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DETERMINING THE UNCERTAINTY IN THE ESTIMATE OF THE LIFETIME OF A PARTICLE BASED ON A SMALL NUMBER OF OBSERVED DECAY EVENTS

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I. INTRODUCTION

Estimation of the parameters of statistical distributions on the basis of small sample statistics, although representing a definite section of mathematical statistics, is extensively applied only in connection with the normal distribution law. As it was shown by the British mathematicians Student (V.Hosset) and R.Fisher, the fluctuations of statistical estimates actually occurring in this case significantly exoeed the errors derived from relations of the classioal Gaussian measurement theory *).

In investigations of radioactivity the methods of olassical measurement theory have also been applied in the case of a random variable with an exponential distribution. The well-known relationships used for determining the variance of the estimated parameter of this distribution for various types of measurement **), however, have been substantiated in a strict manner only for a large number of individually observed decays. At the same time recent high-energy-physics studies of rare generation processes of short-lived particles have made the estimation of lifetimes and the determination of their uncertainties an important problem in the case of a limited number of observed decays of suoh particles. Application of the methods of mathematical statistics in the case of a limited sample belonging

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*) The classical approach turns out to be valid for small sample statistics only in the particular case when a single parameter characterizing the centre of a Gaussian distribution, μ is estimated, while the variance σ^2 is known a priori. Generally, when both parameters, μ and σ^2, are to be estimated on the basis of a sample statistic, one should utilize the Student distribution which differs noticeably from the normal distribution in the case of a small number of measurements.

**) We mean the conditions of measurement of individual lifetimes on the regions of observation of which limits are imposed: upper (see ref. |I|, p. I58) or lower and upper (see ref. |2|, p. I48). We also bear in mind observations of a sequential chain of decaying nonstable states characterized by differing lifetimes (see ref. |I|, p. II3).

to the general statistics of a random value distributed exponentially permits one to obtain precise estimates of the lifetime in the form of confidence intervals of a given reliability α . Truly, such exact results require quite cumbersome computations. Therefore, for practial needs it has sense, on the basis of exact relationships, either to make up numerical tables or to derive approximations that hold for samples that are as small as possible *).

In the present report exact expressions are presented for the probability density function of the mean lifetime $\hat{\tau}_n$ estimated on the basis of n separately observed decay events; the problem of optimal choice of the nonsymmetric relative uncertainties δ_+ and δ_- has been formulated and solved under the condition that the minimal total interval width $\delta_+ + \delta_-$ be obtained for a given reliability α ; tables are also provided of the confidence bounds corresponding to the reliability values of 68.3% and 95.5%. The chosen values are the ones corresponding to the reliabilities of confidence intervals widely applied in experimental physics for errors within one and two standard deviations, respectively, in the case of a normal distribution law. Recommendations are also given in this paper as how to perform approximate computation of uncertainties when the number of measurements is small.

II. ESTIMATION OF THE MEAN LIFETIME BY A CONFIDENCE INTERVAL OF GIVEN RELIABILITY

a) Statistical distribution of the obtained estimate $\hat{\tau}$.

Assume there to be obtained, as a result of measuring the times of decay of unstable particles, n separate values, $t_I, t_2, \ldots, t_n$, and let a statistical conclusion on the mean lifetime of these particles be required to be drawn on the basis of these results. Let us also assume the sample to be sufficiently small and so exclude

*) Naturally, going in pursuit of high precision in determining the uncertainty cannot in itself be justified in the case of interest, when the samples are small and the relative uncertainties are large. For practical applications it is important to exclude the possibility of large errors occurring in determining the uncertainty in the measured quantity. For this reason it is quite legitimate to make use in computing uncertainties of simplifications that provide for a precision higher than 5% in the case of a small number of measurements. At the same time the presence of a systematic deviation in the estimate itself of the particle lifetime is, naturally, undesirable. The unbiasedness of the estimate in the case of a small number of measurements can be provided for rigorously without difficulty.

the possibility of utilizing any approximations admissible when $n \gg I$. For simplicity we shall deal with this problem without considering the restrictions on the region of observable decays that arise in practice. This will allow us to concentrate on the specifics of solving the problem in the case of a small number of measurements.

The average value of the experimentally obtained quantities $t_I, t_2, \ldots, t_n$, to be further denoted by S , represents, in accordance with the maximum likelihood method (MLM), an estimate ($\hat{\tau}$) of the mean lifetime τ of the unstable particles being studied. Indeed, the probability of obtaining the given sample equals (exp- $\frac{I}{\tau}\sum_i^n t_i) \prod^n dt_i$. Thus, the log-likelihood function is $L = I/_\tau \sum_i^n t_i +$ $n \ln \tau$. Hence, from $\frac{dL}{d} = -\frac{I}{\hat{\tau}^2} \sum_I^n t_i + n/_{\hat{\tau}} = 0$ we find

$$\hat{\tau} = S = \frac{1}{n}\sum_1^n t_i \ . \qquad (I)$$

This estimate by the MLM method in our case turns out to be unbiased *). However, in the case of a small number of measurements it is impossible to derive from the likelihood function information of interest on the fluctuations of the obtained estimate $\hat{\tau}$. To this end it is necessary to determine the actual probability density function of the random variable S . Performing n consecutive convolutions of the initial exponential distribution $\varphi(t) = \frac{I}{\tau}\exp - t/_\tau$ as find the sought distribution $\Phi_n(S)$ in the form of the following distribution with the integer-values parameter n :

$$\Phi_n(S) = \frac{1}{(n-1)!}\ \frac{n}{\tau}\left(\frac{nS}{\tau}\right)^{n-1}\exp\left(-\frac{nS}{\tau}\right). \qquad (2)$$

This distribution belongs to the more general family of γ-distributions in which the parameter n may also assume fractional values.

For a given integer value of the parameter n the obtained distribution $\Phi_n(S)$ is characterized by a single scaling parameter τ , which enters into the expressions for all the moments of the distribution. Thus, the expectation value and the variance are respectively

*) Note, however, that this assertion on the unbiasedness of the MLM estimate does not hold when, instead of the lifetime, its inverse is estimated, i.e., the decay constant $\lambda = \tau^{-1}$. For this quantity the MLM yields a biased estimate. The unbiased estimate is represented by the following: $\hat{\lambda} = \frac{n+I}{n} \cdot \frac{I}{S}$ (see ref. |I|, p. I59).

$$E(S) = \tau$$

$$D(S) = \frac{\tau^2}{n} \quad (2a)$$

Thus, the quantity τ to be estimated from the sample data simultaneously determines the mean value and the variance of the distribution of the random quantity S. Therefore the random deviation of the estimate $\hat{\tau}$ from the true value of the parameter derived from a concrete sample also leads to an error in the estimate of the distribution's variance $D = \hat{\tau}^2/n$. This circumstance renders the problem of determining a confidence interval for a single parameter of the exponential distribution law in the case of a small number of measurements to a certain extent equivalent to the problem solved by Student for the normal distribution when two parameters are unknown.

Note also that by introduction of the variable $\chi = 2nS/\tau$ and the parameter $k = 2n$ the obtained distribution $\Phi_n(S)$ can be transformed into the χ^2-distribution

$$\Phi_k(x) = \frac{1}{2(k/2-1)!}\left(\frac{x}{2}\right)^{k/2-1}\exp\left(-\frac{x}{2}\right) \quad (3)$$

for which the expectation value and variance are respectively equal to

$$E(x) = k \quad \text{and} \quad D(x) = 2k. \quad (3a)$$

In this form the scaling parameter τ is included in the variable χ and the distribution is characterized by k degrees of freedom.

b) Determining separately the upper and lower confidence bound

The problem of determining the confidence interval for the random quantity $\hat{\tau}$ reduces to determination from the known distribution $\Phi_n(S)$ (or $\Phi_k(x)$) of the lower and upper limits, $\hat{\tau}_1$ and $\hat{\tau}_2$, outside the range of which the probability for the quantity $\hat{\tau}$ to occur owing to statistical fluctuations is characterized by the sufficiently small value $\beta = I - \alpha$. Therefore, is seems reasonable first to establish separately the upper and lower limits corresponding to given probabilities β_+ and β_-, respectively, of going beyond the indicated limits. Incidentally, when the number of measurements is small, one most often encounters just this problem of determining one of the confidence bounds of reliability $\alpha_{\pm} = I - \beta_{\pm}$. The problem of computing the probabilities β_+ and β_- is solved exactly without knowledge of the true scaling parameter τ, to which end it suffices only to define the confidence bounds $\hat{\tau}_{2j}$ and $\hat{\tau}_{1j}$ is arbitrary units obtained from the samples of the random quantities $\hat{\tau}_j$, i.e. as

$$\hat{\tau}_{2j} = \hat{\tau}_j(1+\delta_+) \quad \text{and} \quad \hat{\tau}_{1j} = \hat{\tau}_j(1-\delta_-), \quad (4)$$

where j stands for the number assigned to the sample.

Conclusions on the probability content of the nonrandom quantity τ, being determined in this case, should apply to the statistical set of repeated samples of given size n, on the basis of the data of which the confidence bounds $\hat{\tau}_{2j}$ and $\hat{\tau}_{1j}$ are determined by the procedure (4). When one deals with a sole concrete sample and the values $\hat{\tau}$, $\hat{\tau}_2$ and $\hat{\tau}_1$ derived from its data, one must clearly bear in mind the randomness of these values and realize that they are bound to undergo changes within a series of consecutive samples. At the same time, when a confidence bound τ_2 and τ_1 is fixed, no conclusion at all can be made on the probability content of the nonrandom quantity τ.

Fixing only the relative value of the bound $\hat{\tau}_2/\tau = I + \delta_+$, or $\hat{\tau}_1/\tau = I - \delta_-$, one can find the corresponding β_+ and β_- from the distribution of the random quantity S. The probability β_+ for the true value of the estimated parameter to occur beyond the chosen upper bound, $\hat{\tau}_2$($\beta_+ = P(\tau > \hat{\tau}_2)$), is then determined by the probability of obtaining small values of S for which the corresponding $x < x_I = \frac{2n}{\tau} S_I$, where the bound S_I is determined from the condition $S_I(I + \delta_+) = \tau$. Thus, the lower bound S_I corresponds to the upper limit $\hat{\tau}_2$ of the estimate of the quantity τ. To underline this peculiarity in the construction of judgments on the probability of the nonrandom quantity τ we have chosen to denote by another letter S the random quantity $\frac{I}{n} \sum t_i$, on the basis of which such judgment are decided upon *).

*) This permutation of confidence bounds was connected in terms of the classical theory of errors with inversion of the probability. Modern formulation of the principal problem of the theory of errors does not imply introduction of the unjustified notion of the statistical distribution of a nonrandom quantity being estimated, but underlines the random nature of the construction of the very judgment on this quantity by calling the probability for the judgment on the estimated quantity to be correct the likelihood. These questions are considered in detail in the author's article "On the interpretation of the principal problems of the theory of errors" in the Supplement to the Russian edition of the book "Statistical Methods in Experimental Physics" (ref. |2|, p. 283).

Consequently, the quantity β_+ can be found integrating the χ^2-distribution (3) from 0 to $x_I = 2n(I + \delta_+)^{-I}$,

$$\beta_+ = P(\tau > \hat{\tau}_2) = \int_0^{x_1} \Phi_k(x)\,dx. \qquad (5)$$

Correspondingly, the reliability of the chosen upper limit $\alpha_+ = P(\tau < \tau_2)$ will be

$$\alpha_+ = 1 - \beta_+ = \int_{x_1}^{\infty} \Phi_k(x)\,dx = P(\chi^2 > x_1). \qquad (5a)$$

Fixing the value α_+ one can find the corresponding $I + \delta_+$ for various n using the tables of extreme values χ_q^2 for the χ^2-distribution (see, for example, ref. |3a|, p. 503 or ref. |3b|, p. 49-55).

The problem of determining the lower bound $\hat{\tau}_1$ for the quantity τ with a given reliability $\alpha_- = P(\tau > \hat{\tau}_1)$ is formulated in a similar way. Owing to the fluctuation of the closely correlated random quantities $\hat{\tau} = S$ and $\hat{\tau}_1 = \hat{\tau}(I - \delta_-)$ the inequality $\tau > \hat{\tau}_1$ is violated starting from $\hat{\tau} \geqslant S_2$, where the upper limit S_2 is determined from the condition $S_2(I - \delta_-) = \tau$. Consequently, $x_2 = 2n\frac{S_2}{\tau}$ and the quantity $\beta_- = I - \alpha_-$ must be represented by the following integral of the χ^2-distribution (3):

$$1 - \alpha_- = P(\tau < \hat{\tau}_1) = \int_{x_2}^{\infty} \Phi_k(x)\,dx .$$

For differing $k = 2n$ the relative values $(I - \delta_-)$, determining the lower bound $\hat{\tau}_1 = \hat{\tau}(I - \delta_-)$, can be found for a given value $I - \alpha_-$ from the tables of extreme values χ_q^2 for the χ^2-distribution.

In Table I there are presented for a 95% reliability level the relative values of the upper and lower bounds $I + \delta_+ = \frac{2n}{\chi_q^2(0.95)}$ and $I - \delta_- = \frac{2n}{\chi_q^2(0.05)}$ for n going from I to I5.

When only a few decays of the investigated particle are registered, one usually makes use only of the lower and upper limits of the mean lifetime in those cases when it is necessary to make a sufficiently reliable conclusion about the discrepancy between the results of the performed experiment and theoretical predictions or experimental results obtained earlier.

c) Optimization of the confidence interval

For estimating the quantity τ by a confidence interval of given reliability α use must be made simultaneously of the upper and lower limits. In this case, however, the required reliability can be obtained in various ways, since the same value $\alpha = I - \beta_+ - \beta_-$ may be obtained for different relationships between the probabilities of going beyond the upper and lower limits established on the basis of the random quantity $\hat{\tau}$. To arrive at a unique solution of this problem an additional condition must be introduced that complies with the general principles of statistical estimation theory (ref. |4|, p. 558).

In the monograph |5| (p. 224) the condition $\beta_+ = \beta_- = \frac{I-\alpha}{2}$ was applied as being natural and self-obvious in considering the example of estimating the mean lifetime of particles by a confidence interval. Actually, this condition is such only in the case of a symmetric statistical distribution. For the case being considered of a random quantity with a nonsymmetric distribution the application of the above condition is not justified. Besides arguments connected with simplifying the solution of the problem, there seem to be no other arguments available to favour this condition.

Table I. Relative values of the upper $\hat{\tau}_2/\tau = I + \delta_+$ and lower $\hat{\tau}_1/\tau = I - \delta_-$ confidence bound on the level of a reliability equal 0.95 for small number of measurements n (from I to I5).

n	$I + \delta_+$	$I - \delta_-$
I	I9.4	0.33
2	5.63	0.42
3	3.68	0.48
4	2.93	0.52
5	2.54	0.55
6	2.3I	0.57
7	2.I2	0.59
8	2.00	0.6I
9	I.9I	0.62
I0	I.84	0.64
II	I.79	0.65
I2	I.74	0.66
I3	I.69	0.67
I4	I.65	0.68
I5	I.62	0.68

From a general standpoint of statistical estimation theory preference should be given to such a relationship between the probabilities β_+ and β_- and the respective δ_+ and δ_- that provides for the minimal total interval $\delta_+ + \delta_-$ for the given reliability α. The latter condition is equivalent to defining the relationship between δ_+ and δ_- that provides for obtaining the maximum value of α for the given total interval $\delta_+ + \delta_- = \frac{\tau}{S_1} - \frac{\tau}{S_2}$.

The α_{max} condition leads to the equation

$$\frac{\partial\alpha}{\partial S_1} = \frac{\partial}{\partial S_1}\int_{S_1}^{S_2} \Phi_n(S)\,dS = 0 \qquad \text{for} \qquad S_2 = [S_1^{-1} - \tau^{-1}(\delta_+ + \delta_-)]^{-1},$$

from which follows

$$\frac{\Phi_n(S_1)}{\Phi_n(S_2)} = \left(\frac{S_2}{S_1}\right)^2. \qquad (6)$$

Thus, the chosen bounds S_I and S_2 of the random quantity S (I) must not only comply with the given reliability

$$\alpha = \int_{S_1}^{S_2} \Phi_n(S)\,dS = 0.683 \text{ or } 0.955,$$

but at the same time it must fulfill relation (6). The minimal value of the interval (δ_+ + δ_-) is thus provided for in the case of a given reliability α .

Using expression (2) for the probability density of a random quantity S we obtain from relation (6) the following:

$$\exp\left[-n\left(\frac{S_2}{\tau} - \frac{S_1}{\tau}\right)\right] = (S_1/S_2)^{n+1}. \qquad (6a)$$

Hence it follows that the extreme values of the corresponding optimal interval are uniquely determined by the ratio $R = S_2/S_I$

$$\frac{S_1}{\tau} = \frac{n+1}{n}\,\frac{1}{R-1}\ln R \qquad \frac{S_2}{\tau} = \frac{n+1}{n}\,\frac{R}{R-1}\ln R. \qquad (7)$$

The bounds S_I/τ and S_2/τ found by these relations define in arbitrary units of τ , for any arbitrary number R , the optimal interval (δ_+ + δ_-) = $(S_I/\tau)^{-I} - (S_2/\tau)^{-I}$ which has the maximum reliability α_{max}. But the maximum reliability value of the interval itself depends on the chosen value of R . To solve the formulated problem it is necessary to find $R_\circ$ for which $\alpha_{max}(R_\circ)$ is equal to the given reliability value α_0. The quantity $R_\circ$ and the extreme values S_I/τ and S_2/τ corresponding to it were determined by the method of successive approximation. The following quantity was computed for values of S_I/τ and S_2/τ that were defined in accordance with (6a), i.e. the chosen number R :

$$\alpha_{max} = \int_{S_1}^{S_2} \Phi_n(S)\,dS = \exp\left(-n\frac{S_1}{\tau}\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left(n\frac{S_1}{\tau}\right)^k - \exp\left(-n\frac{S_2}{\tau}\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left(n\frac{S_2}{\tau}\right)^k. \qquad (8)$$

The program of successive calculations for various R ensured convergence of the result to the given value α_0 with the required precision and determined the corresponding values of the relative uncertainties δ_+ and δ_- . Calculations were performed for n from I to 50 for α_0 = 0.6827 and α_0 = 0.9545 with a precision $\Delta\alpha \leqslant 5\cdot 10^{-5}$. In Table 2 there are presented values of the quantities δ_+ and δ_- for different numbers of measurements. The presented results are rounded off at the third decimal digit which has introduced deviations of the given values of α_0 up to $3\cdot 10^{-4}$. A corresponding truncation is performed also of the reliabilities α of the confidence intervals indicated in the Table.

Table 2. Relative errors δ_+ and δ_- corresponding upper $\hat{\tau}_2(x_I)$ and lower $\hat{\tau}_1(x_2)$ bound of the optimal confidence interval $(\hat{\tau}_2 - \hat{\tau}_1)_{min}$ for two values of the given reliability $\alpha = P(\hat{\tau}_1 < \tau < \hat{\tau}_2)$ equal 0.683 and 0.955 at the differing numbers of independent measurements n .

	α = 0.683		α = 0.955	
n	δ_+	δ_-	δ_+	δ_-
I	I.65	0.829	20.5	0.909
2	0.803	0.684	4.95	0.8I4
3	0.588	0.589	2.84	0.740
4	0.486	0.52I	2.05	0.684
5	0.425	0.472	I.64	0.638
6	0.384	0.433	I.39	0.60I
7	0.353	0.402	I.22	0.570
8	0.330	0.377	I.09	0.543
9	0.3I0	0.355	0.993	0.520
I0	0.294	0.337	0.9I6	0.500
II	0.280	0.32I	0.853	0.482
I2	0.268	0.308	0.80I	0.466
I3	0.258	0.295	0.757	0.45I
I4	0.249	0.284	0.7I9	0.438
I5	0.24I	0.274	0.685	0.426
20	0.2I0	0.237	0.565	0.378
25	0.I88	0.2II	0.49I	0.344
30	0.I73	0.I92	0.439	0.3I8
40	0.I50	0.I65	0.369	0.280
50	0.I35	0.I47	0.324	0.254

Optimization of the confidence interval for $\alpha = 0.683$ leads to a significant violation of the equality $\beta_+ = \beta_-$. The ratio β_-/β_+ is equal to I.8 for $n = 50$, to 2.3 for $n = 25$ and to 6.7 for $n = 5$. This, in turn, means that calculation of the confidence interval on the basis of the condition $\beta_+ = \beta_-$ for small n must lead to a significant excess enhancement of the interval.

It must be especially underlined that the optimal interval of given reliability α, that we have found, is related to the case when the statistical estimate is determined for the mean lifetime of particles, τ. Totally different relationships for optimizing the confidence interval will occur in case the inverse quantity $I/\tau = \lambda$ called the decay constant is estimated. The condition that the interval $\Delta\lambda$ be minimal for a given reliability α, which is equivalent to the condition for obtaining the maximal value α_{max} for the given relative value of the interval $\Delta\hat{\lambda}/\hat{\lambda}$, leads to the relation $\Phi_n(S_I)/\Phi_n(S_2) = I$. In this case, instead of relation (7), we obtain

$$\frac{S_1}{\tau} = \frac{n-1}{n}\,\frac{1}{R-1}\ln R \qquad \frac{S_2}{\tau} = \frac{n-1}{n}\,\frac{R}{R-1}\ln R. \qquad (9)$$

III. COMPARISON WITH APPROXIMATE METHODS OF DETERMINING THE CONFIDENCE INTERVAL

I) Approximate method using symmetrical confidence interval for the inverse quantity I/τ

First of all we shall compare the values of the optimal confidence interval, obtained without any approximations, with the results of the approximate method of determining the confidence interval widely adopted in elementary particle physics without approximate substantiation in the case of a small number of measurements. We mean the method described for the general case in the monograph |2| (p. I96), in which the asymptotically normal behaviour is utilized of the distribution of the following random, for each sample $(x_I, x_2, \ldots, x_n)$, quantity

$$\mu(x_1, x_2, \ldots x_n/\theta) = \frac{\partial L}{\partial\theta}\left[E\left(\frac{\partial^2 L}{\partial\theta^2}\right)\right]^{-\frac{1}{2}}. \qquad (I0)$$

The confidence interval for this quantity is determined from the condition,

$$\left|\left[E\left(\frac{\partial^2 L}{\partial\theta^2}\right)\right]^{-\frac{1}{2}}\frac{\partial L}{\partial\theta}\right| < \mu_0(\alpha_0), \qquad (I0a)$$

where L is the log-likelihood function taken with its sign inverted, $L(x_1, x_2, \ldots x_n/\theta) = -\ln l(x_1, x_2, \ldots x_n/\theta)$; θ is the distribution parameter to be determined, μ_0 is the number of standard deviations being normal provides for the given reliability α_0 of the confidence interval.

For the case under consideration of a random quantity having a negative exponential distribution the condition (I0a) leads to the following simple confidence bounds for the mean particle lifetime $\hat{\tau}_1 < \tau < \hat{\tau}_2$ (see ref. |2|, p. I97):

$$\hat{\tau}_1 = \frac{\hat{\tau}}{1 + \mu_0/n^{1/2}} \quad \text{and} \quad \hat{\tau}_2 = \frac{\hat{\tau}}{1 - \mu_0/n^{1/2}}, \qquad (II)$$

where $\hat{\tau}$ is the estimate of the quantity τ based on the MLM and determined by the relation (I).

For the simplest case under consideration of n observed particle decays this method actually reduces, without any restrictions being imposed on the range of the observed quantities t_1, to utilizing the variance of the random quantity S. The random quantities μ and S are related in a unique way by the simple linear transformation $\mu = n^{1/2}(1 - S/\tau)$. Use of the random quantity μ determined from the sample data is justified for finding the variance of the estimate $\hat{\tau}$ in the essentially more complicated case when restrictions of various kinds are imposed on the range of the observed particle lifetimes *).

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*) R.Peierls (ref. |6|) initially solved the problem of estimating the mean lifetime of particles from observations of the moments of separate particle decays when a common restriction for all the observations is imposed on the time range of the registered decays, $t_i < T$. Then, in I954, M.S.Bartlett (ref. |7|) solved the problem of estimating the quantity τ on the basis of separately observed decays in the general case, comprising entirely different groups of observations, both without restrictions imposed on the range of observed t_i and with restrictions of an individual nature, $t_i < T_i$, and taking into account the information on the particles that did not decay inside the detector. This complicated problem was timely solved specially for determining the lifetime of hyperons. Ref. |8| may serve as an example of the application of this method. Later such methods became widely diffused and they are partly described in monographs (ref. |I|, p. I58 and ref. |2|, p. I48 and p. I96).

The significant progress in solving the complicated problem, that has arised from practical physical reality, of determining the estimate of the mean lifetime of unstable particles with account of systematical corrections due to various restrictions imposed on the range of observable decay moments gave on contribution, however, to altering the essence of the initial formulation of the problem of finding the uncertainty of the obtained estimate. For this reason it has sense to discuss, for greater clarity, the widely utilized method of determining approximately the confidence interval as applied to the simplest idealized case when no restrictions at all are imposed on the range of observable decay moments. After clearing up the essentials of this point and obtaining substantiated recommendations for the approximate approach one can deal with the same problem of determining the confidence interval for the estimated quantity in the more complicated case corresponding to the real situation considered in ref. |7|.

In the case of a confidence interval with a reliability equal in the asymptotic approximation to $\alpha_0 = 0.6827$ ($\mu_0 = I$) the relation (II) for the bounds yields the values

$$\hat{\tau}_1 = \frac{\hat{\tau}}{1 + n^{-1/2}} \quad \text{and} \quad \hat{\tau}_2 = \frac{\hat{\tau}}{1 - n^{-1/2}}, \qquad \text{(IIa)}$$

to which there correspond in the variable S/τ the following respective confidence bounds:

$$S_2/\tau = 1 + m^{-1/2} \quad \text{and} \quad S_1/\tau = 1 - n^{-1/2}, \qquad \text{(IIb)}$$

This means that the considered approximate method of determining the confidence interval reduces to adopting the symmetrically situated bounds (IIb) for the random quantity S/τ with a deviation from unity equal to the square root of the variance $\Delta_0 = (D(S/\tau))^{1/2} = I/\sqrt{n}$. The latter is equivalent to assuming a symmetric relative uncertainty for the inverse quantity $\lambda = \tau^{-1}$, since the bounds (IIa) correspond to the interval $\hat{\tau}^{-1}(1 \pm \Delta_0)$.

On the contrary, the relative uncertainties δ_+ and δ_- expressed in terms of $\hat{\tau}$ turn out to be not equal to each other for small n :

$$\delta_+ = \frac{n^{-1/2}}{1 - n^{-1/2}} \quad \text{and} \quad \delta_- = \frac{n^{-1/2}}{1 + n^{-1/2}}. \qquad \text{(IIc)}$$

Accordingly, for another interval reliability given in the symptotic approximation one must introduce into all the relations (IIa, b, c) the corresponding number μ_0 of standard deviations $\Delta_0 = I/\sqrt{n}$. Thus, the relations (IIc), for instance, in the general case have the form

$$\delta_+ = \frac{\mu_0/n^{1/2}}{1 - \mu_0/n^{1/2}} \quad \text{and} \quad \delta_- = \frac{\mu_0/n^{1/2}}{1 + \mu_0/n^{1/2}} .$$

Presenting this method of determining the uncertainty in the estimate Bartlett himself pointed out is application being justified only in the case of a large number of registered decays (ref. |7|, p. 25I). However, in practice this method without substantiation was applied for handling samples consisting literally of a few registered decays *).

At any rate the warning of Bartlett remained unnoticed, and the absence of symmetry between the errors $\delta_+\hat{\tau}$ and $\delta_-\tau$ obtained at small n lead to the wrong impression, that the applied method accounts for the specifics of handling small-sized samples in reflecting the nonsymmetric character of the initial distribution of the random quantity . Actually, this nonsymmetric character, $\delta_+ > \delta_-$, as we already pointed out, is merely due to our adopting a symmetric uncertainty for the quantity $^{-I}$. Further we shall consider another such as simple approximation consisting in assuming a symmetric uncertainty $\delta_+ = \delta_- = \delta_0 = n^{-I/2}$ for the quantity $\hat{\tau}$ and accordingly a nonsymmetric uncertainty for the inverse quantity $\hat{\tau}^{-1}$:

$$\Delta_+ = \frac{n^{-1/2}}{1 - n^{-1/2}} \quad \text{and} \quad \Delta_- = \frac{n^{-1/2}}{1 + n^{-1/2}} .$$

*) Such a use of this method at the beginning of the investigation of hyperons was caused first of all by the level itself of the first observations of the decays of these particles. In recent years a similar situation was repeated in connection with the study of the decays of charmed particles with even shorter lifetimes. The problem of handling small-sized samples will still be of interest in the near future of elementary particle physics in connection with the search for new short-lived particles predicted theoretically on the basis of newly introduced heavy quarks. Therefore it is of utmost importance to establish the limits for the application of the generally accepted approximate approach.

The expression (2a) made use of for the variance in these approximate approaches is exact for any arbitrarily small number of measurements. However, the probability content of such intervals constructed in the simplest way on the basis of $D^{-1/2}$ is clearly defined only for $n \gg 1$ from the asymptotically normal distribution of the random quantity S .

2) Comparison with the optimal confidence interval

The exact reliability of the interval made use of (IIa) for small n is determined by integrating the distribution (2) within limits given by the relation (IIb). However, the main control must be applied to the deviations of the utilized intervals from the optimal values $\Delta\hat{\tau}_{min}$ for various n .

Differing confidence intervals must be compared with each other first of all on the basis of the total interval value, $\delta_+ + \delta_-$. Another, auxiliary, characteristic of the interval reflects the difference between δ_+ and δ_- , the absence of symmetry of the relative uncertainties. For representing in a clear manner the dependences of these characteristics of confidence intervals on the number n it is convenient to make use of the quantities *)

$$U = \frac{\delta_+ + \delta_-}{2}\,\frac{n^{1/2}}{\mu_0(\alpha)} - 1 \qquad Y = \frac{\delta_+ - \delta_-}{2}\,\frac{n^{1/2}}{\mu_0(\alpha)}\,, \qquad (I2)$$

where the parameter $\mu_0(\alpha)$ equals I for $\alpha = 0.683$ and equals 2 for $\alpha = 0.955$.

The histrograms of the dependences of the quantities (I2) for the discussed confidence intervals are presented in Fig. I. Here the confidence intervals corresponding to the histograms I, 2, 4 correspond exactly to the indicated reliabilities 0.6827 and 0.9545. For the confidence interval (II) (histogram 3) the reliabilities are only approximately equal to these values. Truly, the excess over $\alpha_0 = 0.6827$ amount to less than 10^{-2} for $n > 10$ and to less than $5\cdot 10^{-3}$ for $n > 17$.

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*) Accordingly, the relative uncertainties are expressed through these quantities by the relations $\delta_+ = \frac{\mu_0(\alpha)}{n^{1/2}}(1 + U + Y)$ and $\delta_- = \frac{\mu_0(\alpha)}{n^{1/2}}(1 + U - Y)$.

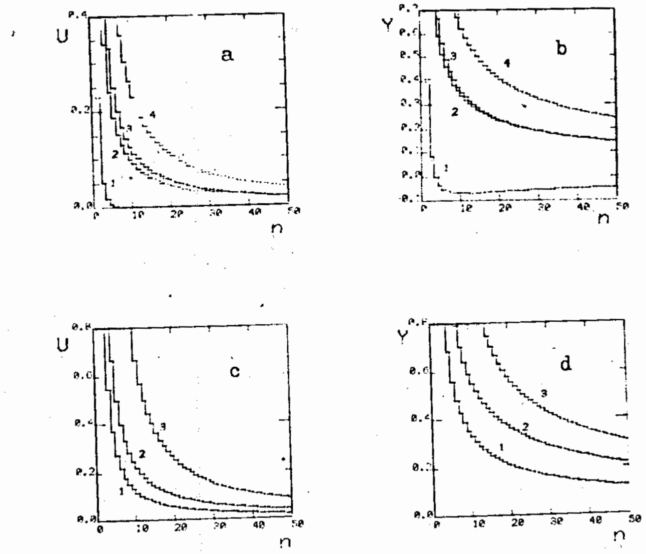

Fig. I. Histograms of the quantities U and Y for confidence intervals corresponding to the following conditions: I – $\Delta\tau_{min}$, 2 – $\beta_+ = \beta_-$, 3 – $\Delta_+ = \Delta_- = \sqrt{n}$, 4 – $\Delta\lambda_{min}$; for the reliabilities $\alpha = 0.683$, a) and b), and $\alpha = 0.955$, c) and d). n – represents the number of measured decay times.

The significant deviation of histogram 3 from histograms I and 4 indicates that the confidence interval determined from the condition that there be introduced equal relative uncertainties $\Delta_+ = \Delta_- = \Delta_0 = 1/\sqrt{n}$ for $\hat{\tau}^{-1}$ essentially exceeds the optimal interval $\Delta\tau_{min}$ and does not coincide with the interval corresponding to the condition of $\Delta\lambda_{min}$. At the same time the histrograms 2 and 3 being close to each other means that the conditions $\Delta_+ = \Delta_- = \Delta_0 = n^{-1/2}$ and $\beta_+ = \beta_- = 0.158$ lead to close intervals.

3) Choice of the best approach from the simply methods

Having utilized for comparison of the earlier discussed confidence intervals the quantities (I2) we have, thus, implicitly introduced one more approximate expression for a confidence interval with extreme values equal in the case of $\alpha_0 = 0.683$ to

$$\hat{\tau}_1 = \hat{\tau}\ (1 - n^{-1/2}) \quad \text{and} \quad \hat{\tau}_2 = \hat{\tau}\ (1 + n^{-1/2}) \qquad (I3)$$

and to which the following symmetric relative uncertainty corresponds:

$$\delta_+ = \delta_- = \delta_0 = \{ D(\hat{\tau}/\tau) \}^{1/2} = n^{-1/2} .$$

It is readily seen that for this confidence interval the introduced quantities (I2) have zero values. Consequently, the interval corresponding to the bounds (I3) is assumed to be the base interval determining the origin for the quantities (I2). For $n >> 1$ this interval coincides with the considered approximate expression for the confidence interval (IIa). However, for small n the optimal confidence interval (histogram I in Fig. I a) turns out to be significantly closer to the approximation (I3), to which corresponds $U = 0$ and $Y = 0$, than to approximation (IIa) (histogram 3 in Fig. Ia). However, this fact is insufficient to draw a conclusion on the advantages of the approximation (I3). The point is that closeness to the optimal interval in the value of the total interval could have been achieved owing to a loss of the probability content of the approximate confidence interval (I3) expressed by a definite integral of the distribution (2) for the random quantity S with limits $S_I = \tau\,(I + I/\sqrt{n})^{-I}$ and $S_2 = \tau\,(I - I/\sqrt{n})^{-I}$.

Calculations of $\alpha(n)$ for the interval (I3) have yielded, however, an unexpected result. The deviations of $\alpha(n)$ from $\alpha_0 = 0.6827$ for this interval turned out to be significantly smaller than for the widely applied approximate confidence interval (IIa). Thus, for $n = I6$ the value $\alpha = 0.6825$, while the condition $|\alpha - \alpha_0| < 5 \cdot I0^{-3}$ is fulfilled starting from $n = 4$.

This means that the approximation (I3) is more precise than (IIa) from all points of view. In other words, the bounds

$$\hat{\tau}_{2,1} = \hat{\tau}\ (1 \mp n^{-1/2})$$

correspond to the optimal interval better than the bounds

$$\hat{\tau}^{-1}_{1,2} = \hat{\tau}^{-1}\,(1 \pm n^{-1/2})$$

of the currently widely diffused approximation.

From the data of Table 2 it is easy to see the sum $\delta_+ + \delta_-$ for $\alpha = 0.683$ with an accuracy higher than $I0^{-2}$ obeys the law $2/\sqrt{n}$ starting from $n = 4$. This means that the approximation

(I3) in the whole region $n > 3$ gives a good description of the total value of the optimal interval. On the other hand, the asymmetry Y for the optimal interval within all this region has a negative value while its absolute value remains less than 0.07. Neglecting this quantity in the approximation (I3) ($Y = 0$) only introduces an insignificant decrease of the probability content of the interval by a value $< 5 \cdot I0^{-3}$. At the same time introduction of nonsymmetric uncertainties $\delta_+ > \delta_-$ in the case of the approximation (IIa) yields more significant deviations from α_0 and from the total value of the optimal interval.

Thus, for a confidence interval of reliability 0.683 one is justified when $n > 3$ in adopting the approximate approach based on the choice of a symmetric relative uncertainty

$$\delta_+ = \delta_- = \delta_0 = n^{-1/2} .$$

From Fig. I c and d one can see that in the case of a reliability $\alpha = 0.955$ both considered approximations do not provide for, small n, a description satisfactory from the point of view of closeness to the results obtained for the optimal confidence interval. However, in this case an approximate approach is not really needed. When the number of registered decays is small, a confidence interval of reliability close to I is utilized only in rare special cases. Turning in such cases to exact tabular values of the optimal confidence interval cannot be considered burdensome, if a guaranteed high reliability of the confidence interval is required.

IV. CONCLUDING RECOMMENDATIONS FOR CALCULATION OF UNCERTAINTIES IN THE COMPLICATED CASES OF REAL OBSERVATIONS

In real experimental conditions the mean lifetime of unstable particles is estimated by simultaneously processing various groups of measurements differing in the restrictions imposed on the range of observations. Together with the n registered decays considered above without restrictions being imposed on the range of observed times, there is included in the joint analysis a group of m decays registered when restrictions are imposed on the ranges of observation: $t_i < T_i \sim \tau$ *). In the joint analysis there is also taken into account one more group of k investigated unstable particles about which one only knows that each one has decayed later than a certain moment of time θ_i. This group, for instance, includes decays that occurred in the detector after the particle had already come to a stop. For such events the quantity θ_i is the time that passed before the stopping of the unstable particle. The same group also includes events of investigated particles travelling through the detector without decaying in it.

The likelihood function for the whole sample consisting of the indicated three groups of events has the following form |8| :

$$L = \sum_{i=1}^{n} (\ln\tau + t_i/_\tau) + \sum_{i=1}^{m} (\ln\tau + t_i/_\tau + \ln[1 - \exp(-T_i/_\tau)]) + \sum_{i=1}^{k} \theta_i/\tau .$$

The estimate of the quantity, is equal to

$$\hat{\tau} = \frac{1}{n+m} \left\{ \sum_{i=1}^{n} t_i + \sum_{i=1}^{m} \left[t_i + \frac{T_i}{\exp(T_i/_\tau) - 1}\right] + \sum_{i=1}^{k} \theta_i \right\} .$$

The random quantity (I0), which is usually utilized for determining the confidence interval, in this general case has the form

$$\mu = \frac{-\tau^{-1} \left[\sum^{n} t_i + \sum^{m} \left[t_i + \frac{T_i}{\exp(T_i/\tau) - 1}\right] + \sum^{k} \theta_i \right\}}{\{n + m + \sum^{m} (T_i/_\tau)^2 \exp(-T_i/_\tau) \, [1 - \exp(-T_i/_\tau)]^{-2}\}^{1/2}} .$$

But, as it was revealed in the preceding section, the procedure proposed by Bartlett |7| for determining the confidence interval does not yield a satisfactory approximation to the optimal confidence interval in those cases, when the uncertainties obtained by applying it turn out to be comparable with the estimated quantity . The more precise approximation $\hat{\tau}_{2,1} = \hat{\tau}\,[I \pm \sqrt{D(\hat{\tau}/_\tau)}\,]$ makes direct use of the same variance of the random quantity $\hat{\tau}/_\tau$ which, however, in the general case considered here cannot be repre-

*) In other words, for each particle decaying inside the detector it is taken into account that its decay would not have been registered in the detector, if it had taken place later than a certain moment T_i. The extreme value of T_i is not constant not only owing to geometric factors of individual character (the point at which the studied particle is created and its angle of departure), but also because of the differences in the particle velocities. This is because the moment when a moving particle decays in its proper time is determined by the measured flight length l_i with account of its velocity V_i and the Lorentz factor γ_i: $t_i = l_i / v_i\gamma_i$.

sented by a simple expression. At the same time the Bartlett procedure gives the interval for the inverse quantity $\hat{\tau}^{-1}$ related to the variance. Therefore it may be applied for a definite variance $D(\hat{\tau}/\tau)$. The random quantity μ is equal to 0 for $\tau = \hat{\tau}$. Its deviations by $\pm I$ from the zero value are associated with certain bounds $\tau_+(\mu=1)$ and $\tau_-(\mu=-1)$ connected with the variance of the ratio $\hat{\tau}/\tau$ by the simple relation

$$\frac{\hat{\tau}}{2}\,(\tau_-^{-1} - \tau_+^{-1}) = [D(\hat{\tau}/\tau)]^{1/2} .$$

Thus, the quantities τ_+ and τ_- are not to be considered bounds of the confidence interval of reliability $\alpha = 0.683$, but bounds of the "variance interval". Now, the bounds of the confidence interval of reliability $\alpha = 0.683$, as it was shown above, with a good precision of approximation to the optimal confidence interval are expressed through the variance of the random quantity $\hat{\tau}/\tau$. by the following relation:

$$\hat{\tau}_{2,1} = \hat{\tau}\,[1 \pm D^{1/2}] = \hat{\tau}\,[1 \pm \frac{\hat{\tau}}{2}\,(\tau_-^{-1} - \tau_+^{-1})] . \qquad (I4)$$

For earlier publications one can determine the more exact value of the confidence interval (I4) $\pm\sigma_0$ directly from the uncertainties σ_+ and σ_- :

$$\sigma_0 = \hat{\tau}\,[D(\hat{\tau}/\tau)]^{1/2} = \frac{\sigma_+ + \sigma_-}{2} \; \frac{1}{(1 + \sigma_+/\hat{\tau})(1 - \sigma_-/\hat{\tau})} .$$

On this basis one may decrease the values of the total interval given in the previous publications. It is important to take into account the possibility of performing such corrections when discussing work devoted to the currently important problem of measuring the lifetime of particles of new family with heavy quarks (see Appendix, item I).

Averaging of the estimates of the mean particle lifetimes obtained in different works must be performed with account of the statistical weight of each estimate. The relative uncertainty of the resulting average value equals $\delta_0 = (\sum_{j=1}^{N} D_j^{-1})^{-1/2}$.

In the general case, when the three indicated groups of measurements are processed simultaneously, one can introduce the concept of the effective number of measurements which corresponds to the found variance:

$$\hat{n}_j = \frac{1}{D_j(\hat{\tau}/\tau)} = 4\,\left(\frac{\hat{\tau}}{\tau_-} - \frac{\hat{\tau}}{\tau_+}\right)^{-2} . \qquad (15)$$

This quantity should be taken as the statistical weight when the results of different experiments are averaged (see Appendix, item 2).

Separately for the second group of events comprising m registered decays with a restriction imposed on the range of observed times $T_1 \sim \tau$, we always have $\hat{n} < m$. In other words, we have $D(\hat{\tau}/\tau) > m^{-1}$. However, when these data are considered simultaneously with the group of k particles travelling through the detector without decaying there should be obtained, on the average according to the results of ref. |5| (p. 193), a variance equal to m^{-1}. This indicates the importance of taking into account the full information when estimating the mean lifetime, including the information on the particles that did not decay. In practice, unfortunately, for particles passing through the detector without decaying not always is there available a reliable proof of their belonging to the investigated class of unstable particles. At the same time the presence in the group of k events of an admixture of background origin leads to a corresponding systematical enhancement of the obtained estimate $\hat{\tau}$.

Rounding off to an integer value the effective number of measurements found in accordance with (15) one can on its basis make use of the results presented in the preceding sections for the simplest case. For instance, from the data of Table 2 one can derive estimates of the relative uncertainties δ_+ and δ_- corresponding to an optimal confidence interval of reliability 0.955 for which there exists no satisfactory approximation based on the variance of the random quantity $\hat{\tau}/\tau$.

I am grateful to O.M.Kuznetsov and V.V.Lyukov for drawing my attention to the problem of determining the uncertainty in the estimate of a mean particle lifetime in the case of a small number of measurements. I also thank greatly G.Pontecorvo for his kind help in translation this article.

APPENDIX

EXAMPLES OF CONFIDENCE INTERVALS FOR EARLIER PUBLISHED RESULTS BEING CORRECTED

1. In one of the first publications, |8|, on the determination of the mean lifetime of the Λ^0-hyperon on the basis of the procedure proposed by Bartlett for processing experimental the following result was presented:

$$\tau = (2.9 {+4.8 \atop -1.1}) \cdot 10^{-10} \text{ s} .$$

In agreement with relation (14a) $\sigma_0 = 1.8 \cdot 10^{-10}$ s and accordingly we obtain the final result

$$\tau = (2.9 \pm 1.8) \cdot 10^{-10}$$

with a confidence interval 1.6 times smaller than the one previously found.

At present this correction is, naturally, of no practical value, since the mean lifetime of the Λ^0-baryon is known with a good precision (10^{-2}). However, in the case of the new particles (the F and B mesons and the Λ_c-baryon), for which only the first measurements of their lifetimes have been performed recently, a similar correction of the confidence interval would be of great topical interest. Thus, for example, in ref. |9| the following result with a nonsymmetric uncertainty is given for the mean lifetime of the $F^{\pm}$-meson:

$$(2.1 {+3.6 \atop -0.8}) \cdot 10^{-13} \text{ s} .$$

In accordance with relation (14a) we find $(2.1 \pm 1.3) \cdot 10^{-13}$ s. The symmetric interval of this more accurate approximation is 1.7 times smaller than the previous confidence interval.

2. For the charged and neutral D-mesons it is now important to correctly perform averaging of the various results of measuring the mean lifetime. The results of individual measurements of the mean lifetime of the $D^{\pm}$ and D^0-mesons adopted in ref. |10| for averaging are presented below in units of 10^{-13} s :

$D^{\pm}$: $2.5^{+2.2}_{-I.I}$; $8.2^{+4.5}_{-2.5}$; $9.5^{+3.I}_{-I.9}$; $8.4^{+3.5}_{-2.2}$; $6.3^{+5.0}_{-2.7}$; $II.5^{+7.5}_{-3.5}$

D^{0} : $3.2^{+2.0}_{-I.6}$; $6.7^{+3.5}_{-2.0}$; $2.3^{+0.8}_{-0.5}$; $4.I^{+I.3}_{-0.9}$; $4.I^{+2.6}_{-I.4}$; $4.2^{+1.6}_{-I.4}$.

The average weighted value given in Tables of particle properties of I984 |I0| was determined by averaging the inverse quantities

$$\hat{\tau} = \frac{\sum_{j=1}^{N} W_j}{\sum_{j=1}^{N} \hat{\tau}_j^{-1} W_j} \quad \text{with the weights} \quad W_j = (\tau_-^{-1} - \tau_{+j})^{-2},$$

where $\tau_{-j} = \hat{\tau}_j - \sigma_{-j}$ and $\tau_{+j} = \hat{\tau}_j + \sigma_{+j}$. The corresponding mean values are equal to $9.2^{+I.7}_{-I.2}$ for the $D^{\pm}$ and to $4.40^{+0.8I}_{-0.60}$ for the D^{0}-meson.

However, in this case of comparatively large measurement errors it is more justified to perform averaging of the $\hat{\tau}_j$ with the weights n_j, in accordance with (I5). Accordingly the confidence interval for the mean weighted value is determined by the symmetric uncertainty $\sigma_0 = \langle\tau\rangle (\sum_{j=1}^{N} \hat{n}_j)^{-1/2}$. In the case of such averaging the final results are obtained equal to $(8.7 \pm I.3) \cdot I0^{-I3}$ s for the $D^{\pm}$ and to $(3.92 \pm 0.56) \cdot I0^{-I3}$ s for the D^{0}-meson.

Thus, the proposed correction of the procedure for averaging results leads to a significant shift of the mean weighted value and to a certain decrease of the total confidence interval. These alterations of the final result must be taken into account, since they are considerably larger than the given uncertainties of fixing the central value and the bounds of the confidence interval.

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Тяпкин А.А. E5-90-351

Определение погрешности оценки среднего времени жизни частиц при малом числе наблюдаемых распадов

Получено точное решение задачи определения минимального доверительного интервала заданной достоверности при малом числе зарегистрированных распадов частиц. Выяснено, что при малом числе измерений определенный на основе дисперсии симметричный интервал $\pm\sqrt{D}$ дает существенно более точное приближение к оптимальному доверительному интервалу по сравнению с получившим в физике широкое распространение приближенным подходом. Рассмотрен случай совместного анализа различных групп измерений.

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Tyapkin A.A. E5-90-351

Determining the Uncertainty in the Estimate of the Lifetime of a Particle Based on a Small Number of Observed Decay Events

The exact solution is obtained for the problem of determining the minimal confidence interval of given reliability in the case of a small number of observed particle decays. It is demonstrated that for a small sample of measurements the symmetric interval $\pm\sqrt{D}$, defined on the basis of the variance, yields a significantly more precise approximation to the optimal confidence interval than is obtained from the approximate approach widely applied in physics. A case of simultaneous analysis of different groups of measurements is examined.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

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