объединенный
институт
ядерных
исследований
дубна

E5-90-12

D.V.Ktitarev

FUNCTIONAL INTEGRAL IN SUPERSYMMETRIC QUANTUM MECHANICS

Submitted to "Letters in Mathematical Physics"

1990

The evolution operator for quantum mechanical systems can be expressed in the form of the Feynman path integral [1]. For some classes of hamiltonians Kac, Ito and other authors defined the path integral as the integral with respect to a measure in some functional space [2,3]. Such formulae named after Feynman, Kac and Ito were considered in numerious works on quantum and statistical physics [3].

The construction of functional measures in supersymmetric theories [4] is a rather more complicated problem [5]. A variant of the Feynman-Kac-Ito formula for some supersymmetric system with the real time was considered by Berezin [6]. The similar results were obtained by Khrennikov for systems with the time belonging to the odd part of a superalgebra [7]. Taking under consideration commuting and anticommuting (1,1)-time, Rogers proved the Feynman-Kac formula for the supersymmetric system corresponding to the imaginary time quantum mechanics [8]. In the present paper we consider the super (1,1)-time square root of the Schrödinger equation that represents the real time quantum mechanics. We obtain the Feynman-Ito formula in the space of superdistributions for some class of superpotentials.

We consider the Cauchy problem for the square root of the Schrödinger equation with (1,1)-supertime:

$$\begin{cases} D_{t,\tau}f(t,\tau,x,\theta)=Qf \\ f(0,0,x,\theta)=g(x,\theta). \end{cases} \tag{1}$$

Here $D_{t,\tau}=\frac{\partial}{\partial\tau} + i\tau\frac{\partial}{\partial t}$ is the supersymmetric time derivative, the supercharge Q has the form $Q= (\Psi\cdot p)/\sqrt{2} + W(t,\tau,x,\theta)$, where Ψ and p represent the operators defined on the space of functions of n commuting and n anticommuting real variables (x_j,θ_j) by

$$\Psi^j= \theta_j+ \frac{\partial}{\partial\theta_j}, \ p_j=-i\frac{\partial}{\partial x_j}.$$

The function W is the superpotential and the function g is the initial condition of the problem (here and further $\iota=\sqrt{-1}$).

We shall construct a solution of the problem (1) in the space of superdistributions. Let $\Lambda=\Lambda_0+\Lambda_1$ be a Banach commutative superalgebra (CSA) over the field of complex numbers $\mathbb{C}$ and $A=A_0+A_1$ be an infinite-dimentional pseudotopological CSA over the field of real numbers $\mathbb{R}$ with nilpotent odd elements (except numbers) and trivial A_1-annulator [7,9]. Denote by $\mathbb{R}_A^{n,n}$ the superspace $A_0^n\times A_1^n$ over the CSA A. Let $\mathcal{D}(\mathbb{R}_A^{n,n},\Lambda)$ be the space of smooth functions defined on $\mathbb{R}_A^{n,n}$, compactly supported over the real directions [7,9]. Denote by $\mathcal{D}'(\mathbb{R}_A^{n,n},\Lambda)$ the space of Λ-linear continuous functionals $F:\mathcal{D}\to\Lambda$.

The super-pseudodifferential operator (S-PDO) $P(x,\theta,-\iota\partial,\iota D)$ acting on $\mathcal{D}'(\mathbb{R}_A^{n,n},\Lambda)$ with smooth pq-symbol $P:\mathbb{R}_A^{n,n}\times\mathbb{R}_A^{n,n}\to\Lambda$ is defined by [7]

$$\langle PF,\phi\rangle=\langle F,P^*\phi\rangle,\quad F\in\mathcal{D}',\quad \phi\in\mathcal{D},\text{and}$$

$$P^*(x,\theta,-\iota\partial,\iota D)\phi(x,\theta)=\frac{1}{(2\pi)^n\iota^n}\int_{\mathbb{R}^{2n}\times A_1^{2n}} d^np\times d^nq\times d^n\xi\times d^n\eta\times$$

$$\times P(q,\xi,-p,-\eta)\phi(x,\theta)\exp[\iota(p,q)-\iota(p,x)+\iota\xi(\eta-\theta)]$$

(we consider left derivatives of anticommuting variables throughout this paper).

Let us now consider the Cauchy problem (1) for differentiable functions $f:\ A\longrightarrow\mathcal{D}'(\mathbb{R}_A^{n,n},\Lambda)$ so that $(t,\tau)\in A$, $(x,\theta)\in\mathbb{R}_A^{n,n}$.

Note that the free super charge $Q_0=(\Psi\cdot p)/\sqrt{2}$ of the righthand side of the equation (1) may be defined as a S-PDO acting on the space $\mathcal{D}'$ with the symbol $Q_0(\theta,p,\eta)=\sum_{j=1}^{n}(\theta_j+\eta_j)p_j$.

Next, let us suppose that the potential $W:\ A\times\mathbb{R}_A^{n,n}\to\Lambda$ is the Fourier transform of the family of Borel measures $\tilde{W}$ depending on the parameters $(t,\tau)\in A,\eta\in A_1^n$:

$$W(t,\tau,x,\theta)=\int_{\mathbb{R}^n\times A_1^n} \exp[i(x,p)+i(\theta,\eta)]\tilde{W}(t,\tau,d^np,\eta)d^n\eta,$$

$\tilde{W}(t,\tau,d^np,\eta)=\sum_{|\alpha|\leq n}\tilde{W}_\alpha(t,\tau,d^np)\eta^\alpha,\quad \eta^\alpha=\eta_1^{\alpha_1}\ldots\eta_n^{\alpha_n}$ and the measures $\tilde{W}_\alpha(t,\tau,d^np)$ have uniformly bounded supports and uniformly bounded variations: $\forall t,\tau,\alpha\ \operatorname{supp}\tilde{W}_\alpha(t,\tau,d^np)\subset B_R,\ B_R=\{x\in\mathbb{R}^n: \|x\|\leq R\}$,

$$\int d\sigma\int_0^T ds\int_{\mathbb{R}^n}\|\tilde{W}_\alpha(s,\sigma,d^np)\|<C,\ C>0. \tag{2}$$

We shall also suppose the same conditions for the initial value $g(x,\theta)$ are hold:

$$g(x,\theta)=\int_{\mathbb{R}^n\times A_1^n}\exp[i(x,p)+i(\theta,\eta)]\tilde{g}(d^np,\eta)d^n\eta,$$

$$\tilde{g}(d^np,\eta)=\sum_{|\beta|\leq n}\tilde{g}_\beta(d^np)\eta^\beta,\ \forall\beta\ \operatorname{supp}\tilde{g}_\beta(d^np)\subset B_R,$$

$$\int_{\mathbb{R}^n}\|\tilde{g}_\beta(d^np)\|<C,\ C>0. \tag{3}$$

Finally, define the symbol $\tilde{Q}_0$ by $\tilde{Q}_0(\xi,p,\eta)=\sum_{j=1}^n(\eta_j\xi_jp_j-ip_j)$.

THEOREM. The super-distribution

$$f(t,\tau,x,\theta)=\sum_{m=0}^\infty \frac{1}{i^m}\int_0^\tau\int_0^t DT_m\int_0^{\tau_m}\int_0^{t_m}DT_{m-1}\ldots\int_0^{\tau_2}\int_0^{t_2}DT_1\times\sum_{l=0}^m\ \sum_{1\leq k_1<\ldots<k_l\leq m}$$

$$\int\overset{(m)}{\ldots}\int\tilde{W}(t_m,\tau_m,dp_m,\xi_m)\ \ldots\ \tilde{Q}_0\Big(\xi_{k_l},\sum_{\substack{j=0\\ j\neq k_1,\ldots,k_{l-1}}}^{k_l-1}p_j,\sum_{\substack{j=0\\ j\neq k_1,\ldots,k_{l-1}}}^{k_l-1}\eta_j\Big)\times$$

$$\times\ \ldots\ \tilde{Q}_0\Big(\xi_{k_1},\sum_{j=0}^{k_1-1}p_j,\sum_{j=0}^{k_1-1}\eta_j\Big)\ \ldots\ \tilde{W}(t_1,\tau_1,dp_1,\xi_1)\ \tilde{g}(dp_0,\xi_0)\times$$

$$\times\exp\Big[i\Big(x,\sum_{\substack{j=0\\ j\neq k_1,\ldots,k_l}}^m p_j\Big)+i\Big(\theta,\sum_{\substack{j=0\\ j\neq k_1,\ldots,k_l}}^m\eta_j\Big)\Big]d\xi_m\ldots d\xi_0 \tag{4}$$

is the solution of the Cauchy problem (1).

The proof of the theorem is similar to the proof of its analogue in the paper [10]. First, using the estimates (2), (3) one needs to show that the series (4) converges in the space $\mathcal{D}'(R_A^{n,n},\Lambda)$ for each moment (t,τ). Then, differentiating the series and using the superanalogue of the formula for differentiation with respect to the limits of integration (see [8])

$$D_{t,\tau}\left[\frac{1}{\iota}\int_0^{\tau}\int_0^{t} DSf(s,\sigma)\right]=f(t,\tau),$$ where for $f(s,\sigma)=f_0(s)+f_1(s)$, $(s,\sigma)=S$

the integral is defined by $\frac{1}{\iota}\int_0^{\tau}\int_0^{t}DS[f_0(s)+\sigma f_1(s)]=\frac{1}{\iota}\int_0^{t}f_1(s)ds+\tau f_0(t)$, one may deduce the proposition of the theorem. Note that the theorem remains true for a wider class of S-PDO, considered in [10].

To conclude, let us discuss the formula (4). We can consider it as the functional integral with respect to generalized measure in the space of superpaths for the evolution operator of a supersymmetric quantum mechanical system (cf. [8]):

$$\langle 0,0|\exp(-\iota tH-\tau Q)|x,\theta\rangle=$$

$$=\int DyD\eta\ \mathrm{T}\text{-}\exp\left\{\frac{1}{\iota}\int_0^{\tau}\int_0^{t}dS\left[\frac{\partial y(s,\sigma)}{\partial s}\eta(s,\sigma)+W(s,\sigma,x+y(s,\sigma),\theta+\eta(s,\sigma))\right]\right\}.$$

The formula (4) may be regarded also as the definition of the chronological exponent of the symbol of the S-PDO(cf.[10]). This result shows that the methods of functional integration in the bosonic case may be applied to the fermionic and sypersymmetric cases, though the construction of the corresponding measure in the space of paths is more complicated.

The author expresses gratitude to A.Yu.Khrennikov and O.K. Pashaev for helpful discussions.

REFERENCES

1. Feynman R.P., Hibbs A. *Quantum mechanics and path integrals*. McGraw Hill, New York, 1965.
2. Kac M.- *Proc. 2nd Berkeley symp. math. stat. prob.*, Univ. Calif. press, Berkeley, 1951, p. 189.

   Ito K.- *Proc. 4th Berkeley symp. math. stat. prob.*, Univ. Calif. press, Berkeley, 1961, p. 227; *Proc. 5th Berkeley symp. math. stat. prob.*, Univ. Calif. press, Berkeley, 1967, p. 145.
3. Daletskiĭ Yu.L.,Fomin S.V. *Measures and differential equations in infinite-dimentional spaces*. Nauka, Moscow, 1983 (in russian); see also references to ch. VI.
4. Berezin F.A. *Introduction to superanalysis*. D.Reidel, Dordrecht, 1986.

   Rogers A.- *Journ. Math. Phys.*, 1981, **22**, 939 (1981).

   DeWitt B. *Supermanifolds*. Cambridge Univ. press,Cambridge, 1984.

   Vladimirov V.S., Volovich I.V.- *Theor. Math. Fiz.*, **59**, 3; **60**, 169 (1984).
5. Smolyanov O.G., Shavgulidze E.T.- *Doklady Acad. Nauk USSR*, **299**, 816 (1988).
6. Berezin F.A.- *Theor. Math. Fiz.*, **6**, 194 (1971).
7. Khrennikov A.Yu.- *Theor. Math. Fiz.*, **73**, 420 (1987); *Usp. Matem. Nauk*, **43**, 87 (1988), *Diff. Uravn.*, **24**, 2144 (1988); **25**, 314 (1989).
8. Rogers A.- *Phys. Lett. B*, **193**, 48 (1987); **213**, 37 (1988); *Comm. Math. Phys.*, **113**, 353 (1987).
9. Nagamachi S., Kobayashi Y.- *Lett. Math. Phys.*, **14**, 15 (1987); **15**, 17 (1988).
10. Ktitarev D.V. - *Lett. Math. Phys.*, **18**, 325 (1989).

Received by Publishing Department
on January 9, 1990.