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KNEADING SEQUENCES OF PIECEWISE LINEAR BIMODAL MAPS

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1. Introduction

We study maps with two turning points, i.e. bimodal maps. One of their essential features, to our opinion , is that such maps in general are described by two parameters. So here we encounter some problems which or don't appear at all in unimodal case, or are rather trivial. For example, dynamics of bimodal maps is defined by two kneading sequences (itineraries of two turning points). So if we want to prove the theorem about realization of all possible pairs of kneading sequences (the corresponding theorem in unimodal case is proved relatively simply), then we need to consider the question how a set in a parameter plane where an itinerary of one of the turning points is constant, looks like. It is also intersting whether there is some connection between relation of parameters and relation of kneading sequences. We will study these questions in the case of piecewise linear bimodal maps. These maps generalize "skew tent" maps which were considered in /1/. In fact, this work inspired our investigation. Bimodal maps were intensively studied by R.Mackay and C.Tresser /2/,/3/. They used kneading theory to describe the bifurcation structure and to find the boundary of topological chaos.

We consider the continuous piecewise linear maps $F:[-1,1]\to[-1,1]$ which are given by the formula

$$F_{\lambda,\mu}(x)=\begin{cases}\lambda x+\lambda-1, & -1\le x<c\\ \kappa x+\alpha, & c<x\le d\\ \mu x+1-\mu, & d<x\le 1.\end{cases} \tag{1.1}$$

where

$$K = \frac{\mu d + 2 - \mu - \lambda c - \lambda}{d - c} \quad , \alpha = \frac{\lambda c d - c\mu d + \lambda d + c\mu - c - d}{d - c}$$

$d > 0$, $c < 0$. We shall consider the maps with the following properties:

1. $F(x)$ is strictly increasing on $[-1,c) \cup (d,1]$ and strictly decreasing on (c,d).
2. F is a mapping of $[-1,1]$ into itself (then $F(c) \leq 1$, $F(d) \geq 1$).
3. $F(c) \geq d, F(d) \leq c$ i.e. we consider "essentially" bimodal maps rather than monotone or unimodal maps.

We shall assume $|c| = d = 1/2$ (in fact, it is important for us only the equality $|c| = d$, but this always can be achieved by some monotone differentiable change of coordinates, for example quadratic). The value $d = 1/2$ we choose just for simplicity.
Then

$$K = 2 - \frac{\mu + \lambda}{2} \quad , \quad \alpha = \frac{\lambda - \mu}{4}$$

and according to 2.,3. λ, μ, K vary in the range $3 \leq \lambda \leq 4$, $3 \leq \mu \leq 4$, $-2 \leq K \leq -1$.

The main result of this paper is that the map from some subset in the parameter plane to some subset of kneading sequences pairs is 1-1 and onto. We shall show that a set in the parameter plane where an initerary of one of the turning points equals to a given sequence from this subset, is a continuos increasing curve $\lambda(\mu)$. The plan of our paper is as follows. In section 2 we recall some notions of kneading theory and give necessary definitions In section 3 we shall prove some estimates. In section 4 we shall prove monotonicity of the kneading sequences, in section 5 - intermediate value theorem, and finally, in section 6 - the theorem about realizability of a given kneading sequences pair.

2. Some definitions and statement of results

We shall consider symbolic dynamics of our maps. The basic notions of kneading theory can be find in /4/, and especially for bimodal maps in /3/.

If we consider bimodal maps $f(x)$ with turning points c, d then to each point x of $[-1,1]$ one can associate the itinerary defined to be the sequence of symbols L, C, M, D, R and constructed by the next rule:

$$I_i(x) = \begin{cases} L & -1 \le f^i(x) < c \qquad \text{(or we shall write } f^i(x)\in L) \\ C & f^i(x) = c \\ M & d < f^i(x) < c \qquad (\text{or } f^i(x) \in M\ ) \\ D & f^i(x) = d \\ R & d < f^i(x) \le 1 \qquad (\ \text{or } f^i(x) \in R\ ) \end{cases}$$

$\underline{I}(x)$ is or infinite sequence of L's, M's, R's, or finite sequence of L's, M's, R's followed by C or D. Two itineraries can be compared. First, $L < C < M < D < R$. We say $\underline{A} = \underline{B}$, if $A_i = B_i$, $i = 0,1,\ldots,K$ if A is finite, and $A_i = B_i$, $i = 0,1,2,\ldots$ if $\underline{A}$ is infinite. If $\underline{A} \ne \underline{B}$ then there is an index i, for which $A_i \ne B_i$. Let $m = \min i$. Then we say $\underline{A} < \underline{B}$, if either

1. There are even number of M's in $A_0 \ldots A_{m-1}$ and $A_m < B_m$, or
2. There are odd number of M's in $A_0 \ldots A_{m-1}$ and $A_m > B_m$

It is eady to check that $\underline{I}(x) < \underline{I}(y) \Rightarrow x < y$ and $x < y \Rightarrow \underline{I}(x) \le \underline{I}(y)$ (for our maps we have $x < y \Longleftrightarrow \underline{I}(x) < \underline{I}(y)$)

Now we use these notations for maps $F_{\lambda,\mu}(x)$. Let $\underline{I}(F_{\lambda,\mu}(-\frac{1}{2})) = \underline{I}^+(\lambda,\mu)$, $\underline{I}(F_{\lambda,\mu}(\frac{1}{2})) = \underline{I}^-(\lambda,\mu)$.

Since $F_{\lambda,\mu}(-1/2) = \max\limits_{x\in[-1,1]} F_{\lambda,\mu}(x)$ then $\underline{I}^+(\lambda,\mu)$ is maximal (see /4/), i.e.

$$J^K \underline{I}^+(\lambda,\mu) \le \underline{I}^+(\lambda,\mu) \qquad \forall K = 1,2,\ldots$$

where J denotes a shift $J\underline{A} = A_1 A_2 \ldots$, and similarily

$$J^K \underline{I}^-(\lambda,\mu) \ge \underline{I}^-(\lambda,\mu) \qquad \forall K = 1,2,\ldots$$

It is obvious that $\underline{I}^+ > \underline{I}^-$ and moreover

$$J^K \underline{I}^+(\lambda,\mu) \ge \underline{I}^-(\lambda,\mu) \qquad J^K \underline{I}^-(\lambda,\mu) \le \underline{I}^+(\lambda,\mu).$$

The standard way of comparing $\underline{I}^+$ and $\underline{I}^-$ doesn't seem to be the best one. It is more natural to "make" from $\underline{I}^+(\lambda,\mu)$ minimal sequence (or from $\underline{I}^-(\lambda,\mu)$ maximal) and then to compare them. So we give

Definition 1. If $\underline{A} = A_0 A_1 A_2 \ldots$, then

$$\underline{A}^* = A_0^* A_1^* A_2^* \ldots \qquad \text{where } L^* = R,\ R^* = L,\ M^* = M,\ C^* = D.$$

This construction will be very convenient for maps under consideration, since if $x \in R$, then $-x \in L$, and if $x \in M$, then $-x \in M$; $x = c \Rightarrow -x = d$. Hence if $\underline{A} = \underline{I}_F(x)$ then $\underline{A}^* = I(-x)\, I(-F(x)) \ldots I(-F^K(x)) \ldots$ Note that $(\underline{A}^*)^* = \underline{A}$. It is also trivial to prove that if $\underline{A}$ is maximal, then $\underline{A}^*$ is minimal and vice versa. We have the next relation between $(\underline{I}^+(\lambda,\mu))^*$ and $\underline{I}^-(\lambda,\mu)$

Theorem A. Let $(\lambda,\mu) \in \mathcal{D}$, where $\mathcal{D} = \{3 \le \lambda \le 4,\ 3 \le \mu \le 4,$ $\lambda + \mu \ge \frac{20}{3}$ Then

$$\underline{I}^+(\lambda,\mu) > (\underline{I}^-(\lambda,\mu))^* \iff \lambda > \mu .$$

Futher we would like to prove a theorem about realizability of a given kneading sequnces pair. For this purpose fist we define class of maps. Since we consider "everywhere expanding" maps (i.e. $|DF(x)| > 1 \ \forall x \in [-1,1]$) , two points x, y $x \neq y$ will noticeably separate (under repeated action of F) It means that $F(x)$ hasn't stable periodic orbits, and this imposes some restrictions to possible form of kneading sequences. For example, if for some values λ_0, μ_0, $F_{\lambda_0,\mu_0}(x)$ is superstable, $\underline{I}^+(\lambda_0,\mu_0) = \underline{A}C$, then for every $(\lambda,\mu) \neq (\lambda_0,\mu_0)$ (no matter how small is difference $max\{|\lambda-\lambda_0|, |\mu-\mu_0|\}$) this superstable cycle will be destroyed. (This is not the case for smooth maps where there is some neighbourhood U of (λ_0,μ_0) such that $\forall(\lambda,\mu) \in U\ \underline{I}^+(\lambda,\mu) = (\underline{A}F)^\infty,\ F = M, L, C)$

Now we define class Π of kneading sequences pairs. First, we shall say that $\underline{A} \in \Pi_a$, if

1A. $\underline{A}$ is maximal, $T^k\underline{A} \le \underline{A}$, $k = 1, 2, \ldots$, $D \le \underline{A} \le R^\infty$

2A. If $\underline{A} = \hat{\underline{A}}C$, then $\underline{A}_E = (\hat{\underline{A}}C)^\infty$, where $\underline{A}_E$ denotes extended itinerary (see /4/)

3A. $\underline{A}$ can not be represented as $\underline{Q} * \underline{F}$, where $\underline{Q}$ is a finite maximal sequence $\underline{Q} > D$, and $\underline{F}$ is a maximal sequence, containing symbols L, M, C ; and

$\underline{Q} * \underline{F} = \underline{Q}F_0\underline{Q}F_1 \ldots$ if $\underline{Q}$ is even, and

$\underline{Q} * \underline{F} = \underline{Q}\check{F}_0\underline{Q}\check{F}_1 \ldots$ if $\underline{Q}$ is odd , where $\check{L} = M, \check{M} = L, \check{C} = C$

Remark. This definition is similar to $*$ -product in unimodal case /4/. Of course it is not complete, but sufficient for our purposes. It is easy to prove (analogously as in /4/) that defined in this way $\underline{Q} * \underline{F}$ is maximal. Moreover, if $\underline{A}$ can not be represented as $\underline{Q} * \underline{F}$, then for every finite maximal $\underline{Q} > D$ we must have one of the inequalities $\underline{A} > \underline{Q} * ML^\infty$ or $\underline{A} = \underline{Q} * \underline{L}^\infty$ (otherwise one can prove that there is $\underline{F}$ such that $\underline{A} = \underline{Q} * \underline{F}$ in above sense) Similar sequences in unimodal case are called primary.

We shall say that $\underline{B} \in \Pi_B$ if

1B. $\underline{B}$ is minimal $T^k\underline{B} \ge \underline{B}$, $k = 1, 2, \ldots$ $L^\infty \le \underline{B} \le C$

2B. If $\underline{B} = \hat{\underline{B}}D$, then $\underline{B}_E = (\hat{\underline{B}}D)^\infty$

3B. $\underline{B}$ can not be represented as $\underline{P} * \underline{K}$, where $\underline{P}$ is a finite minimal sequence $\underline{P} < C$, and $\underline{K}$ is a minimal containing symbols R, M, D , and

$\underline{P} * \underline{K} = \underline{P} K_0 \underline{P} K_1 \dots$ if $\underline{P}$ is even

$\underline{P} * \underline{K} = \underline{P} \check{K}_0 \underline{P} \check{K}_1 \dots$ if $\underline{P}$ is odd.

$\underline{P} * \underline{K}$ is minimal; if $\underline{B}$ can be of form $\underline{P} * \underline{K}$ then for every finite minimal $\underline{P} < C$ we must have $\underline{B} > \underline{P} * R^{\infty}$ or $\underline{B} < \underline{P} * MR^{\infty}$

We shall say that $(\underline{A}, \underline{B})$, if $\underline{A} \in \Pi_a$, $\underline{B} \in \Pi_b$ and

1C. If $\underline{A} = \hat{\underline{A}} D$, then $\underline{A}_E = \hat{\underline{A}} D B$; if $\underline{B} = \hat{B} C$ then $\underline{B}_E = \hat{\underline{B}} C \underline{A}$

2C. If $J^{K-1}\underline{A} \neq D$, then $J^K \underline{A} > \underline{B}$, $K = 1, 2, \dots$

If $J^{K-1}\underline{B} \neq C$, then $J^K \underline{B} < \underline{A}$, $K = 1, 2, \dots$

The following lemma shows that these conditions really define possible kneading sequences.

Lemma C. Let $3 \leq \lambda \leq 4$, $3 \leq \mu \leq 4$ Then

$$(\underline{I}^+(\lambda,\mu), \underline{I}^-(\lambda,\mu)) \in \Pi .$$

And finally we describe the main result of this paper.

Theorem C. Let $(\lambda,\mu) \subset \widetilde{\mathcal{D}} = \{\lambda \geq 4 - \frac{1}{\mu^2} \cap \mu \leq 4 - \frac{1}{\mu^2}\}$

Let $\Pi \supset \widetilde{\Pi} = \{(\underline{A}, \underline{B}) \in \Pi, \underline{A} \geq RRD, \underline{B} \leq LLC\}$

Let $(\underline{A}, \underline{B}) \in \widetilde{\Pi}$. Then there is an unique $(\lambda,\mu) \in \widetilde{\mathcal{D}}$ such that

$$\underline{I}^+(\lambda,\mu) = \underline{A} \ , \ \underline{I}^-(\lambda,\mu) = \underline{B} .$$

3. Some estimates

Here we shall obtaine some estimates of partial derivatives. We use the technique developed in /1/. Let

$$x_n = F^n_{\lambda,\mu}(-\tfrac{1}{2}), y_n = F^n_{\lambda,\mu}(\tfrac{1}{2}) \ , \ a_n = \frac{\partial x_n}{\partial \lambda}, \ b_n = \frac{\partial x_n}{\partial \mu}, \ p_n = \frac{\partial y_n}{\partial \lambda}, \ q_n = \frac{\partial y_n}{\partial \mu}.$$

These derivatives exist if $x_i \neq \pm 1/2$, $y_i \neq \pm 1/2$ for all $i < n$. It means that $I^+_i \neq C, D$ for all $i < n$, and $I^-_i \neq C, D$ for all $i < n$.

Now we define some notations as in /1/. If $I^+_1 I^+_2 \dots I^+_n$ doesn't contain C, D , then we define θ^+_n as the number of M's in $I^+_1 \dots I^+_n$ and set $\varepsilon_n = (-1)^{\theta^+_n}$. Similarily, if $I^-_1 I^-_2 \dots I^-_n$ doesn't contain C, D , then θ^-_n is the number of M's in this sequence, and $\gamma_n = (-1)^{\theta^-_n}$. If $I^+_K = C, D$ (or $I^-_K = C, D$) , then $\varepsilon_n = 0$ for all $n \geq K$ ($\gamma_n = 0$ for all $n \geq K$).

Lemma 1B. Let $(\lambda,\mu) \in \mathcal{D}$ If $\varepsilon_n \neq 0$ $\forall n \geq 1$ then $\lim_{n\to\infty} |a_n| = \infty$ and $a_n \varepsilon_n > 0$ $\forall n \geq 1$

If $\gamma_n \neq 0$ $\forall n \geq 1$ then $q_n \gamma_n < 0$ and $\lim_{n\to\infty} |q_n| = \infty$.

Proof. We shall prove the estimate for a_n (and the estimate for q_n can be obtained in the same way).

We recall that for simplicity we assume $|c| = d = 1/2$. Then from

(1.1) one can obtain the recursive formulas:

$$a_0 = 0, \ a_1 = 1/2$$

$$a_{n+1} = \begin{cases} x_n + \lambda a_n + 1 & \text{if } -1 \le x_n < -\frac{1}{2} \ (x_n \in L) \\ -\frac{1}{2} x_n + \kappa a_n + \frac{1}{4} & \text{if } -\frac{1}{2} \le x_n < \frac{1}{2} \ (x_n \in M) \\ \mu a_n & \text{if } \frac{1}{2} < x_n \le 1 \ (x_n \in R) . \end{cases} \tag{3.1}$$

Then we can obtain the next estimates:

$$\begin{aligned} \lambda a_n \le a_{n+1} < \lambda a_n + \tfrac{1}{2} \quad & \text{if} \quad x_n \in L \\ \kappa a_n < a_{n+1} < \tfrac{1}{2} + \kappa a_n \quad & \text{if} \quad x_n \in M \\ a_{n+1} = \mu a_n \quad & \text{if} \quad x_n \in R . \end{aligned} \tag{3.2}$$

Let K be the first index for which $I_K^+ = M \, (K > 1)$. From (3.2) we have

$$0 < a_2 < a_3 < \ldots < a_K \tag{3.3}$$

and in fact $a_K = \mu^j \lambda^{K-j-1} a_1$ where j is the number of R'S in $I_1^+ \ldots I_{K-1}^+$, $K-j-1$ is the number of L'S in $I_1^+ \ldots I_{K-1}^+$ To provide a constant sign of a_{n+1} when $x_n \in M$ we must have

$$\kappa a_n + \frac{1}{2} < 0 \tag{3.4}$$

and for increasing of $|a_n|$ we must have

$$\kappa a_n + \frac{1}{2} \le - a_n . \tag{3.5}$$

If $|\kappa a_2| = |\kappa \frac{\mu}{2}| > \frac{1}{2}$ and $|\kappa + 1| a_2 > \frac{1}{2}$ then by (3.3) (3.4) and (3.5) hold. Since $\mu > 3$, $|\kappa| > 1$, the first inequality is satisfied, and the second is true if $\mu|\kappa+1| \ge 1$, but this is the case when $(\lambda, \mu) \in \mathcal{D}$. Hence the sign of a_n is changed to opposite on M

Since $\lambda a_2 = \lambda \mu/2 > 1/2$, the sign of a_n is reserved on L, and if a_n comes to L with negative sign, then since $(\lambda - 1)\mu/2 > 1$, $|a_{n+1}| > |a_n|$. So the sign of a_n changes only on M, hence $\varepsilon_n a_n > 0$. We also have

$$|a_2| \le |a_3| \le \ldots \le |a_n| .$$

Moreover

$$|a_n| \ge \frac{\mu}{2} \mu^{N_R} |\kappa|^{\frac{N_M}{2}} 2^{N_L} ,$$

where N_R is the number of R'S in $I_2^+ \ldots I_{n-1}^+$
N_L is the number of L'S in $I_2^+ \ldots I_{n-1}^+$
N_M is the number of M'S in $I_2^+ \ldots I_{n-1}^+$.

So the assertion of the lemma is obvious.

Lemma 2B. Let $3 \le \mu \le 4$ and $\lambda \ge 4 - 1/\mu^2$. If $\varepsilon_n \ne 0 \ \forall n \ge 1$ then $\varepsilon_n b_n < 0$ and $\lim_{n \to \infty} |b_n| = \infty$.

Proof. Note the condition $\lambda \ge 4 - 1/\mu^2$ means that $\underline{I}^+(\lambda, \mu) \ge RRD$ and $|\kappa| \ge 13/9$

Let us write the recursive formulas for $b_n = \frac{\partial x_n}{\partial \mu}$

$$b_{n+1} = \begin{cases} \lambda b_n, & x_n \in L \\ -\frac{1}{2} x_n + \kappa b_n - \frac{1}{4}, & x_n \in M \\ x_n + \mu b_n - 1, & x_n \in R \end{cases} \tag{3.6}$$

$b_0 = b_1 = 0$, $b_2 = \lambda/2 - 2$.

From (3.6) we obtain:

$$\begin{gathered} b_{n+1} = \lambda b_n, \quad x_n \in L \\ \kappa b_n - \tfrac{1}{2} < b_{n+1} < \kappa b_n, \quad x_n \in M \\ \mu b_n - \tfrac{1}{2} < b_{n+1} < \mu b_n, \quad x_n \in R. \end{gathered} \tag{3.7}$$

Let N be the number of R's in $\underline{I}^+$ from the beginning, i.e. $\underline{I}^+ = \underbrace{R \dots R}_{N} \dots$ $(N > 2)$ N can be find as follows. Let $\frac{\lambda}{2} - 2 = - \varepsilon$, $\varepsilon > 0$ Then $x_1 = \lambda/2 - 1 = 1 - \varepsilon$, $x_2 = 1 - \mu\varepsilon \dots$ $x_n = 1 - \mu^{n-1}\varepsilon$, $n = 2, \dots, N+1$. From (3.6) we obtain:

$$b_2 = -\varepsilon, \quad b_3 = 1 - \mu\varepsilon - \mu\varepsilon - 1 = -2\mu\varepsilon \quad \dots$$

$$b_{N+1} = -N\mu^{N-1}\varepsilon = -\frac{N}{\mu}\left[1 - x_{N+1}\right]. \tag{3.8}$$

Let us consider the minimal value $N = 3$, $\underline{I}^+ = RRR \dots$ (It is obvious that if the assertion is true for $N = 3$, then by (3.8) it is also true for $N > 3$)

If $\underline{I}^+ = RRRL \dots$ then since $x_4 < 1/2$, $|b_4| > \frac{3}{2\mu} > \frac{3}{8}$, and by (3.6) $|b_5| > 3\lambda/8 > 1$. Later we'll see that this value is sufficient to provide the necessary behaviour.

Now let $\underline{I}^+ = RRRM \dots$ Let j be the number of M's, i. e. $\underline{I}^+ = RRR\underbrace{M \dots M}_{j}$

1. $j = 1$, $\underline{I}^+ = RRRMR \dots$ Then $x_4 < 0$ and $|b_4| > 3/4$, $|b_5| > 7/4$
2. $j = 1$, $\underline{I}^+ = RRRML \dots$ Here we need some more exact estimates.

We have:

$$\kappa = 2 - \frac{\mu + \lambda}{2} = -\frac{\mu}{2} + \varepsilon, \quad b_5 = -\frac{1}{2}x_4 + \kappa b_4 - \frac{1}{4} = -\frac{3}{4} + 2\mu^3\varepsilon - 3\mu^2\varepsilon^2.$$

Since $1 - \mu^3\varepsilon < \frac{1}{2}$, $|b_5| > \frac{1}{4} + O(\varepsilon^2)$, ε is small $(x_1 < x_4 < \frac{1}{2}$

where $F(x_1)=-\frac{1}{2}$, $x_1 \in M$, $x_1 = \frac{-1/2 - \alpha}{\kappa} > \frac{2-2\varepsilon}{8}$. So $\varepsilon < \frac{3}{4\mu^3-1} < \frac{3}{107}$).

3. $j = 2$ Let $x_4 = x^* + \Delta$, where x^* is a fixed point, i.e. $\kappa x^* + \alpha = x^*$ (some calculations show that under the conditions of the lemma $-\frac{1}{88} < x^* < \frac{1}{10}$) Since $x_5 = x^* + \kappa\Delta \in M$, $x^* + \kappa\Delta > -\frac{1}{2}$ and

$$x_6 = x^* + \kappa^2\Delta > \frac{1}{2}. \qquad (3.9)$$

From (3.6) we have

$$b_6 = (-\tfrac{1}{2}x^* - \tfrac{1}{4})(\kappa+1) - \kappa\Delta + \kappa^2 b_4.$$

Using (3.9) we obtain

$$|b_6| > \kappa^2|b_4| - \frac{(\frac{1}{2} - x^*)}{|\kappa|} - (\tfrac{1}{2}x^* + \tfrac{1}{4})|\kappa+1| > 0{,}2, \quad |b_7| > 0{,}6 .$$

The case $x = x^* - \Delta$ can be considered similarily, and in fact we'll obtain here the greater value of $|b_7|$.

4. $j > 2$ We'll show that if $|b_4| \geq 3/8$ then

$$|b_5| < |b_7| < |b_9| < \dots \qquad (3.10)$$

By (3.7) we have

$$|b_6| > |b_5|, \quad |b_8| > |b_7|, \dots$$

so $|b_n|$ increase. We have from (3.7)

$$b_5 > \kappa b_4 - 1/2, \quad \kappa b_5 < \kappa^2 b_4 - \tfrac{1}{2}\kappa, \quad b_6 < \kappa b_5 < \kappa^2 b_4 - \tfrac{1}{2}\kappa$$

$$b_7 > \kappa b_6 - \tfrac{1}{2} > \kappa^3 b_4 - \tfrac{1}{2}\kappa^2 - \tfrac{1}{2}.$$

Thus if $\kappa^3 b_4 - \frac{1}{2}\kappa^2 - \frac{1}{2} > \kappa b_4$, then $b_7 > b_5$. The inequality

$$\kappa^3 b_4 - \tfrac{1}{2}\kappa^2 - \kappa b_4 - \tfrac{1}{2} > 0 \qquad (\kappa < 0,\ b_4 < 0)$$

holds as it is easy to see if $|b_4| > 3/8$, and $13/9 \leq |\kappa| \leq 2$. We can repeat these arguments, and show that $|b_9| > |b_7|$, ... and so on.

Now we see that in all cases we'll start our considerations with a value $|b_i|$ which is greater than an "initial" one. $|b_i|$ will not decrease at the most "dangerous" part M , and in fact it is easy to show, using (3.6), that if $J^\kappa \underline{I}^+ = M^\infty$, then $\lim_{n\to\infty} |b_n| = \infty$ In the remaining cases this is obvious.

Remark. Unfortunately, we can not consider a smaller domain in the conditions. For example, if we allow itinerary $\underline{I}^+ = RR\dots < RRD$ then for $\underline{I}^+ = RRML\dots$ as one can see from (3.6) $|b_4|$ becomes very small, and for iterary $\underline{I}^+ = RML\dots$ even the condition $b_4\varepsilon_4 < 0$ doesn't hold. Of course, for some itineraries (e.g. $\underline{I}^+ = RRLL\dots$) we have $\varepsilon_n b_n < 0$, $\lim_{n\to\infty} |b_n| = \infty$.

Lemma 3B. Let $3 \le \lambda \le 4$ and $\mu \le 4 - 1/\lambda^2$ ($\underline{I}^- \le LLC$)
If $\gamma_n \ne 0$ $\forall n \ge 1$, then $\mu_n p_n > 0$ and $\lim\limits_{n\to\infty} |p_n| = \infty$.

Proof is the same as the previous one.

Let now $c_n = a_n - b_n$, $d_n = a_n + b_n$, $z_n = p_n - q_n$, $\ell_n = p_n + q_n$; $\theta_n^+, \theta_n^-, \varepsilon_n, \gamma_n$ are as before. Then we have

Lemma 4B. Let $\lambda + \mu \ge 5 + \sqrt{3}$. If $\varepsilon_n \ne 0$, $\forall n \ge 1$ and $\gamma_n \ne 0$ $\forall n \ge 1$ then
$\varepsilon_n c_n > 0$, $\varepsilon_n d_n > 0$, $\gamma_n z_n > 0$, $\gamma_n \ell_n < 0$, $\lim\limits_{n\to\infty}|c_n| = \lim\limits_{n\to\infty}|d_n| = \lim\limits_{n\to\infty}|z_n| = \lim\limits_{n\to\infty}|\ell_n| = \infty$

Proof. We'll write only the recursive formulas for c_n and d_n the further proof is similar to previous one.

$$c_{n+1} = \begin{cases} \lambda c_n + x_n + 1, & x_n \in L \\ \kappa c_n + \frac{1}{2}, & x_n \in M \\ \mu c_n - x_n + 1, & x_n \in R \end{cases}$$

$$c_2 = 2 + \frac{\mu - \lambda}{2}, \quad c_2 > 3/2$$

$$d_{n+1} = \begin{cases} x_n + \lambda d_n + 1, & x_n \in L \\ -x_n + \kappa d_n, & x_n \in M \\ x_n - 1 + \mu d_n, & x_n \in R \end{cases}$$

$$d_2 = \frac{\mu + \lambda}{2} - 2 = -\kappa, \quad d_2 \ge \frac{1 + \sqrt{3}}{2}.$$

Finally, we shall prove a lemma in the same spirit.

Let $x_n = F^n_{\lambda,\mu}(-\frac{1}{2})$, $y_n = F^n_{\lambda,\mu}(\frac{1}{2})$. Let $I(x_i) = I^*(y_i)$, $i=1,\dots,K$ (Recall that $R^* = L$, $L^* = R$, $M^* = M$, $C^* = D$) Set $\delta_n = x_n + y_n$.

Lemma A. If $(\lambda,\mu) \in \mathcal{D}$, $\lambda > \mu$, $\varepsilon_n \ne 0$, $\forall n = 1,\dots,K$ then

$$\varepsilon_n \delta_n > 0 \quad \forall n \le K; \quad \lim_{K\to\infty} |\delta_K| = \infty.$$

Proof.

$$\delta_1 = \frac{\lambda - \mu}{2} > 0$$

$$\delta_{n+1} = \begin{cases} (\lambda - \mu) + \mu\delta_n + (1-\mu)x_n, & x_n \in L\,(y_n \in R) \\ \kappa\delta_n + (\lambda-\mu)/2, & x_n \in M\,(y_n \in M) \\ (1-\mu) + \mu\delta_n + (\lambda-\mu)y_n, & x_n \in R\,(y_n \in L). \end{cases} \tag{3.11}$$

From (3.11) we obtain

$$\begin{aligned} \mu\delta_n \le \delta_{n+1} &\le \frac{\lambda-\mu}{2} + \mu\delta_n \\ \delta_{n+1} &= \kappa\delta_n + \frac{\lambda-\mu}{2} \\ \mu\delta_n \le \delta_{n+1} &\le \frac{\lambda-\mu}{2} + \mu\delta_n. \end{aligned} \tag{3.12}$$

If $\mu(\kappa+1) \geq 1$ then by (3.12) $\varepsilon_n \delta_n > 0 \quad \forall n = 1, \dots, \kappa$

$$|\delta_1| < |\delta_2| < \dots < |\delta_\kappa|$$

$$|\delta_n| \geq 2^{N_R + N_L} |\kappa|^{N_M/2} \left(\frac{\lambda - \mu}{2}\right),$$ where N_R, N_M, N_L are the same as in lemma 1B. Thus $\lim_{\kappa \to \infty} |\delta_\kappa| = \infty$.

4. Monotonicity of the kneading sequences

Using the estimates, one may prove the following

Theorem B. Let $(\lambda_1, \mu_0), (\lambda_2, \mu_0) \in \mathcal{D}$, $\lambda_1 > \lambda_2$. Then

$$\underline{I}^+(\lambda_1, \mu_0) > \underline{I}^+(\lambda_2, \mu_0). \tag{4.1}$$

If $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{D}_1 = \{3 \leq \mu \leq 4, \lambda \geq 4 - \frac{1}{\mu^2}\}$ and $\lambda_1 > \lambda_2$, $\mu_1 < \mu_2$ ($\lambda \geq \lambda_2$, $\mu_1 < \mu_2$ or $\lambda_1 > \lambda_2$, $\mu_1 \leq \mu_2$ also is possible), then

$$\underline{I}^+(\lambda_1, \mu_1) > \underline{I}^+(\lambda_2, \mu_2). \tag{4.2}$$

Let $(\lambda_0, \mu_1), (\lambda_0, \mu_2) \in \mathcal{D}$, $\mu_1 < \mu_2$. Then

$$\underline{I}^-(\lambda_0, \mu_1) > \underline{I}^-(\lambda_0, \mu_2). \tag{4.3}$$

Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{D}_2 = \{3 \leq \lambda \leq 4, \mu \leq 4 - \frac{1}{\lambda^2}\}$, $\lambda_1 > \lambda_2$ $\mu_1 < \mu_2$ (one of these inequalities may be non-sharp)

then

$$\underline{I}^-(\lambda_1, \mu_1) > \underline{I}^-(\lambda_2, \mu_2). \tag{4.4}$$

Proof. Since it is very similar to the corresponding proof in /1/, we shall not give it here. The idea is that, if, for example, $\lambda_1 > \lambda_2$ and $\underline{I}^+(\lambda_1, \mu_0) \neq \underline{I}^+(\lambda_2, \mu_0)$ then according to lemma 1B $\varepsilon_n \frac{\partial x_n}{\partial \lambda} > 0$, thus $\underline{I}^+(\lambda_1, \mu_0) > \underline{I}^+(\lambda_2, \mu_0)$, and since $\lim_{n\to\infty} \left|\frac{\partial x_n}{\partial \lambda}\right| = \infty$ the equality $\underline{I}^+(\lambda_1, \mu_0) = \underline{I}^+(\lambda_2, \mu_0)$ is impossible, if $\lambda_1 \neq \lambda_2$. (In the case of finite sequences it is also impossible, because $\frac{\partial x_n}{\partial \lambda} \neq 0$).

From lemma 4B we also obtain the similar proposition about monotonicity of the kneading sequences along lines $\lambda + \mu = const$, $\lambda - \mu = const$.

Proposition B. Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{D}_3 = \{3 \leq \lambda \leq 4, 3 \leq \mu \leq 4, \lambda + \mu \geq \sqrt{3} + 5$ Set $u_i = \frac{\lambda_i + \mu_i}{2}$, $v_i = \frac{\lambda_i - \mu_i}{2}$, $i = 1, 2$, If $u_1 > u_2$, $v_1 > v_2$ (one of these inequalities may be non-sharp) then

$$\underline{I}^+(\lambda_1, \mu_1) > \underline{I}^+(\lambda_2, \mu_2)$$

If $u_1 < u_2$, $v_1 > v_2$ (or $u_1 \leq u_2, v_1 > v_2$; $v_1 \geq v_2, u_1 < u_2$) then

$$\underline{I}^-(\lambda_1,\mu_1) > \underline{I}^-(\lambda_2,\mu_2).$$

Proof is the same.

We say that $(\lambda_1,\mu_1)\neq(\lambda_2,\mu_2)$ if at least one of the inequalities $\lambda_1\neq\lambda_2$, $\mu_1\neq\mu_2$ holds. Similarily $(\underline{A}_1,\underline{B}_1)\neq(\underline{A}_2,\underline{B}_2)$ if either $\underline{A}_1\neq\underline{A}_2$ or $\underline{B}_1\neq\underline{B}_2$ Note that $\tilde{\mathcal{D}}\subset\mathcal{D}_1\subset\mathcal{D}_3\subset\mathcal{D}$, $\tilde{\mathcal{D}}\subset\mathcal{D}_2\subset\mathcal{D}_4\subset\mathcal{D}$ Then using theorem B and proposition B one can easily prove the following

Corollary B. If $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\tilde{\mathcal{D}}$ and $(\lambda_1,\mu_1)\neq(\lambda_2,\mu_2)$ then $(\underline{I}^+(\lambda_1,\mu_1),\underline{I}^-(\lambda_1,\mu_1))\neq(\underline{I}^+(\lambda_2,\mu_2),\underline{I}^-(\lambda_2,\mu_2))$.

In other words, kneading sequences pairs corresponding to distinct points at the parameter plane are distinct.

Proof. We separate the domain around the point (λ_1,μ_1) by the lines $\lambda=const$, $\mu=const$, $\lambda+\mu=const$. (see fig. 1)

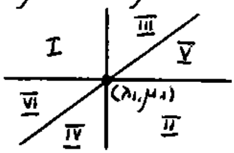

Fig. 1

According to theorem B we have:

$$\underline{I}^+(\lambda,\mu) > \underline{I}^+(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{I}$$

$$\underline{I}^-(\lambda,\mu) < \underline{I}^+(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{II}$$

and by proposition B :

$$\underline{I}^+(\lambda,\mu) > \underline{I}^+(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{III}$$

$$\underline{I}^+(\lambda,\mu) < \underline{I}^+(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{IV}$$

$$\underline{I}^-(\lambda,\mu) > \underline{I}^-(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{VI}$$

$$\underline{I}^-(\lambda,\mu) < \underline{I}^-(\lambda_1,\mu_1) \quad \text{if} \quad (\lambda,\mu)\in \underline{V}.$$

So whichever point (λ,μ) distict from (λ_1,μ_1) we take, at least one of the itineraries $\underline{I}^+(\lambda,\mu)$ or $\underline{I}^-(\lambda,\mu)$ will be different from $\underline{I}^+(\lambda_1,\mu_1)$, or correspondingly from $\underline{I}^-(\lambda_1,\mu_1)$.

Finally in the same spirit using lemma we may prove the theorem A.

Proof of theorem A. It is similar to the proof of the theorem B. If $\lambda>\mu$, $\varepsilon_k\neq 0$ then according to the lemma A $\varepsilon_n\delta_n>0$ So if there is the smallest m such that $I(x_m)\neq I^*(y_m)$, then $x_m>-y_m$ if the sequence $I(x_1)\dots I(x_{m-1})$ is even, and $x_m<-y_m$ if $I(x_1)\dots I(x_{m-1})$ is odd. But this means that

$$\underline{I}^+(\lambda,\mu)=I(x_1)\dots I(x_m)\dots > I^*(y_1)I^*(y_2)\dots I^*(y_m)\dots = (\underline{I}^-(\lambda,\mu))^*.$$

Such m exists, becauce if $\underline{I}^+$ is infinite, then $\lim_{k\to\infty} |\delta_k| = \infty$. If $\underline{I}^+(\lambda,\mu)$ is finite, then m exists, since $\delta_n \neq 0$ (see /1/). The inverse assertion is obvious ($\underline{I}^+(\lambda,\mu) = (\underline{I}^-(\lambda,\mu))^*$ only if $\lambda = \mu$).

5. Intermidiate value theorem

We consider some connected set at the parameter plane, for example a continuos curve $\lambda(t), \mu(t)$. Let $t \in [t_0, t_1]$, $\lambda(t_0) = \lambda_0, \mu(t_0) = \mu_0$ and $\lambda_1 = \lambda(t_1), \mu_1 = \mu(t_1)$, $\underline{I}^+(\lambda_0,\mu_0) = \underline{A}_0$, $\underline{I}^+(\lambda_1,\mu_1) = \underline{A}_1$, $\underline{I}^-(\lambda_0,\mu_0) = \underline{B}_0$, $\underline{I}^-(\lambda_1,\mu_1) = \underline{B}_1$. Suppose that $\underline{A}_1 > \underline{A}_0$, $\underline{B}_1 > \underline{B}_0$. Then we have the following theorem.

Theorem D. Let $\underline{A}$ be maximal sequence,and $\underline{A}_0 < \underline{A} < \underline{A}_1$, $\underline{A} \in \Pi_a$. Let $\forall t \in [t_0, t_1]$, $\forall k \geq 1$, $J^k \underline{A} > \underline{I}^-(\lambda(t), \mu(t))$ if $J^{k-1}\underline{A} \neq D$. Then there is $t^* \in (t_0, t_1)$ such that $\underline{I}^+(\lambda(t^*), \mu(t^*)) = \underline{A}$.

If $\underline{B} \in \Pi_B$, and $\underline{B}_0 < \underline{B} < \underline{B}_1$, $\forall k \geq 1$, $\forall t \in [t_0, t_1]$, $J^{k-1}\underline{B} < \underline{I}^-(\lambda(t), \mu(t))$ if $J^{k-1}\underline{B} \neq C$ then there is $t^* \in (t_0, t_1)$ such that $\underline{I}^-(\lambda(t^*), \mu(t^*)) = \underline{B}$.

Proof. It is enough to consider the case of $\underline{A}$, the other one being similar. To prove realizability of $\underline{A}$ we define by standart way the following sets:

$$L_{\underline{A}} \equiv \{ t : t \in [t_0, t_1] \quad \text{and } \underline{I}^+(\lambda(t), \mu(t)) < \underline{A} \}$$

$$R_{\underline{A}} \equiv \{ t : t \in [t_0, t_1] \quad \text{and } \underline{I}^+(\lambda(t), \mu(t)) > \underline{A} \}.$$

We need to show that they are open. Let us consider $R_{\underline{A}}$ (the case of $L_{\underline{A}}$ is similar) Assume $\hat{t} \in R_{\underline{A}}$. We shall show that there is some neighborhood $V_\delta = (\hat{t} - \delta, \hat{t} + \delta)$ such that $\forall t \in V_\delta$, $\underline{I}^+(\lambda(t), \mu(t)) > \underline{A}$.

For simplicity we denote $\underline{I}^+(\lambda(t), \mu(t)) \equiv \underline{I}^+(t)$, $\underline{I}^-(\lambda(t), \mu(t)) \equiv \underline{I}^-(t)$. Let n be the first index for which $I_n^+(\hat{t}) \neq A_n$. We may assume that the sequence $I_0 \ldots I_{n-1}$ doesn't contain C, D , since otherwise $\underline{I}^+(\hat{t}) = \underline{A}$ (due to the "stopping" rule). The further consideration is just a careful enumaration of various possibilities.

1. If $I_n \neq C, D$ then by continuity of we can preserve the equalities $I_0^+(\hat{t}) = I_0^+(t) \ldots \quad I_n^+(\hat{t}) = I_n^+(t)$ in some neighbourhood V_δ of $\hat{t}$, $\forall t \in V_\delta$, $t \in R_{\underline{A}}$.

2. If $I_n(\hat{t}) = C$ then $\underline{I}_E(\hat{t}) = (\hat{\underline{A}}C)^\infty$ and $\underline{A} = \hat{\underline{A}}T\ldots$ There is a neighbourhood V of $\hat{t}$ such that if $t \in V$, then $\underline{I}^+(t) = \hat{\underline{A}}F$, where $F = M$ or L .

2.1. Let $\hat{\underline{A}}$ be even. Since $\hat{A}C > \hat{A}T$, $T = L$. If $F = M$, then $\underline{I}^+(t) = \hat{\underline{A}}M\ldots > \hat{\underline{A}}L\ldots$ so let us consider the case $F = L$. Since $\underline{A} \in \Pi_a$ we must have or $\underline{A} < \hat{\underline{A}} * L^\infty$ or $\underline{A} > \hat{\underline{A}} * ML^\infty$ (but $\underline{A} = \hat{\underline{A}}L\ldots$ so this is impossible) Therefore there is k such that $\underline{A} = (\hat{\underline{A}}L)^k \hat{\underline{B}}\ldots$, where $|\hat{\underline{B}}| = |\hat{\underline{A}}L|$ and $\hat{\underline{B}} < \hat{\underline{A}}L$. Then choosing sufficiently small V we can provide $\underline{I}^+(t) = (\hat{\underline{A}}L)^{k+1}\ldots$

for $t \in V$ and hence $\underline{I}^+(t) > \underline{A}$ for $t \in V$.

2.2. If $\hat{\underline{A}}$ is odd, then $T = M, R, D$, and the only non-trivial case is $T = M$, $F = M$ Again we must have or $\underline{A} > \hat{\underline{A}} * ML^\infty$ (this is impossible), or $\underline{A} < \hat{\underline{A}} * L^\infty = (\hat{\underline{A}} M)^\infty$. Then there is K such that $\underline{A} = (\hat{\underline{A}} M)^K \hat{\underline{B}} \ldots$ where $|\hat{\underline{B}}| = |\hat{\underline{A}} M|$, $\hat{B} < \hat{\underline{A}} M$. Therefore we again can find V such that $\underline{I}^+(t) = (\hat{\underline{A}} M)^{K+1} \ldots > (\hat{A} M)^K \hat{\underline{B}} \ldots$ for t.

3. Now we consider the case $I_n^+(t) = D$, i.e. $\underline{I}_E^+ = \hat{\underline{A}} D \underline{I}^-(\hat{t})$ $\underline{A} = \hat{\underline{A}} T \ldots$ and for some neghbourhood $\underline{I}^+(t) = \hat{\underline{A}} F \ldots$, $F = M, R$ for $t \in V$. If $\hat{\underline{A}}$ is even, then $T = L, M, C$ The cases $T = C, L$, $F = R$ are trivial. So let $\underline{I}^+(t) = \hat{\underline{A}} M \underline{P}$, $\underline{A} = \hat{\underline{A}} M \hat{\underline{B}}$. According to the conditions if $J^{K-1} \underline{A} \neq D$ then $\forall t \in [t_0, t_1]$, $J^K \underline{A} > \underline{I}^-(t)$ and in particular $\hat{\underline{B}} > \underline{I}^-(\hat{t})$. Let m be the first index for which $\hat{B}_m \neq I_m^-(\hat{t})$. Here again we have several possibilities.

3.1. $I_m^-(\hat{t}) \neq C, D$. Then by continuity of F we can find a neibourhood $V_1 \subset V$ such that for $t \in V_1$

$$I_{n+1}^+(t) = P_1 = I_1^-(\hat{t}) = B_1, \ldots, I_{m+n}^+(t) = P_m = I_m^-(\hat{t}) \neq B_m.$$

Thus for $t \in V_1$, $\underline{I}^+(t) = \hat{\underline{A}} M \underline{P} > \hat{\underline{A}} M \hat{\underline{B}}$ (since $\hat{\underline{A}} M$ is odd)

3.2. $I_m^-(\hat{t}) = D$, i.e. $\underline{I}_E^+(\hat{t}) = \hat{\underline{A}} D \hat{\underline{B}} D$, $\underline{I}_E^-(\hat{t}) = (\underline{B} D)^\infty$. Then $\underline{A} = \hat{\underline{A}} M \hat{\underline{Q}} F \ldots$ If $\underline{Q}$ is even, then $F = R$. By the above arguments for some $V_1 \subset V$ when $t \in V_1$, $\underline{I}^+(t) = \hat{\underline{A}} T \underline{Q} K \ldots$ where $T = M$, $K = M, R$. We consider only the case $K = R$ (the other one being trivial). In the next section we'll show that $\forall$ (λ, μ) $(\underline{I}^+(\lambda,\mu), \underline{I}^-(\lambda,\mu)) \in \Pi$. So $\underline{I}^-(t) = \underline{Q} R \ldots > \underline{Q} * R^\infty$ or $\underline{I}^-(t) = \underline{Q} R \ldots < \underline{Q} * MR^\infty$ (this is impossible). By the conditions

$$J^n \underline{A} = \underline{Q} R \ldots > \underline{I}^-(t) > \underline{Q} * R^\infty.$$

Therefore there is K such that $\underline{A} = \hat{\underline{A}} M (\underline{Q} R)^K \hat{\underline{B}} \ldots$, $|\hat{\underline{B}}| = |\underline{Q} R|$, $\hat{\underline{B}} > \underline{Q} R$ Then choosing $V_2 \subset V_1 \subset V$ we can provide for $t \in V_2$

$$I^-(t) = (\underline{Q} R)^{K+1} \ldots, I^+(t) = \hat{\underline{A}} M (\underline{Q} R)^{K+1} \ldots > \hat{\underline{A}} M (\underline{Q} R)^K \hat{\underline{B}} \ldots$$

The remaining possibilities when $\underline{Q}$ is odd , or $\underline{A}$ is odd, or some other combinations are considered analogously.

3.3. $I_m^-(\hat{t}) = C$, $\underline{I}_E^+(\hat{t}) = (\hat{\underline{A}} D \hat{\underline{B}} C)^\infty$ (so called double cycle). Let $\hat{\underline{A}}$, $\hat{\underline{B}}$ be even. Then we consider some neighbourhood V_1 , where $\underline{I}^+(t) = \hat{\underline{A}} M \hat{\underline{B}} M \ldots$, $\underline{A} = \hat{\underline{A}} M \hat{\underline{B}} M \ldots$ (the other possibilities are trivial). If t is sufficiently close to $\hat{t}$ then

$$\underline{I}^{+}(t) = (\underline{\hat{A}}M\underline{\hat{B}}M)^{p}\ldots$$

(and only this combination is possible, if $\underline{\hat{A}}$ and $\underline{\hat{B}}$ are even!) Note that $\underline{\hat{A}}M\underline{\hat{B}}$ is maximal and odd. Then since $\underline{A} \in \Pi_a$ or $A > (\underline{\hat{A}}M\underline{\hat{B}}) * ML^{\infty}$, or $\underline{A} < (\underline{\hat{A}}M\underline{\hat{B}}) * L^{\infty} = (\underline{\hat{A}}M\underline{\hat{B}}M)^{\infty}$. The first inequality is impossible. Thus there is k, such that

$$\underline{A} = (\underline{\hat{A}}M\underline{\hat{B}}M)^{k}\underline{Q}\ldots \qquad \text{where } |\underline{Q}| = |\underline{\hat{A}}M\underline{\hat{B}}M|,\ \underline{Q} < \underline{\hat{A}}M\underline{\hat{B}}M.$$

Therefore we can find $V_2 \subset V_1$ such that

$$\underline{I}^{+}(t) = (\underline{\hat{A}}M\underline{\hat{B}}M)^{k+1}\ldots > (\underline{\hat{A}}M\underline{\hat{B}}M)^{k}\underline{Q}\ldots$$

If $\underline{\hat{A}}$ is even, $\underline{\hat{B}}$ is odd, then one should consider $\underline{I}^{+}(t) = \underline{\hat{A}}M\underline{\hat{B}}L\ldots$, $\underline{A} = \underline{\hat{A}}M\underline{\hat{B}}L\ldots$. The arguments in this case, as in the remaining cases are similar. This in fact completes the proof, since $L_A^{\perp}$ and $R_A^{\perp}$ closed non-empty, and $[t_0, t_1]$ is connected. Thus $L_A^{\perp} \cap R_A^{\perp}$ is non-empty, so there is $t^* \in (t_0, t_1)$ such that $\underline{I}^{+}(t^*) = \underline{A}$

6. Proof of theorem C

Proof of lemma C. We must check whether the conditions 1A,2A,3A 1C,2C are satisfied by $\underline{I}^{+}(\lambda,\mu)$ and correspondingly 1B,2B,3B, 1C, 2C - by $\underline{I}^{-}(\lambda,\mu)$. 1A,B,C and 2B,A,C follow immediately from the definition of $\underline{I}^{+}(\lambda,\mu)$ and $\underline{I}^{-}(\lambda,\mu)$, and in fact the only condition which we have to consider, is 3A (3B).

Suppose on the contrary that for some (λ,μ) for example $\underline{I}^{+}(\lambda,\mu)$ can be represented as $\underline{Q} * \underline{P}$ in sense of the defini- toin given in 3A. First note that if $\underline{I}^{+}(\lambda,\mu)$ doesn't contain C, D it can not be periodic, since otherwise two points $F(C)$ and $F^{n+1}(C)$ (where n is a period) have equal itineraries (this is impossible because our maps are everywhere expanding, so two arbitrary points will be separated.)

Let now for definitness $\underline{Q}$ be even, $\underline{I}^{+}(\lambda,\mu) = \underline{Q}M\underline{Q}\ldots$, $|\underline{Q}| = n$. Then $F^{n+1}(c) = c + \delta_1$, $\delta_1 > 0$. We consider

$$F^{2(n+1)}(c) = F^{n+1}(c+\delta_1) = c + \delta_1 + \delta_1 k\alpha,$$

where

$$\alpha = \prod_{j=1}^{n} DF(F^{j}(c)).$$

Since $\lambda,\mu > 3$, $|k| > 1$, $\alpha > 3$. Thus

$F^{2(n+1)}(c) = c - \delta_1[1 - |K|\alpha] = c - \delta_2$ and $\delta_2 > \delta_1$.

We have

$F^{3(n+1)}(c) = c - \delta_3$, $|\delta_3| > |\delta_2| > \delta_1$ and so on (we have supposed that $\underline{I}^+ \neq \underline{Q} * \underline{P}$ so we can continue thos process infinitely).

Then

$$\delta_m = c - F^{m(n+1)} \quad , \ \delta_m > f(m)\,\delta_1 \quad \text{where} \quad \lim_{m\to\infty} f(m) = \infty .$$

When δ_m increase sufficiently, the sequence $\underline{Q}$ will be destroyed, so $\underline{I}^+(\lambda,\mu)$ can not be of form $\underline{Q} * \underline{P}$. The treatment of the case, when $\underline{Q}$ is odd, and of $\underline{I}^-(\lambda,\mu)$ is similar.

Proof of theorem C. Let $(\underline{A},\underline{B}) \in \Pi$ and $\underline{A} \geq RRD$, $\underline{B} \leq LLC$. The assertion is obvious if $\underline{A} = R^\infty, \underline{B} = L^\infty$ ($\underline{I}^+(4,4) = R^\infty, \underline{I}^-(4,4) = L^\infty$). It is also obvious if $\underline{A} = R^\infty$, $\underline{B} \neq L^\infty$ or $\underline{B} = L^\infty$, $\underline{A} \neq R^\infty$. (Since for example the set $= \{(\lambda,\mu) \in \tilde{\mathcal{D}}, \underline{I}^+(\lambda,\mu) = R^\infty\}$ is the line $\lambda = 4$ and $J^K\underline{B} < R^\infty$ if $J^{K-1}\underline{B} \neq C$ and $L^\infty < \underline{B} < D$, we can apply theorem C).

So let $\underline{A} \neq R^\infty$, $\underline{B} \neq L^\infty$. According to the just given arguments there is a μ_0 such that $\underline{I}^-(4,\mu_0) = \underline{B}$, $\mu_0 < 4$. Let us consider the domain $G_0 = \{(\lambda,\mu) \in \tilde{\mathcal{D}}, \mu \geq \mu_0, \lambda < 4\}$. By theorem B $\forall (\lambda,\mu) \in G_0$ $\underline{I}^-(\lambda,\mu) < \underline{I}^-(4,\mu_0) = \underline{B}$. Hence

$$J^K\underline{A} > \underline{B} > \underline{I}^-(\lambda,\mu) \qquad \forall\ (\lambda,\mu) \in G_0 .$$

Since $D < \underline{A} < R^\infty$, we can apply theorem C. So for every $\mu \geq \mu_0$ there is a $\lambda(\mu)$ such that $\underline{I}^+(\lambda(\mu),\mu) = \underline{A}$. Moreover, by theorem B this value $\lambda(\mu)$ is unique. $\lambda(\mu)$ is increasing and continuos by the intermidiate value theorem: if $\mu_1 < \mu_2$ then $\lambda(\mu_1) < \lambda(\mu_2)$. If $\lambda(\mu_1) < \lambda < \lambda(\mu_2)$, then $\underline{I}^+(\lambda,\mu_2) < \underline{A} < \underline{I}^+(\lambda,\mu_1)$, hence there is a μ, $\mu_1 < \mu < \mu_2$ such that $\underline{I}^+(\lambda,\mu) = \underline{A}$).

Let $\lambda(\mu_0) = \lambda_0$. Let us consider now the domain $G_1 = \{(\lambda,\mu) \in \tilde{\mathcal{D}}, \lambda \geq \lambda_0 , \mu < 4\}$. By the similar arguments for every $\lambda \geq \lambda_0$ there is an unique $\mu(\lambda)$ such that $\underline{I}^-(\lambda,\mu(\lambda)) = \underline{B}$. $\mu(\lambda)$ is continuos and increasing. Set $\mu(\lambda_0) = \lambda_1$, $\mu(\lambda_0) < \mu(4) = \mu_0$. Then we consider $G_2 = \{(\lambda,\mu) \in \tilde{\mathcal{D}}, \mu \geq \mu_1, \lambda < 4\}$. We can continue the curve $\lambda(\mu)$ (where $\underline{I}(\lambda(\mu),\mu) = \underline{A}$) for $\mu_1 \leq \mu \leq \mu_0$. Let $\lambda(\mu_1) = \lambda_1$,.. and so on.

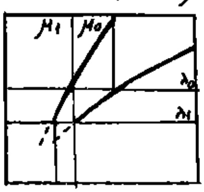

Fig. 2

We obtain the sequences

$$\lambda_0 = \lambda(\mu_0),\ \mu_1 = \mu(\lambda_0),\ \lambda_1 = \lambda(\mu_1),\ \dots,\ \mu_n = \mu(\lambda_{n-1}),\ \lambda_n = \lambda(\mu_n)$$

$$\underline{A}_0 = \underline{I}^+(\lambda_0,\mu_0),\ \underline{B}_0 = \underline{I}^-(\lambda_0,\mu_1),\ \underline{A}_1 = \underline{I}^+(\lambda_1,\mu_1),\ \dots,\ A_n = \underline{I}^+(\lambda_n,\mu_n),\ B_n = \underline{I}^-(\lambda_{n-1},\mu_n).$$

The sequences $\{\lambda_n\}$, $\{\mu_n\}$ are decreasing and bounded (the same is true for $\underline{A}_n$, $\underline{B}_n$). Hence

$$\lim_{n\to\infty} \lambda_n = \lambda^*,\quad \lim_{n\to\infty} \mu_n = \mu^*$$

exist, and $\mu^* = \mu(\lambda^*)$, $\lambda(\mu^*) = \lambda^*$. By definition of the curves

$$\underline{I}^+(\lambda^*,\mu^*) = \underline{A},\quad \underline{I}^-(\lambda^*,\mu^*) = \underline{B}.$$

Due to corollary B (λ^*,μ^*) is unique. Note also, that $(\lambda^*,\mu^*) \in \tilde{\mathcal{D}}$ since otherwise or $\underline{I}^+(\lambda^*,\mu^*) < RRD$ or $\underline{I}^-(\lambda^*,\mu^*) > LLC$. This completes the proof.

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