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ERROR ESTIMATE FOR DISCRETIZATION IN TIME TO NONHOMOGENEOUS PARABOLIC EQUATIONS WITH ROUGH INITIAL DATA

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1. Introduction. Let $\mathfrak{X}$ be a Banach space with norm $\| \ \|$. Let A be a sectorial operator in $\mathfrak{X}$ with the domain $D(A)$, where $\operatorname{Re} \sigma(A) > \sigma_0 > 0$. The problem we are considering is the parabolic evolution equation

$$u'(t) + Au(t) = f(t,u(t)) \qquad (1.1)$$
$$u(0) = v \in \mathfrak{X}.$$

Here the function $f : \mathbb{R} \times \mathfrak{X} \to \mathfrak{X}$ satisfies

$$\|f(t,x) - f(s,y)\| \le C_0 \left(|t-s|^{\theta} + \|x-y\| \right) \qquad (1.2)$$
$$\forall x,y \in \mathfrak{X};\ \forall t,s \in \mathbb{R};\ 0 < \theta \le 1.$$

It is well known that for $f \equiv 0$ the solution of homogeneous problem is defined in this way

$$u(t) = T(t)v = (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t} (\lambda + A)^{-1} v \, d\lambda , \qquad (1.3)$$

where Γ is a curve in $\rho(-A)$ (the resolvent set of $-A$) such that $\arg \lambda \to \pm \varphi$ as $|\lambda| \to \infty$ for any fixed $\varphi \in (\pi/2, \pi)$.

In the case when $f \not\equiv 0$ the solution of nonhomogeneous problem is described by this formula

$$u(t) = T(t)v + \int_0^t T(t-s)\, f(s,u(s))\, ds. \tag{1.4}$$

The aim of this paper is to give the error estimate for discretization in time (Rothe's method, backward Euler's method) to the problem (1.1) when the initial element v is assumed to be only in $\mathbb{X}$. The main result is formulated in the Theorem 1. It is easy to see that such an estimate is reasonable only for $0 < t \le T$. For large values of t we can suppose smooth initial data in virtue of smoothing effect for parabolic equation.

Error estimates for semidiscrete Galerkin method applied to parabolic problems have been derived by many authors, cf. [1], [3-5], [7] and references therein. Completely discrete methods have been studied in [4], [7] under some restrictive assumptions on the right-hand side f.

Our work differs from the ones mentioned above in three aspects. First, we work in Banach space only. Second, we deal with more general operator. Third, the right-hand side is nonlinear.

Remark 1. C denotes a generic positive constant independent of τ.

2. Main result. Let us consider the following discretization in time

$$(u_i - u_{i-1})/\tau + Au_i = f(t_i, u_{i-1}), \qquad u_0 = v \tag{2.1}$$

where $\tau > 0$ is the time step ($\tau \le \tau_0$), $t_i = i\tau$, $i = 1,2,\dots$.

One can prove that

$$u_i = (I + \tau A)^{-i} v + \sum_{k=0}^{i-1} (I + \tau A)^{-(i-k)} f(t_{k+1}, u_k)\tau. \tag{2.2}$$

Lemma 1. *(i) Let u be the solution of* (1.1) *defined by* (1.4). *Then*

$$\|u(t)\| \le C \qquad \forall t \le T.$$

(ii) Let u_i $(i = 1,2,\dots)$ be the solutions of (2.1) *defined by* (2.2). *Then*

$$\|u_i\| \le C \qquad \forall i = 1,2,\dots .$$

Proof : In virtue of (1.2) we have

$$\|f(t,x)\| \le C\,(1 + \|x\|). \tag{2.3}$$

(i) Using

$$\|T(t)\| \le C \qquad \forall t \in \langle 0,T\rangle ,$$

(1.4) and (2.3) we get

$$\|u(t)\| \le C + C\int_0^t \|u(s)\|\, ds.$$

The rest of the proof is a consequence of Gronwall's lemma.

(ii) From [6,Th.2] we deduce

$$\|(I+\tau A)^{-i}\| \le C \qquad \forall i = 1,2,\dots . \tag{2.4}$$

Applying this fact to (2.2) one can find

$$\|u_i\| \le C + C\sum_{k=0}^{i-1} \|u_k\|\, \tau.$$

The assertion (ii) follows from the last inequality and the discrete version of Gronwall's lemma. □

Lemma 2. *Let u be the solution of* (1.1) *defined by* (1.4). *Then*

$$\|u(t)-u(s)\| \le C\tau \left(1 + t^{-1} + \ln(t\tau^{-1})\right)$$

for $0 < \tau \le t < s \le t + \tau \le T$.

Proof : Let us denote $h = s-t \le \tau$. We can write

$$u(t+h) - u(t) = (T(h)-I)T(t)v + \int_t^{t+h} T(t+h-z)f(z,u(z))\,dz +$$

$$+ \int_0^t (T(h)-I)T(t-z)f(z,u(z))\,dz = I_1 + I_2 + I_3.$$

Let us estimate I_1, I_2 and I_3. For any $t > 0$ we have $T(t)v \in D(A)$ and [2, Th.1.4.3, Th.1.3.4] yield

(2.5) $$\|I_1\| \le Ch\,\|AT(t)v\| \le Cht^{-1}\,\|v\| \le C\tau t^{-1}.$$

For the second term we get

(2.6) $$\|I_2\| \le \int_t^{t+h} \|T(t+h-z)f(z,u(z))\|\,dz \le$$

$$\le C \int_t^{t+h} \|f(z,u(z))\|\,dz \le Ch \le C\tau.$$

We estimate the last term in this way. If $t = \tau$ it is easy to find that

(2.7) $$\|I_3\| \le C\tau.$$

If $t > \tau$ then

$$\|I_3\| \le \int_0^{t-\tau} \|[T(h)-I]T(t-z)f(z,u(z))\|\,dz +$$

$$+ \int_{t-\tau}^{t} \|[T(h)-I]T(t-z)f(z,u(z))\|\,dz \le C\tau +$$

$$+ Ch \int_0^{t-\tau} \|AT(t-z)f(z,u(z))\|\,dz \le C\tau \left[1 + \int_0^{t-\tau} (t-z)^{-1}\,dz \right].$$

From this we conclude

(2.8) $$\|I_3\| \le C\tau \left[1 + \ln (t\tau^{-1}) \right].$$

The rest of the proof follows from (2.5) - (2.8). □

Theorem 1. *Let u be the solution of (1.1) defined by (1.4). Let u_i be the solutions of (2.1) defined by (2.2). Then*

$$\|u(t_i) - u_i\| \le C \left[\tau + \tau^{\theta} + \tau t_i^{-1} + \tau \ln(t_i\tau^{-1})\right],$$

where $C = C(T,v,C_0)$, $i = 1,2,\ldots$.

Proof : Let us denote

$$T_\tau(t_i) = (I + \tau A)^{-i}.$$

Then we can write

(2.9) $$u(t_i) - u_i = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = (T(t_i)-T_\tau(t_i))v,$$

$$I_2 = \sum_{k=0}^{i-1} T(t_i-t_k)\left[f(t_{k+1},u(t_k)) - f(t_{k+1},u_k)\right] \tau,$$

$$I_3 = \sum_{k=0}^{i-1} \left[T(t_i-t_k) - T_\tau(t_i-t_k)\right] f(t_{k+1},u_k)\,\tau,$$

$$I_4 = \int_\tau^{t_{i-1}} T(t_i-s)f(s,u(s))\,ds - \sum_{k=1}^{i-2} T(t_i-t_k)f(t_{k+1},u(t_k))\,\tau,$$

$$I_5 = \int_0^{\tau} T(t_i-s)\,f(s,u(s))\,ds + \int_{t_{i-1}}^{t_i} T(t_i-s)\,f(s,u(s))\,ds -$$

$$- \left[T(t_i)f(\tau,v) + T(\tau)f(t_i,u(t_{i-1}))\right] \tau.$$

Now we estimate $I_1,\dots,I_5$. From [6,Th.2] we get

(2.10) $$\|I_1\| \le C\, \tau t_i^{-1}.$$

It is easy to see that

(2.11) $$\|I_5\| \le C\tau.$$

Further

(2.12) $$\|I_2\| \le C \sum_{k=0}^{i-1} \|f(t_{k+1},u(t_k)) - f(t_{k+1},u_k)\| \tau \le$$

$$\le C \sum_{k=0}^{i-1} \|u(t_k)-u_k\| \tau = C \sum_{k=1}^{i-1} \|u(t_k)-u_k\| \tau .$$

The third term can be estimated in this way

$$\|I_3\| \le \sum_{k=1}^{i-1} \|(T(t_i-t_k)-T_\tau(t_i-t_k))f(t_{k+1},u_k)\| \tau +$$

$$+ \|(T(t_i)-T_\tau(t_i))f(\tau,v)\| \tau \le$$

$$\le C \left(\sum_{k=1}^{i-1} (i-k)^{-1}\|f(t_{k+1},u_k)\| \tau + \tau \right) \le C \tau \left(1 + \sum_{k=1}^{i-1} (i-k)^{-1}\right) =$$

$$= C \tau \left(1 + \sum_{k=1}^{i-1} k^{-1}\right).$$

So, we can write

(2.13) $$\|I_3\| \le C \tau \left(1 + \ln(t_i\tau^{-1}) \right),$$

because of

(2.14) $$\sum_{k=1}^{n} k^{-1} \le 2 \ln(n+1).$$

In order to estimate the fourth term we rewrite it into the following form

$$I_4 = S_1 + S_2,$$

where

$$S_1 = \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} T(t_i-s) \left[f(s,u(s))-f(t_{k+1},u(t_k))\right] ds,$$

$$S_2 = \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \left[T(t_i-s)- T(t_i-t_k)\right] f(t_{k+1},u(t_k)) \, ds.$$

Using (1.2), (2.14) and Lemma 2 we obtain

$$\|S_1\| \le C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \left\|\left[f(s,u(s))-f(t_{k+1},u(t_k))\right]\right\| ds \le$$

$$\le C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \left[|s-t_{k+1}|^\theta + \|u(s)-u(t_k)\| \right] ds \le$$

$$\le C \sum_{k=1}^{i-2} \left[\tau^{1+\theta} + \int_{t_k}^{t_{k+1}} \tau \left[t_k^{-1} + 1 + \ln k \right] ds \right] \le$$

$$\le C \sum_{k=1}^{i-2} \tau \left[\tau^\theta + \tau + k^{-1} + \tau \ln k\right] \le$$

$$\le C \left[\tau^\theta + \tau + \tau \ln(t_i\tau^{-1})\right].$$

Applying [2,Th.1.4.3,Th.1.3.4] one can get

$$\|S_2\| \le \left\| \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \left[T(s-t_k)-I\right] T(t_i-s)\ f(t_{k+1},u(t_k))\ ds \right\| \le$$

$$\le C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} (s-t_k) \left\| A\ T(t_i-s)\ f(t_{k+1},u(t_k)) \right\| ds \le$$

$$\le C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \tau\ (t_i-s)^{-1}\ ds \le\ C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \tau\ (t_i-t_{k+1})^{-1}\ ds \le$$

$$\le C \sum_{k=1}^{i-2} k^{-1}\tau \le C\ \tau\ \ln\ (t_i\tau^{-1}).$$

So, we have proved

$$\|I_4\| \le C \left[\tau^{\theta} + \tau + \tau\ \ln(t_i\tau^{-1}) \right]. \qquad (2.15)$$

In the end, using (2.10)-(2.13) and (2.15), we conclude

$$\|u(t_i) - u_i\| \le C \left[\tau + \tau^{\theta} + \tau t_i^{-1} + \tau\ \ln(t_i\tau^{-1}) + \sum_{k=1}^{i-1} \|u(t_k)-u_k\|\tau\right].$$

The assertion of the Theorem 1 follows from this fact applying Gronwall's lemma. □

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