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N-MONOPOLE-SOLITON-TYPE SOLUTIONS OF THE SELF-DUAL EQUATIONS FOR AN SU(2) GAUGE THEORY IN MINKOWSKI SPACE-TIME

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1. *The technical preliminaries.* Let us introduce in the Minkowski space-time (M4) the orthonormal basis of three space-like vectors k^i_μ ($i = 1, 2, 3$, $\mu = 0, 1, 2, 3$), $k^i_\mu k^j_\mu = -\delta^{ij}$ and a time-like vector $n_\mu = \frac{1}{6}\epsilon_{\mu\nu\alpha\gamma}\epsilon^{ijk} k^i_\nu k^j_\alpha k^k_\gamma$, $n^2_\mu = 1$; $n_\mu k^i_\mu = 0$. These definitions imply some useful identities, such as $\epsilon^{ijk} k^j_\alpha k^k_\gamma = \epsilon_{\mu\nu\alpha\gamma} n_\mu k^i_\nu$, the completness condition $k^i_\mu k^i_\nu = n_\mu n_\nu - g_{\mu\nu}$, and we may use them to construct the following important objects:

a) The antisymmetric tensor $R^i_{\mu\nu} = -\epsilon^{ijk} k^j_\mu k^k_\nu$, its dual $\tilde{R}^i_{\mu\nu} = J^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\gamma} R^i_{\alpha\gamma} = n_\mu k^i_\nu - n_\nu k^i_\mu$, and self- (antiself-) dual tensors $\eta^{i\pm}_{\mu\nu} = R^i_{\mu\nu} \pm i J^i_{\mu\nu}$. It is not difficult to check that $\eta^{i\pm}_{\mu\nu}$ satisfy in M4 the identities introduced by t'Hooft [1] for his tensor; in the standard references frame ($k^i_\mu = -\delta^i_\mu$, $n_\mu = (1,0,0,0)$) our tensors $\eta^{i\pm}_{\mu\nu}$ coincide with t'Hooft's tensors. So $\eta^{i\pm}_{\mu\nu}$ are t'Hooft's tensors for an arbitrary reference frame (the covariant form of t'Hooft's tensors);

b) Scalar variable $w = \sqrt{(k^i_\mu \tilde{x}_\mu)^2} = \sqrt{(n_\mu \tilde{x}_\mu)^2 - \tilde{x}^2_\mu} = \sqrt{-\xi^2_\mu}$; $\xi_\mu = n_\mu (n\tilde{x}) - \tilde{x}_\mu$; $\tilde{x}_\mu = x_\mu - x_{0\mu}$; $x_{0\mu}$ = const. In the standard frame we have $w = \tilde{r} = |\vec{r} - \vec{r}_0| = \sqrt{(x_i - x_{0i})^2}$. We need w to construct the spherical-symmetric functions in covariant form (for an arbitrary frame). The derivative $w_\mu = \xi_\mu / w$ is a unit vector: $w^2_\mu = -1$; $w_{\mu\mu} = -2w^{-1}$.

2. Here we shall construct the regular spherical-symmetric solutions of the complex self-dual equations for Yang-Mills tensor $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu$:

$$F^i_{\mu\nu} = i\tilde{F}^i_{\mu\nu} \left(= \frac{i}{2}\epsilon_{\mu\nu\alpha\gamma} F^i_{\alpha\gamma}\right) \qquad (1)$$

in an arbitrary frame M4 for the algebra SU(2) (or of the real equations for sl(2,c) [2]). Instead of Eq.(1) we shall use the equivalent [3] equation

$$\eta^{k-}_{\mu\nu} F^i_{\mu\nu} = 0, \qquad (2)$$

where $\eta^{k-}_{\mu\nu}$ is introduced in sec.1. Let us seek to determine the solution of (2) in the form

$$A^i_\mu = R^i_{\mu\nu} p_\nu(x) + iJ^i_{\mu\nu} f_\nu(x) \qquad (p_\nu = \partial_\nu p, \dots) . \tag{3}$$

Inserting Eq.(3) into (2), we obtain

$$\begin{gathered} -2\delta^{ik}(p_{\mu\mu} + p_\mu f_\mu) + 2\delta^{ik} n_\mu n_\nu (p_{\mu\nu} + f_{\mu\nu} + f_\mu p_\nu + f_\mu f_\nu) \\ -k^k_\mu k^i_\nu (2p_{\mu\nu} + 2f_{\mu\nu} + f_\mu p_\nu + f_\nu p_\mu + 2p_\mu p_\nu) + \\ + 2i\epsilon^{ijk} k^j_\mu n_\nu (p_{\mu\nu} + f_{\mu\nu} + f_\mu f_\nu + p_\mu f_\nu) = 0 . \end{gathered} \tag{4}$$

By requiring that the coefficients at the independent tensor structures be equal to zero, we arrive at the over-determined system for p_ν, f_ν. This system can be simplified in two ways:

a) Let $f(x)$ and $p(x)$ depend only on the variable $w = \sqrt{(k^i_\mu \tilde x_\mu)^2}$ and identify the set of vectors $(\check k^i_\mu, \check n_\mu)$ entering in w, with the set (k^i_μ, n_μ) on which $R^i_{\mu\nu}$, $J^i_{\mu\nu}$, depend (in general these sets are independent). Then Eq.(4) implies ($f' = \partial_w f, \dots$)

$$\begin{gathered} \delta^{ik}[-p'' + w^{-1}(f' - p') - p'f'] + \\ + (k^k x)(k^i x)[p'' + f'' - w^{-2}(f' + p') + f'p' + p'^2] w^{-2} = 0 , \end{gathered} \tag{5}$$

and we arrive at the system

$$\begin{gathered} f'' - 2w^{-1} p' + p'^2 = 0 , \\ p'' + w^{-1}(p' - f') + p'f' = 0 . \end{gathered} \tag{6}$$

The substitution $p' = \alpha + w^{-1}$, $f' = \beta - w^{-1}$ implies

$$\beta' + \alpha^2 = 0 ,$$

$$\alpha' + \alpha\beta = 0 ,$$

these equations yield $\alpha^2 - \beta^2 = c = \text{const}$. Integrating, we have finally

$$\begin{gathered} p' = w^{-1} \pm c\,\mathrm{csh}[c(w - w_0)] , \\ f' = -w^{-1} + c\,\mathrm{cth}[c(w - w_0)] , \qquad w_0 = \text{const} . \end{gathered} \tag{7}$$

The regularity condition yields $c \neq 0$, $w_0 = 0$ and the "-" sign in the 1-st equation. For A^i_ν we get

$$A^i_\nu = w^{-1}[n_\mu(n\tilde x) - \tilde x_\mu][R^i_{\mu\nu} p'(w) + iJ^i_{\mu\nu} f'(w)] . \tag{8}$$

In the rest frame $n_\mu = (1,0,0,0)$, $w = \tilde r = |\vec r - \vec r_0|$ we have $A^i_j = -\epsilon^{ijk} \tilde x^k \tilde r^{-1} p'(\tilde r)$, $A^i_0 = -i\tilde x^i \tilde r^{-1} f'(\tilde r)$. This static solution was obtained in ref. /4/ by solving the Yang-Mills equations for the sl(2,c) algebra.

b) The second way of the simplification of Eq.(4) is the imposing of the reduction $f = -p$ (in this case the sets $(\check k^i_\mu, \check n_\mu)$, (k^i_μ, n_μ) are generically independent). Then we have a single equation $p_{\mu\mu} - p^2_\mu = 0$; the substitution $p = -\ln\phi$ implies the d'Alambert equation

$$\phi_{\mu\mu} = 0 \tag{9}$$

Substitution $\phi = \phi(w)$ yields the singular solution $\phi \sim w^{-1}$. Let us search for the particular solution of Eq.(9) in the more general form

$$\phi = \alpha(w)\,\beta(s) , \qquad s = (n_\mu \tilde x_\mu) . \tag{10}$$

Then we have

$$\phi_{\mu\mu} = -\alpha''\beta - 2w^{-1}\alpha'\beta + \alpha\ddot\beta = 0 \quad (\alpha' = \partial_w \alpha,\ \ \dot\beta = \partial_s \beta) ,$$

and

$$\frac{\ddot\beta}{\beta} = \frac{\alpha''}{\alpha} + \frac{2}{w}\frac{\alpha'}{\alpha} = k = \text{const} .$$

Putting $\alpha = w^{-1}\gamma$ we arrive at the system

$$\begin{gathered} \gamma'' - k\gamma = 0 , \\ \ddot\beta - k\beta = 0 . \end{gathered} \tag{11}$$

We choose $k = m^2 > 0$. The general solution of Eq.(11) is $\beta = c_1 e^{\tilde s} + c_2 e^{-\tilde s}$, $\gamma = c_3 e^{\tilde w} + c_4 e^{-\tilde w}$ ($\tilde w = mw$, $\tilde s = ms$); regularity of ϕ yields $c_4 = -c_3$, and we put $c_2 = 0$ (or $c_1 = 0$) to avoid the "tachionic exponent" in $(\ln\phi)_\mu$. Then we have $\phi = c\tilde w^{-1}\mathrm{sh}\,\tilde w\, e^{\epsilon \tilde s}$ ($\epsilon = \pm 1$), and, finally

$$A^i_\nu = \bar\eta^i_{\nu\mu}[\tilde w_\mu(\mathrm{cth}\,\tilde w - \tilde w^{-1}) + \epsilon m n_\mu] . \tag{12}$$

This is a regular localized solution with the centre in an arbitrary point of space, $\vec{r}_0$. It moves in an arbitrary direction with a speed $\vec{v} = \vec{n}/n_0$ $(n_\mu = (n_0, \vec{n}))$.

The general solution of eq.(5) of the form (10)

$$\phi = \sum_{a=1}^{N} c_a\, \alpha(\tilde{w}^a)\, \beta(\hat{s}^a) + c_0 ,$$

$$\tilde{w}^a = m^a \sqrt{[\,n^a_\mu (x_\mu - x^a_{0\mu})]^2 - (x_\mu - x^a_{0\mu})^2}\,; \quad \tilde{s}^a = \epsilon^a m^a [\,n^a_\mu (x_\mu - x^a_{0\mu})],$$
$$\epsilon^a = \pm 1 .$$

produces the final N-soliton-type expression:

$$A^i_\nu = \eta^i_{\nu\mu} \frac{\sum_{a=1}^{N} c_a [\, w^a_\mu w^{a\,-1} (\mathrm{ch}\,\tilde{w}^a - \tilde{w}^{a\,-1}\, \mathrm{sh}\,\tilde{w}^a) + m^a \epsilon^a n^a_\mu\, \mathrm{sh}\,\tilde{w}^a]\, e^{\tilde{s}^a}}{c_0 + \sum_{a=1}^{N} c_a \tilde{w}^a\, \mathrm{sh}\,\tilde{w}^a e^{\tilde{s}^a}} . \quad (13)$$

This solution depends on 8N parameters such as c_a/c_0 (or c_a for $c_0 = 0$), x^a_{0i}, m^a, and 3N "angles" which parameterize the vectors n^a_μ. There is also a set of discrete parameters, $\epsilon^a = \pm 1$.

The analytical investigation of the general solution is extremely complicated; the computer analysis of the simplest "head-on" collision ($N = 2$, $n^1_0 = n^2_0$, $n^1_i = -n^2_i$, $x^1_0 = -x^2_0$, $m^1 = m^2$, $c_0 = 0$, $c_1 = c_2$) gives a picture which is rather far from the standard one for two-dimensional elastic scattering (and is essentially distinct for $\epsilon^1 = \epsilon^2$ and $\epsilon^1 = -\epsilon^2$). The detailed analysis will be published elsewhere.

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Гетманов Б.С. E5-89-826
Решения N-монополь-солитонного типа уравнений самодуальности в пространстве Минковского

Предложен аппарат для представления анзаца Ву-Янга, тензора Хуфта и сферически-симметричных функций в ковариантной форме /в произвольной системе отсчета/. С помощью этого аппарата получены в ковариантной форме монопольные решения /в том числе N-солитонного типа/ уравнений самодуальности в пространстве Минковского.

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Getmanov B.S. E5-89-826
N-Monopole-Soliton-Type Solutions of the Self-Dual Equations for an SU(2) Gauge Theory in Minkowski Space-Time

The techniques for representation of Wu-Yang ansatz, t'Hooft tensor, and spherical-symmetric functions in a covariant form (for an arbitrary frame) is introduced. Monopole solutions (including N-soliton-type solutions) of the self-dual equations in Minkowski space-time are constructed in a covariant form.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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