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INSTABILITIES AND SOLITON STRUCTURES IN THE DRIVEN NONLINEAR SCHROEDINGER EQUATION

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1. Introduction

Small-amplitude breathers of the easy-axis ferromagnet and the long Josephson junction placed in the h.f. alternating field, may be described within the frame of the damped driven NLS equation:

$$i\psi_t + \psi_{xx} + 2\psi \mid \psi \mid^2 = -Fe^{i\Omega t} + i\gamma\psi. \tag{1}$$

The same equation arises in various problems of nonlinear optics and plasma physics. The wide spectrum of applications stimulated intensive mathematical studies of eq.(1), mainly of its soliton solutions (see [1,2] and refs. therein).

Most of these studies were devoted to the evolution of the initial condition in the form of the "pure" ($\gamma = F = 0$) NLS soliton. Speaking otherwise, the formulation of problem was as what happens to the unperturbed NLS soliton under the action of pumping and friction. As a rule, the r.h.s. of (1) was considered as a small perturbation, and the treatment was reduced to the analysis of the adiabatic change of the soliton's parameters.

Here we consider eq.(1) from the different point of view. Namely, we address ourselves to a question of what are the basic nonlinear constituents (i.e., asymptotic states) of this equation. In other words, it is the exact soliton solutions of the full eq.(1) that will be the object of our interest here. To discriminate between the effects of pumping and friction, at the first stage we confine ourselves to the damping-free case, $\gamma = 0$.

2. Solitons and the associated linearized system

The transformation $\psi(x,t) = \phi(x,t)e^{i\Omega t}$ takes eq.(1) to

$$i\phi_t + \phi_{xx} - \Omega\phi + 2\phi \mid \phi \mid^2 = -F. \tag{2}$$

Given a solution $\phi(x,t)$, $\tilde{\phi}(x,t) = k\phi(kx, k^2 t)$ is also a solution, this time corresponding to $\tilde{F} = k^3 F$ and $\tilde{\Omega} = k^2\Omega$. Consequently, any solution to eq. (2) is characterized, up to a simple scaling, by a single combination $h = F\Omega^{-3/2}$.

It is not difficult to find two different soliton solutions of eq.(2):

$$\phi_\pm(x,t) = \phi_\pm(x) = \phi_0 \left(1 + \frac{2\sinh^2\alpha}{1 \pm \cosh\alpha\cosh(Ax)}\right). \tag{3}$$

Here α is the monotonously decreasing function of h:

$$h = \sqrt{2}\cosh^2\alpha(1 + 2\cosh^2\alpha)^{-3/2}, \tag{4}$$

$A/2$ is the "area" of both ϕ_+ and ϕ_- solitons :

$$A = 2\int\{\phi_\pm^2(x) - \phi_0^2\}dx = 2\sinh\alpha\phi_0, \tag{5}$$

and ϕ_0 is their asymptotics:

$$\phi_\pm(x) \to \phi_0, \qquad \phi_0^3 = \frac{F}{4\cosh^2\alpha}.$$

Without loss of generality we shall accept that F is > 0 and therefore ϕ_0 is > 0. We also remark that from the very beginning Ω was chosen to be > 0 since otherwise the homogeneous solution $\phi(x,t) \equiv \phi_0$ would be unstable.

To examine the stability of the solitons, we write $\phi(x,t) = \phi_\pm(x) + \delta\phi(x,t)$ and linearize eq.(2) w.r.t. small perturbation $\delta\phi$. Denoting $f = Re\delta\phi$ and $g = Im\delta\phi$, we have

$$f_t = L_0 g \equiv (-d^2/dx^2 + \Omega - 2\phi^2)g \tag{6}$$

$$-g_t = L_1 f \equiv (-d^2/dx^2 + \Omega - 6\phi^2)f, \tag{7}$$

where $\phi = \phi_\pm(x)$. The following properties of L_0 and L_1 appear to be rather essential.

A. L_1 has a single negative eigenvalue, with the corresponding eigenfunction being nodeless.

The proof is standard.

B. When $\phi = \phi_+$ the operator L_0 is positive definite.

Proof. It is straightforward to observe that

$$\tilde{L}_0(\phi - \phi_0) = 0, \tag{8}$$

where $\tilde{L}_0 \equiv L_0 - \phi_0(\phi - \phi_0)$. For $\phi - \phi_0$ is nodeless, eq.(8) implies that the minimum eigenvalue of $\tilde{L}_0$ is zero. And since $\phi_0(\phi_+ - \phi_0)$ is > 0 we have that the minimum eigenvalue of the operator $L_0 = \tilde{L}_0 + \phi_0(\phi_+ - \phi_0)$ is positive.

C. When $\phi = \phi_-$, the operator L_0 has a single negative eigenvalue and no zero eigenvalues.

Proof. Denote the minimal eigenvalue through μ_0. It is useful to fix, without loss of generality, $\Omega = 1$. Then $\mu_0 = \mu_0(F)$. When $F = 0$, $\mu_0 = 0$ holds. If F becomes small but finite, then the perturbation theory yields that μ_0 becomes < 0. It is straightforward then to verify that $\mu_0(F)$ can never vanish so it should stay negative for all F's. Lastly, the assumption that L_0 has another non-positive eigenvalue contradicts the fact that L_1, $L_1 = L_0 - 4\phi^2$ has only one negative eigenvalue.

3. Stability

The analysis of the stability of ϕ_+ is elementary. Since L_0 is positive definite we can pass from (6), (7) to

$$L_0^{-1} f_{tt} = -L_1 f. \tag{9}$$

Equations of this type were analysed extensively in literature (see e.g. [3]-[6]). An immediate consequence of that L_1 has a negative eigenvalue, is instability of the zero solution of (9) and thereby of the soliton ϕ_+.

The situation with ϕ_- is less trivial, however. In this case L_0 is also invertible, but in equation (9) both L_1 and L_0^{-1} operators have negative eigenvalues so that all standard stability criteria [3] -[6] are inapplicable.

Assuming that

$$f(x,t) = f(x)e^{\lambda t}, \qquad g(x,t) = g(x)e^{\lambda t}, \tag{10}$$

with λ real, eqs.(6,7) reduce to

$$L_0 g = \lambda f, \qquad L_1 f = -\lambda g. \tag{11}$$

It is well known that when $F = 0$ the eigenvalue problem (11) has a doubly degenerate eigenvalue $\lambda_0 = \lambda_1 = 0$ corresponding to one even and one odd eigenfunction. When F is deviated from zero, the degeneracy breaks down. The odd eigenfunction continues to correspond to $\lambda_1 = 0$ whereas the even one pertains now to λ_0 imaginary. Since there is no other eigenvalues for small F, this fact implies that solutions of the form (10) do not exist in the mentioned limit. Surprisingly, this property remains valid in the general case.

Proposition. Eqs. (11) do not have real eigenvalues λ .

Proof. Rewrite (11) as

$$L_1 f = -\lambda^2 L_0^{-1} f \tag{12}$$

and consider an auxiliary problem

$$(L_1 - \delta)f = -\lambda^2(L_0^{-1} + \gamma)f, \tag{13}$$

with $\delta \geq 0, \gamma > 0$. Suppose eq.(13) has an eigenvalue $-\lambda^2 = -\lambda^2(\delta,\gamma)$. Then the necessary condition that this $(-\lambda^2)$ be simultaneously an eigenvalue for eq.(12) is clearly

$$-\lambda^2(\delta,\gamma) = -\delta/\gamma. \tag{14}$$

Eigenvalues of eq. (13) exist for any $\delta \geq 0$ and $\gamma > \mid \mu_0^{-1} \mid$ where μ_0 is the negative eigenvalue of L_0. The minimum eigenvalue, $-\lambda_0^2$ can be found as

$$-\lambda_0^2 = \min_{\xi} \frac{< \xi \mid L_1 - \delta \mid \xi >}{< \xi \mid L_0^{-1} + \gamma \mid \xi >}. \tag{15}$$

The corresponding eigenfunction is obviously nodeless. There is also an eigenvalue $-\lambda_1^2$ pertaining to the one-node eigenfunction. It is given by

$$-\lambda_1^2 = \min_{odd\ \xi} \frac{< \xi \mid L_1 - \delta \mid \xi >}{< \xi \mid L_0^{-1} + \gamma \mid \xi >}. \tag{16}$$

By properly choosing test functions it can be demonstrated that $-\lambda_0^2(\delta,\gamma) < -\delta/\gamma$, $-\lambda_1^2(\delta,\gamma) \geq -\delta/\gamma$, with the equality being attained only when $\delta = 0$. Thus for $\delta > 0$ we have

$$-\lambda_0^2(\delta,\gamma) < -\delta/\gamma < -\lambda_1^2(\delta,\gamma) < -\lambda_2^2(\delta,\gamma) < \ldots \tag{17}$$

Consequently, eq. (14) does not have solutions and nonzero eigenvalues of (12) do not exist. Q.E.D.

So, if unstable perturbations exist then they have a different form from (10). A more general possibility is

$$\begin{aligned} f(x,t) &= \{f_R(x)cos\lambda_I t - f_I(x)sin\lambda_I t\}e^{\lambda_R t}, \\ g(x,t) &= \{g_R(x)cos\lambda_I t - g_I(x)sin\lambda_I t\}e^{\lambda_R t}, \end{aligned} \tag{18}$$

with $f_R, f_I, g_R, g_I, \lambda_R, \lambda_I$ real. Feeding (18) into (6,7) and making combinations $f = f_R + if_I$, $g = g_R + ig_I$, $\lambda = \lambda_R + i\lambda_I$, we are led to the eigenvalue problem (11) again, but this time with f, g, and λ complex.

An eigenvalue λ of eq. (11) satisfies $\lambda = A^2\tilde{\lambda}$, with A as in (5) and $\tilde{\lambda}$ depending solely on α eq.(4). The numerical analysis of the set (11) revealed that the non-vanishing eigenvalue $\tilde{\lambda} = \tilde{\lambda}(\alpha)$ exists for any α. There is a certain $\alpha = \alpha_c$ such that $\tilde{\lambda}$ is pure imaginary for $\alpha \geq \alpha_c$ and acquires a positive real part for $\alpha < \alpha_c$ (Fig.1). Consequently, we may conclude that the ϕ_- soliton is stable for $\alpha > \alpha_c$ and unstable otherwise, or, equivalently, stable for h less than some h_c and unstable for $h > h_c$. Numerically, $\alpha_c = 2.5327$, and the corresponding h is given by eq.(4): $h_c = 0.07749$.

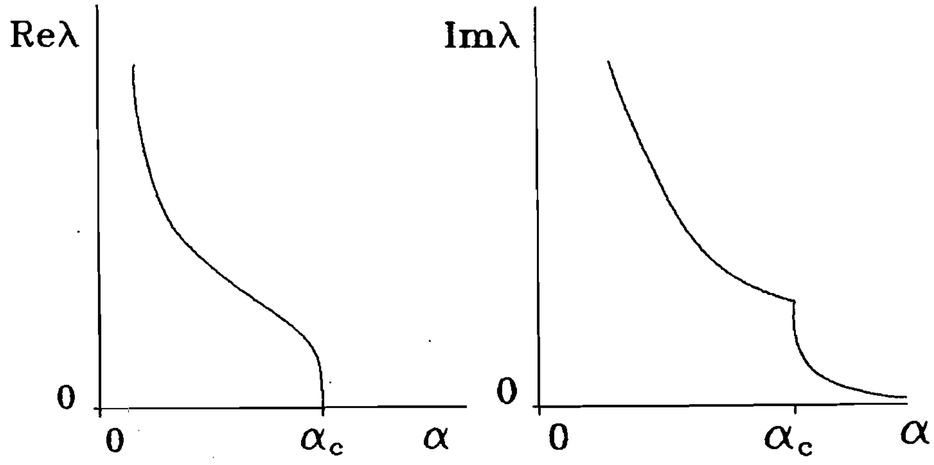

Fig. 1. *The real and imaginary part of the eigenvalue of eq. (11):* $\lambda = A^2\tilde{\lambda} = A^2(\tilde{\lambda}_R + i\tilde{\lambda}_I)$.

4. Numerical simulation

So,we have shown that in the region $h < h_c$ the basic nonlinear constituent is the soliton ϕ_-. Accordingly, the question arises of what are the asymptotic states for $h > h_c$. The fact that the unstable eigenvalue possesses an imaginary part there indicates that the stable configuration should be time-dependent. We have verified this hypothesis in the direct numerical simulation, and here are the preliminary results of these studies.

In the region $h > h_c$ (i.e., $\alpha < \alpha_c$) we used ϕ_- as initial condition and the observed evolution was the same for each h. The soliton was destroyed, with energy being radiated away in the form of hump-like solitary waves. (These waves could be taken for moving solitons unless they were gradually being dispersed). Instead of the initial static configuration a pulsating localized structure (a kind of a breather) was appearing at the origin. The period of pulsations was constant with high accuracy while the amplitude of the breather was not seen to decrease. Consequently, this structure is a natural candidate for the role of the asymptotic state for large values of the control parameter h.

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