s)F(X_1>\delta s)\| \in L^1(\mathbb{R}^1_+,ds). \tag{2.20}$$

We have

$$\|\tilde{V}_1(t+s)F(X_1>\delta s)\| \le \tag{2.21}$$

$$M_1 \sup_{x_1\ge\delta s} (1 + |x_1 + v_{11}(t+s) - x_{11}(t+s))^{-1-\varepsilon}.$$

If $s > \varrho_1/\delta$ we find the estimate

$$\sup_{t\ge 0} \|\tilde{V}_1(t+s)F(X_1>\delta s)\| \le M_1(1 + \delta s - \varrho_1)^{-1-\varepsilon} \tag{2.22}$$

which obviously yields (2.20).

The relation (2.17) can be proved in the same manner.■

Furthermore, in the following we need a modification of Lemma 2.3.

LEMMA 2.4. - *Let* (1.4), (1.7) *and* (1.8) *be satisfied and let* $g \in C^\infty(\mathbb{R}^1)$ *and* $g' \in C^\infty_o(\mathbb{R}^1)$.
If supp $g \subset \Delta_N$, *then for every* $l = 1,2,\dots,N$ *we have*

$$\sup_{t>0} \|V_1(t+s)e^{-isH_o} g(P_1)F(X_1< v_{11}t)\| \in L^1(\mathbb{R}^1_-,ds). \tag{2.23}$$

If supp $g \subset \Delta_o$, *then for every* $l = 1,2,\dots,N$ *we have*

$$\sup_{t>0} \|V_1(t+s)e^{-isH_o} g(P_1)F_1(X_1>v_{1N}t)\| \in L^1(\mathbb{R}^1_-,ds). \tag{2.24}$$

Proof. - On account of (2.14) we get

$$\|V_1(t+s)e^{-isH_o} g(P_1)F(X_1< v_{11}t)\| = \tag{2.25}$$

$$\|\tilde{V}_1(t+s)e^{-isH_o} g(P_1+ v_{11})F(X_1 + v_{11}t < v_{11}t)\|.$$

Hence, we will prove (2.23) if we show that

$$\sup_{t>0} \|\hat{V}_1(t+s)e^{-isH_o} \tilde{g}(P_1)F(X_1 < 0)\| \in L^1(\mathbb{R}^1_-,ds) \tag{2.26}$$

with supp $\tilde{g} \subset (0,+\infty]$. From (2.3) we obtain the estimate

$$\|F(X_1> \delta s)e^{-isH_o} g(P_1)F(X_1 < 0)\| \le C_k(1-s)^{-k}, \tag{2.27}$$

$s < 0$, where $\delta = \frac{1}{2}$ dist$(0,\text{supp } \tilde{g})$. Using this estimate and repeating previous proof arguments, we immediately prove (2.26). Similarly, we establish (2.24).■

At the end we establish a simple fact.

LEMMA 2.5. - *If* supp $g \subset (v_o,+\infty]$, *then*

$$\operatorname*{s-lim}_{t\to+\infty} F(X_1< v_ot)e^{-itH_o} g(P_1) = 0. \tag{2.28}$$

If supp $g \subset [-\infty,v_o)$, *then*

$$\operatorname*{s-lim}_{t\to+\infty} F(X_1> v_ot)e^{-itH_o} g(P_1) = 0. \tag{2.29}$$

Proof. - Since supp $g \subset (v_o,+\infty]$ there is a $v_o' > v_o$ such that supp $g \subset (v_o',+\infty]$. Applying (2.2) we obviously find

$$\lim_{t\to+\infty} F(X_1 < a + v_o' t)e^{-itH_o} g(P_1)F(a < X_1 < b)f = 0, \quad (2.30)$$

$f \in \mathfrak{h}$. Since $v_o' > v_o$ there is a t_o such that

$$F(X_1 < v_o t)F(X_1 < a + v_o' t) = F(X_1 < v_o t) \quad (2.31)$$

for $t > t_o$ which yields

$$\lim_{t\to+\infty} F(X_1 < v_o t)e^{-itH_o} g(P_1)F(a<X_1< b)f = 0. \quad (2.32)$$

But $\{F(a<X_1< b)f: f \in \mathfrak{h}, a,b \in \mathbb{R}^1\}$ is a dense subset of $\mathfrak{h}$. Consequently, (2.32) implies (2.28).

Similarly we prove (2.29).■

3. EXISTENCE

We start with some general remarks which allow the existence and completeness problem to be simplified.

REMARK 3.1. - Introducing the family $U(t) = U(t,0)$, $t \in \mathbb{R}^1$, and using for the propagator of Eq.(1.1) the representation

$$U(t,s) = U(t)U(s)^*, \quad t,s \in \mathbb{R}^1, \quad (3.1)$$

it is not hard to see that it is enough to consider the case $s = 0$.

REMARK 3.2.- Defining the family $\hat{H}(t) = H(-t)$, $t \in \mathbb{R}^1$, and denoting by $\{\hat{U}(t,s)\}_{(t,s)\in\mathbb{R}^2}$ the corresponding propagator, one can prove that the propagators $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ and $\{\hat{U}(t,s)\}_{(t,s)\in\mathbb{R}^2}$ are related by

$$JU(t,s) = \hat{U}(-t,-s)J, \quad t,s \in \mathbb{R}^1, \quad (3.2)$$

where J denotes the operator of complex conjugation, i.e $(Jf)(x) = \overline{f(x)}$, $f \in \mathfrak{h}$. On account of (3.2) now it is easy to carry over the existence and completeness problem for $W_- = W_-(0)$ to $\hat{W}_+ = \text{s-}\lim_{t\to+\infty} \hat{U}(t)^* e^{-itH_o} = W_-$ where, of course, we have set $\hat{U}(t) = \hat{U}(t,0)$, $t \in \mathbb{R}^1$. Hence, it is enough to consider the time direction $t \to +\infty$.

REMARK 3.3. - Since The Schrödinger equation (1.1) is a local one, its propagator $\{U(t,s)\}_{(t,s)\in\mathbb{R}^1}$ is not influenced for $t \geq s$ by the behavior of $\{H(t)\}_{t\in\mathbb{R}^1}$ for $t \leq s$. Consequently, having other trajectories $\hat{x}_j(.) \in C^1_{loc}(\mathbb{R}^1,\mathbb{R}^n)$ such that

$$x_j(t) = \hat{x}_j(t), \quad t \geq 0, \quad (3.3)$$

and denoting by $\{\hat{U}(t,s)\}_{(t,s)\in\mathbb{R}^2}$ the propagator of the Schrödinger equation whose potentials q_j move along the trajectories $\hat{x}_j(.)$, the propagators $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ and $\{\hat{U}(t,s)\}_{(t,s)\in\mathbb{R}^2}$ coincide for $t,s \geq 0$, in particular, we have $U(t) = \hat{U}(t)$ for $t \geq 0$. Therefore, modifications of the trajectories $x_j(.)$ for $t \leq 0$ have no influence on the wave operators W_+. Hence, it is quite possible to modify the trajectories $x_j(.)$ in such a manner that the wave operators W_+ are not influenced and the conditions $\lim_{t\to-\infty} \frac{1}{t} x_j(t) = v_j^- = v_j^+$, $j = 1,2,...,N$, are fulfilled.

Now we are going to show the existence of the wave operators.

PROPOSITION 3.4. – *If the conditions* $q_j \in C^1_{loc}(\mathbb{R}^1)$, $|\nabla q_j|, \dot{q}_j \in L^\infty(\mathbb{R}^{n+1})$, $j = 1,2,\dots,N$, *as well as* (1.4) *and* (1.7) *are satisfied, then for every* $s \in \mathbb{R}^1$ *the wave operators* $W_\pm(s)$ *exist and obey*

$$\mathcal{R}(W_\pm(s)) \subseteq \mathfrak{h}^{ac}_\pm(s). \tag{3.4}$$

Proof. – On account of the previous remarks we consider only the case $s = 0$, $t \longrightarrow +\infty$, and $v^+_j = v^-_j \equiv v_j$, $j = 1,2,\dots,N$. Furthermore, we assume that the set $\{v_j\}^N_{j=1}$ is ordered by (2.4).

Obviously, the set $\{g(P_1)F(a<X_1<b)f:\ g \in C^\infty(\mathbb{R}^1\setminus \bigcup_{j=1}^{N}\{v_{1j}\}),\ g' \in C^\infty_0(\mathbb{R}^1),\ a,b \in \mathbb{R}^1,\ f \in \mathfrak{h}\}$ is dense in $\mathfrak{h}$. Moreover, we have

$$U(t)^* e^{-itH_0} g(P_1)F(a<X_1<b)f = g(P_1)F(a<X_1<b)f + \tag{3.5}$$
$$i\sum_{j=1}^{N}\int_0^t ds\ U(s)^* V_l(s) e^{-isH_0} g(P_1)F(a<X_1<b)f.$$

Applying Lemma 2.2 we immediately get the existence of W_+. It remains to show (3.4). Since $W_+ = \text{s-}\lim_{t\to+\infty} U(t)^* e^{-itH_0}$ we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \|U(t)W_+f - e^{-itH_0}f\|^2 = 0 \tag{3.6}$$

which yields

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(|X|<R)e^{ix_j(t)P}\{U(t)W_+f - e^{-itH_0}f\}\|^2 = 0, \tag{3.7}$$

$f \in \mathfrak{h}$, for every $j = 1,2,\dots,N$ and every $R > 0$. Consequently, condition (1.10) is satisfied if we can prove the relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(|X|<R)e^{ix_j(t)P} e^{-itH_0} f\|^2 = 0, \tag{3.8}$$

$f \in \mathfrak{h}$, for every $j = 1,2,\dots,N$ and every $R > 0$. Taking into account the formulas

$$e^{ix_j(t)P} e^{-itH_0} = e^{i\frac{x_j(t)^2}{2t}} \cdot e^{i\frac{1}{t}x_j(t)X} e^{-itH_0} e^{-i\frac{1}{t}x_j(t)X} \tag{3.9}$$

and

$$\text{s-}\lim_{t\to\infty} e^{-i\frac{1}{t}x_j(t)X} = e^{-iv_jX} \tag{3.10}$$

we immediately see that (3.8) is fulfilled if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(|X|<R)e^{-itH_0} e^{-iv_jX} f\|^2 = 0,\quad f \in \mathfrak{h}, \tag{3.11}$$

holds for every $j = 1,2,\dots,N$ and every $R > 0$. But the last fact is obvious for the free Hamiltonian $H_0 = -\frac{1}{2}\Delta$.

Since $W_+ = \text{s-}\lim_{t\to+\infty} U(t)^* e^{-itH_0}$ for every $\varepsilon_j > 0$ and every $\eta_j > 0$ there are $\tau_j > 0$ such that

$$\|F(|P-v_j|<\eta_j)\{U(t)W_+f - e^{-itH_0}f\}\| < \frac{1}{2}\varepsilon_j,\quad f \in \mathfrak{h}, \tag{3.12}$$

for $j = 1,2,\dots,N$ and $t > \tau_j$. Furthermore, there is a $\eta_j > 0$ such that $\|F(|P - v_j| < \eta_j)f\| < \varepsilon_j/2$ for every $t \in \mathbb{R}^1$. Hence, by the estimate

$$\|F(|P - v_j| < \eta_j)U(t)W_+f\| \le \tag{3.13}$$

$$\|F(|P-v_j|< \eta_j)(U(t)W_+f - e^{-itH_o} f)\| + \|F(|P-v_j|< \eta_j)f\|$$

we get $\sup_{t>\tau_j} \|F(|P - v_j| < \eta_j)U(t)W_+f\| < \varepsilon_j$, $j = 1,2,\ldots,N$, which proves (1.11).■

4. COMPLETENESS

In this section we show that the inclusion (3.4) can be replaced by an equality.

THEOREM 4.1. - *If the* Assumptions P *and* T *are satisfied, then*

$$\mathfrak{R}(W_\pm(s)) = \mathfrak{h}_\pm^{ac}(s), \quad s \in \mathbb{R}^1. \tag{4.1}$$

Proof. - Again in accordance with the previous remarks we restrict the considerations to $s = 0$, $t \longrightarrow +\infty$ and $v_j^+ = v_j^- \equiv v_j$, $j = 1,2,\ldots,N$.

Let us assume that (4.1) is violated. Consequently, there is a nontrivial $f \in \mathfrak{h}_+^{ac} \ominus \mathfrak{R}(W_+)$, $\mathfrak{h}_+^{ac} \equiv \mathfrak{h}_+^{ac}(0)$. The aim will be to show that necessarily $f = 0$. In order to show this we establish that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\|U(t)f\| = 0. \tag{4.2}$$

Let $v_{11} \le v_{12} \le \ldots \le v_{1N}$ and let the intervals Δ_j, $j = 0,1,\ldots,N$, be defined as before. At first, we assume that $g \in C^\infty(\mathbb{R}^1)$, $g' \in C_o^\infty(\mathbb{R}^1)$ and supp $g \subset \Delta_j$ for some $j = 0,1,2,\ldots,N$. Since $W_+^*f = 0$ the representation

$$F(\frac{X_1}{t} \in \Delta_j)g(P_1)U(t)f =$$
$$F(\frac{X_1}{t} \in \Delta_j)g(P_1)e^{-itH_o} (e^{itH_o} U(t) - W_+^*)f, \quad t \ge 0, \tag{4.3}$$

holds. A simple computation proves the formula

$$(U(t)^*e^{-itH_o} - W_+)e^{itH_o} g(P_1)F(\frac{X_1}{t} \in \Delta_j)h =$$
$$-i \sum_{l=1}^N \int_0^\infty ds\ U(t+s)^*V_l(t+s)e^{-isH_o} g(P_1)F(\frac{X_1}{t} \in \Delta_j)h, \tag{4.4}$$

$h \in \mathfrak{h}$. Applying Lemma 2.3 we see that the integrals of the right-hand side of (4.4) converges in the operator norm uniformly in $t > 0$. Since

$$\|F(\frac{X_1}{t} \in \Delta_j)g(P_1)e^{isH_o} V_l(t+s)U(t+s)f\| \le$$
$$\|V_l(t+s)e^{-isH_o} g(P_1)F(\frac{X_1}{t} \in \Delta_j)\|\ \|f\|, \quad f \in \mathfrak{h}, \tag{4.5}$$

the integral $\int_0^\infty ds\|F(\frac{X_1}{t} \in \Delta_j)g(P_1)e^{isH_o} V_l(t+s)U(t+s)f\|ds$ conver- ges and by (4.3) we have the estimate

$$\|F(\frac{X_1}{t} \in \Delta_j)g(P_1)U(t)f\| \le$$
$$\sum_{l=1}^N \int_0^\infty ds\|F(\frac{X_1}{t} \in \Delta_j)g(P_1)e^{isH_o} V_l(t+s)U(t+s)f\|\ ds, \quad t > 0. \tag{4.6}$$

Since $\sup_{t>0} \|F(\frac{X_1}{t} \in \Delta_j)g(P_1)e^{isH_o} V_l(t+s)U(t+s)f\| \in L^1(\mathbb{R}_+^1,ds)$ we obtain the estimate

$$\frac{1}{T}\int_0^T dt\ \|F(\frac{X_1}{t}\in\Delta_j)g(P_1)U(t)f\| \le \tag{4.7}$$
$$\sum_{l=1}^{N}\int_0^\infty ds\ \frac{1}{T}\int_0^T dt\ \|F(\frac{X_1}{t}\in\Delta_j)g(P_1)e^{isH_o}\ V_l(t+s)U(t+s)f\|.$$

On account of (1.4) for every $\varepsilon > 0$ there is a $R > 0$ such that

$$\sup_{t>0}\|V_l(t)e^{-ix_l(t)P}\ F(|X|>R)\| < \varepsilon/2. \tag{4.8}$$

Taking into consideration Definition 1.1 we find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \|F(|X|<R)e^{ix_j(t+s)P}\ U(t+s)f\| = 0, \tag{4.9}$$

$s > 0$. Therefore, by the estimate

$$\frac{1}{T}\int_0^T dt\ \|F(\frac{X_1}{t}\in\Delta_j)g(P_1)e^{isH_o}\ V_l(t+s)U(t+s)f\| \le$$
$$\|g(P_1)\|\ \|f\|\ \sup_{t>0}\|V_l(t)e^{-ix_l(t+s)P}\ F(|X|>R)\| + \tag{4.10}$$
$$\|g(P_1)\|\ \sup_{t>0}\|V_l(t)\|\ \frac{1}{T}\int_0^T dt\|F(|X|<R)e^{ix_l(t+s)P}\ U(t+s)f\|,$$

$s > 0$, and the relations (4.8) and (4.9) we find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t}\in\Delta_j)g(P_1)e^{isH_o}\ V_l(t+s)U(t+s)f\| = 0 \tag{4.11}$$

for every $l = 1,2,\dots,N$ and $j = 0,1,2,\dots,N$. Lemma 2.3 allows one to apply the dominated convergence theorem which yields

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \|F(\frac{X_1}{t}\in\Delta_j)g(P_1)U(t)f\| = 0 \tag{4.12}$$

for supp $g \subset \Delta_j$ and every $j = 0,1,2,\dots,N$.

We note that on account of (1.7) – (1.9) the trajectories have the properties

$$\sup_{t\in\mathbb{R}^1}|x_j(t) - \dot{x}_j(t)t| < +\infty,\ j = 1,2,\dots,N. \tag{4.13}$$

If $j = 1,2,\dots,N-1$ and supp $g \subset \Delta_j$, then obviously we have $g \in C_0^\infty(\mathbb{R}^1)$ and, consequently, g has a summable Fourier transform. Applying Corollary 4.5 of [2] we find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|(g(\frac{X_1}{t}) - g(P_1))U(t)f\| = 0. \tag{4.14}$$

Notice that the Assumptions T and P are stronger than the corresponding ones of [2]. Using (4.14) we immediately get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t}\in\mathbb{R}^1\backslash\Delta_j)g(P_1)U(t)f\| =$$
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t}\in\mathbb{R}^1\backslash\Delta_j)(g(P_1) - g(\frac{X_1}{t}))U(t)f\| = 0, \tag{4.15}$$

$j = 1,2,\dots,N-1$. Summarizing (4.12) and (4.15) we find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\|g(P_1)U(t)f\| = 0 \tag{4.16}$$

for supp $g \subset \Delta_j$, $j = 1,2,\dots,N-1$.

In order to extend (4.16) to $j = 0$ and $j = N$ we have to use a different method. Our first aim will be to show that supp $g \subset \Delta_N$ yields

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \Delta_o)g(P_1)U(t)f\| = 0 \qquad (4.17)$$

and that supp $g \subset \Delta_o$ implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \Delta_N)g(P_1)U(t)f\| = 0. \qquad (4.18)$$

Proving (4.17) we use the representation

$$F(\frac{X_1}{t} \in \Delta_o)g(P_1)U(t)f =$$

$$F(\frac{X_1}{t} \in \Delta_o)g(P_1)e^{-itH_o}(e^{itH_o} U(t) - W_-^*)f + \qquad (4.19)$$

$$F(\frac{X_1}{t} \in \Delta_o)e^{-itH_o} g(P_1)W_-^*f.$$

On account of Lemma 2.5 the last summand of the right-hand side tends to zero as $t \longrightarrow +\infty$. Furthermore, we have the formula

$$\{U(t)^*e^{-itH_o} - W_-\}e^{itH_o} g(P_1)F(\frac{X_1}{t} \in \Delta_o)h =$$
$$(4.20)$$
$$i\sum_{l=1}^{N}\int_{-\infty}^{0} ds\ U(t+s)^*V_l(t+s)e^{-isH_o} g(P_1)F(\frac{X_1}{t} \in \Delta_o)h,$$

$h \in \mathfrak{H}$, $t > 0$. Taking into account Lemma 2.4 the representation (4.20) immediately yields the estimate

$$\frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \Delta_o)g(P_1)e^{-itH_o}(e^{itH_o} U(t)-W_-^*)f\| \le$$
$$(4.21)$$

$$\sum_{l=1}^{N}\int_{-\infty}^{0} ds\ \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \Delta_o)g(P_1)e^{isH_o}\ V_l(t+s)U(t+s)f\|\ ds,$$

$t > 0$. As before, we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t}\in\Delta_o)g(P_1)e^{isH_o}\ V_l(t+s)U(t+s)f\| = 0, \qquad (4.22)$$

$s < 0$. On account of Lemma 2.4 we can apply the dominated convergence theorem. Thus, we find (4.17). Similarly we prove (4.18).

Taking into account (4.12) the relations (4.17) and (4.18) can be summarized as follows:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \Delta_o \cup \Delta_N)g(P_1)U(t)f\| = 0 \qquad (4.23)$$

for supp $g \subset \Delta_o \cup \Delta_N$. Choosing g so that it equals one in a neighbourhood of $+\infty$ and $-\infty$ ($g' \in C_o^\infty(\mathbb{R}^1)$!) we obviously have $1-g \in C_o^\infty(\mathbb{R}^1)$. Hence, $1-g$ possesses a summable Fourier transform. Applying again Corollary 4.5 of [2] we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|\{g(\frac{X_1}{t}) - g(P_1)\}U(t)f\| =$$
$$(4.24)$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|\{(1 - g(\frac{X_1}{t})) - (1 - g(P_1))\}U(t)f\| = 0.$$

But (4.24) immediately yields

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(\frac{X_1}{t} \in \mathbb{R}^1\setminus\Delta_o\cup\Delta_N)g(P_1)U(t)f\| = 0. \qquad (4.25)$$

But from (4.23) and (4.25) we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|g(P_1)U(t)f\| = 0 \tag{4.26}$$

for supp $g \subset \Delta_0 \cup \Delta_N$ and $g = 1$ in a neighbourhood of $+\infty$ and $-\infty$.

Summing up (4.16) and (4.26) we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|g(P_1)U(t)f\| = 0 \tag{4.27}$$

for $g \in C^{\infty}(\mathbb{R}^1 \setminus \bigcup_{j=1}^{N}\{v_{1j}\})$, $g' \in C_0^{\infty}(\mathbb{R}^1)$ and $g = 1$ in neighbourhoods of $+\infty$ and $-\infty$.

Obviously, the same can be done for all other axes $x_2, x_3, \ldots, x_n$. Doing so, we find

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|g(P)U(t)f\| = 0 \tag{4.28}$$

for $g \in C^{\infty}(\mathbb{R}^1 \setminus \bigcup_{j=1}^{N}\{v_j\})$ and $g = 1$ in a neighbourhood of infinity. Since the relation (4.28) holds for every such a g, we get that for every $\eta > 0$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(|P - v_j| \geq \eta)U(t)f\| = 0, \tag{4.29}$$

$j = 1,2,\ldots,N$. But on account of (1.11) for every $f \in \mathfrak{H}_+^{ac}$ and every $\varepsilon > 0$ there is a $\eta > 0$ and a $\tau > 0$ such that

$$\|F(|P - v_j| < \eta)U(t)f\| < \varepsilon, \quad j = 1,2,\ldots,N, \tag{4.30}$$

for $t > \tau$. Taking into account (4.29) and (4.30) we obviously obtain

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T dt\|U(t)f\| \leq$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(|P - v_j| \geq \eta)U(t)f\| + \tag{4.31}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\|F(|P - v_j| < \eta)U(t)f\| < \varepsilon,$$

$j = 1,2,\ldots,N$. Hence (4.2) is fulfilled which immediately yields $f = 0$.■

COROLLARY 4.2. - *If the* Assumption P *is satisfied and the trajectories* $x_j(.) \in C^1_{loc}(\mathbb{R}^1,\mathbb{R}^n)$ *obey*

$$\sup_{t\in\mathbb{R}^1} |x_j(t)| < +\infty, \quad j = 1,2,\ldots,N, \tag{4.32}$$

(which yields $v_j^+ = v_j^- = 0$, $j = 1,2,\ldots,N$*), then* (4.1) *holds.*

Proof. - We note that in this case it is not necessary to use Corollary 4.5 of [2]. Hence it is not necessary to satisfy condition (4.13) which allows one to drop condition (1.9).■

REFERENCES

[1] H.Neidhardt, *Moving Potentials and the completeness of wave operators. Part I: The Propagator*, to appear.ear.

[2] H.Neidhardt, *Moving Potentials and the completeness of wave operators. Part II: Propagating observables on scattering state*, to appear.

[3] V.Enss, *Commun. Math. Phys.*, t. 89, 1983, p. 245.

[4] D.Ruelle, *Nuovo Cimento*, t. 59 A, 1969, p.655.

[5] W.O.Amrein, V.Georgescu, *Helv. Phys. Acta*, t. 46, 1973, p. 653.653.
[6] H.Kitada, K.Yajima, *Duke Math. J.*, t. 49, 1982, p.341.
[7] V.Enss, K.Veselic, *Ann. Inst. Henric Poincare*, t. 39 A, 1983, No. 2, p.159.
[8] D.R.Yafaev, *Dokl. Akad. Nauk SSSR*, t. 251, 1980, No. 4, p.812.
[9] D.R.Yafaev, *Funkz. Analysis Pril.*, t. 14, 1980, No. 4, p.91.
[10] D.R.Yafaev, *Dokl. Akad. Nauk. SSSR*, t. 243, 1978, No. 5, p.1150.
[11] D.R.Yafaev, *Mat. Sbornik*, t. 111, 1980, No. 2, p.187.
[12] H.Kitada, K.Yajima, *Duke Math. J.*, t. 50, 1983, p.1005.
[13] K.Yajima, *Commun. Math. Phys.*, t. 75, 1980, p.153.
[14] G.M.Graf, *Phase space analysis of the charge transfer model*, preprint, ETH-Zürich 1988.988.
[15] G.A.Hagedorn, *Ann. Inst. Henric Poincare*, t. 38 A, 1983, p.59.
[16] U.Wüller, *Asymtotische Vollständigkeit des Charge-Transfer-Modells*, Ph. D. Thesis, Berlin 1988.988.
[17] U.Wüller, *Geometric Methods in Scattering Theory of the Charge Transfer Model*, preprint.
[18] V.Enss, *J. Functional Analysis*, t.52, 1983, p.219.

Received by Publishing Department
on May 31, 1990.

Найдхардт Х. E5-90-370

Движущиеся потенциалы и полнота волновых операторов. Существование и полнота

Для обобщенного заряда переносящей модели, точнее, для движущихся и временно зависящих короткодействующих потенциалов, показано существование и полнота волновых операторов, которая определяется подходящим образом.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1990

Neidhardt H. E5-90-370

Moving Potentials and Completeness of Wave Operators. Existence and Completeness

For the generalized charge transfer model, i.e. for moving and time-dependent short range potentials the existence and completeness, defined in a suitable manner, of the wave operators are shown.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1990