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BREATHER TYPE SOLUTIONS OF THE VECTOR NONLINEAR SCHRÖDINGER EQUATION WITH QUASI-CONSTANT BOUNDARY CONDITIONS

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In this paper two-soliton solutions of breather type of the vector Schrödinger equation with cubic nonlinearity (VS3) are considered. The method applied was proposed in /1/. In fact, this method is a version of the basic algebro-geometrical approach for finding exact solutions of nonlinear equations of the mathematical physics. An excellent instructive review of this method may be found in /2/.

We proceed to formulate the mathematical problem under consideration. We have S3 equation in one space dimension

$$i\frac{\partial}{\partial t}\Phi(x,t) = \frac{\partial^2}{\partial x^2}\Phi(x,t) - \mathcal{U}(x,t)\Phi(x,t) \tag{1}$$

for the complex two-component vector function

$$\Phi(x,t) = (\Phi_1(x,t), \Phi_2(x,t))^T.$$

Here $\mathcal{U}(x,t)$ is selfconsistent, real and nonsingular "potential" which is expressed by the following formula:

$$\mathcal{U}(x,t) = 2\left[\varepsilon_1|\Phi_1|^2 + \varepsilon_2|\Phi_2|^2 - \varepsilon_1\varrho_1^2 - \varepsilon_2\varrho_2^2\right] \tag{2}$$

with constants ε_j and ϱ_j^2 being real. We shall suppose $|\varepsilon_j| = 1$ and the signature of ε_j to be determinated by the choice of the internal symmetry of the model.

Boundary conditions at infinity are quasi-constant, i.e. we have the following asymptotics in x

$$\Phi_j(x,t) \xrightarrow[|x|\to\infty]{} \varrho_j\, e^{i\kappa_j(x+\kappa_j t) + i\eta_j}, \tag{3}$$

where κ_j and η_j are real constants. The constant phases η_j, generally speaking, are different at $x=+\infty$ and $x=-\infty$.

Possible physical interpretation of the model and the constants κ_j, ϱ_j^2, ε_j, η_j one can find e.g. in /3/, where the mathematical model (1)-(3) is used to describe two-component Bose-gas with pair point like interaction at zero temperature.

According to /1/, the solution of (1)-(3) can be found by using auxiliary scalar function $\Psi(x,t,\kappa)$ which depends on the complex spectral parameter κ. This function has pole type singularities at some finite points $\varkappa_1, \varkappa_2, \ldots, \varkappa_N$ an essential singularity at the infinity $\kappa=\infty$ of the following form

$$\Psi(x,t,\kappa)\Big|_{\kappa=\infty} = e^{i\kappa(x+\kappa t)}. \tag{4}$$

By use of the function $\Psi(x,t,\kappa)$, which is the so-called Baker-Akhiezer function /3/, $\Phi(x,t)$ can be represented by the expression

$$\Phi_j(x,t) = \varrho_j\,\Psi(x,t,\kappa)\Big|_{\kappa=\kappa_j} \qquad j=1,2. \tag{5}$$

The real constants κ_j determine the first order poles of some auxiliary function $E(\kappa)$. The number of the poles κ_j defines the number of oscillating components of the solution $\Phi(x,t)$, whereas the number of the poles of $\Psi(x,t,\kappa)$ defines a kind of solutions. In what follows we shall consider a problem with two poles, such kind solutions will be called the two-soliton solutions.

To solve the problem (1)-(3), as in /1/, we use the following formula for the function

$$\Psi(x,t,\kappa) = \frac{\det \hat{M}(x,t,\kappa)}{\det M(x,t)}\, e^{i\kappa(x+\kappa t)}. \tag{6}$$

To obtain the breather solutions one must take the matrix $M(x,t)$ in the form

$$M(x,t) = \begin{vmatrix} \dfrac{e^{i(\bar\omega_1-\omega_1)}}{\bar\varkappa_1-\varkappa_1}, & \dfrac{\varrho + e^{i(\bar\omega_1-\omega_2)}}{\bar\varkappa_1-\varkappa_2} \\ \dfrac{\bar\varrho + e^{i(\bar\omega_2-\omega_1)}}{\bar\varkappa_2-\varkappa_1} & \dfrac{e^{i(\bar\omega_2-\omega_2)}}{\bar\varkappa_2-\varkappa_2} \end{vmatrix}. \tag{7}$$

Here $\varrho = |\varrho|e^{i\theta}$ is an arbitrary complex constant and $\omega_j = \varkappa_j x + \varkappa_j^2 t$. The matrix $\hat{M}(x,t,\kappa)$ in (6) is defined by the formulae

$$\hat{M}_{00} = 1, \quad \hat{M}_{ij} = M_{ij} \qquad i,j=1,2$$
$$\hat{M}_{0i} = \frac{e^{-i\omega_i}}{\kappa-\varkappa_i}, \quad \hat{M}_{i0} = e^{i\bar\omega_i} \qquad i=1,2. \tag{8}$$

Let one of the poles $\varkappa_1$ and $\varkappa_2$ lies in the upper half-plane, and the other pole lies in the lower half-plane. As we shall see later such a disposition of the poles and the chosen form (7) for $M(x,t)$ will lead to the breather type solutions.

Substituting (7) and (8) in (6) and using (5) we obtain the formula describing two-soliton solutions

$$\Phi_j(x,t) = \mathcal{C}_j \left[1 + \frac{z_1^j e^{2\beta x+4\mu t} + e^{\beta x+2\mu t}\left[z_2^j e^{-i(\alpha x+gt)} + z_3^j e^{i(\alpha x+gt)}\right]}{z_4^j e^{2\beta x+4\mu t} + z_5^j e^{\beta x+2\mu t}\cos(\alpha x+gt+\gamma) + z_6^j}\right] e^{i\kappa_j(x+\kappa_j t)} . \qquad (9)$$

Here

$$\alpha = \alpha_1 = Re\,\varkappa_1 , \; \beta_1 = Im\,\varkappa_1 , \; \beta_2 = Im\,\varkappa_2 \qquad (10)$$

$$\beta = \beta_1 + \beta_2 , \quad \mu = \alpha\beta_1 , \quad g = \alpha^2 + \beta_2^2 - \beta_1^2 .$$

The form of the constants z_ℓ^j is very complicate, so we shall not write them here, one may find them in /4/.

Solution (9) can be in fact of the soliton type as well as quasiperiodic depended on the order of the poles $\varkappa_1$, $\varkappa_2$, κ_1 and κ_2.

More exactly:

1. If $\alpha \neq 0$, $\beta = 0$ then the solution is quasiperiodic in x.

$$\Phi_j\left(x + \frac{2\pi}{\alpha}, t\right) = \Phi_j(x,t)\, e^{2\pi i \kappa_j/\alpha} .$$

when κ_j and α are commensurate, $\Phi_j(x,t)$ is periodic in x.

2. If $\alpha = 0$, $\beta \neq 0$ then the solution is quasiperiodic in t.

$$\Phi_j\left(x, t + \frac{2\pi}{\beta_2^2-\beta_1^2}\right) = \Phi_j(x,t)\, e^{2\pi i \frac{\kappa_j^2}{\beta_2^2-\beta_1^2}} .$$

If κ_j^2 and $\beta_2^2-\beta_1^2$ are commensurate then the solution is periodic in t.

3. If $\alpha = 0$, $\beta = 0$ then $\Phi_j(x,t) = \mathcal{C}_j e^{i\kappa_j(x+\kappa_j t)}$ i.e. the solution is a conventional plane wave. This fact is not suprising, since when $\alpha = \beta = 0$ the potential $U(x,t)$ vanishes identically in x, t and (1) reduces to a system of two nonconnected linear Schrodinger equations.

4. The case $\alpha \neq 0$, $\beta \neq 0$ is the most interesting. Namely, here a new type of breather like two-soliton solutions arise. To investigate their properties it is convinient to introduce a new variable

$$\zeta = x + \frac{2\mu}{\beta} t . \qquad (11)$$

Then the solution (9) can be written in the following form

$$\Phi_j'(\zeta,t) = \mathcal{C}_j \left[1 + \frac{z_1^j e^{2\beta\zeta} + e^{\beta\zeta}\left(z_2^j e^{-i(\alpha\zeta+\tilde{g}t)} + z_3^j e^{+i(\alpha\zeta+\tilde{g}t)}\right)}{z_4^j e^{2\beta\zeta} + z_5^j e^{\beta\zeta}\cos(\alpha\zeta+\tilde{g}t+\gamma) + z_6^j}\right] e^{i\kappa_j(\zeta+\kappa_j' t)} \quad j=1,2, \qquad (12)$$

where

$$\tilde{g} = g - \frac{2\alpha\mu}{\beta} , \qquad \kappa_j' = \kappa_j - \frac{2\mu}{\beta} . \qquad (13)$$

One can see from (13) that both components of the solutions are exponentially localized in ζ, i.e. these solutions are really soliton solutions. This soliton moves with the velocity $V_B = \frac{2\mu}{\beta}$, the same for both components. However the most interesting property of this solution is that it is periodical in t with period

$$T_B = \frac{2\pi}{\tilde{g}} = \frac{2\pi}{g - \alpha V_B} . \qquad (14)$$

To our opinion, due to existence of internal degree of freedom which is characterized by a frequency $\tilde{g}$, we shall call the soliton obtained the breather of VS3 by analogy with breathers of SG equation.

Depended on the order of the poles $\varkappa_1$ and $\varkappa_2$ of the Baker--Akhiezer function and the poles κ_1 and κ_2 of the auxiliary function we have solitons with different kind of unitary symmetry. It is convinient to reformulate the conditions for the parameters $\varkappa_j$ and κ_ℓ, which lead to different kinds of symmetry, in terms of some critical velocities and the velocity of the breather. Namely, when

$$V_B < V_1 \qquad \text{or} \qquad V_B > V_3 \qquad (15)$$

we obtain solutions with $\mathcal{U}(1,1)$ symmetry. Here we denote the critical velocities by

$$V_1 = \frac{2\kappa_1}{a} , \qquad V_3 = \frac{2\kappa_2}{a} , \qquad (16)$$

where

$$a = (p-1)/p(p+1) \qquad \text{and} \qquad p = \beta_1/|\beta_2| .$$

If

$$V_1 < V_B < V_3 \qquad (17)$$

then we obtain the solutions with $\mathcal{U}(2,0)$ and $\mathcal{U}(0,2)$ symmetries. Transition from the solution with $\mathcal{U}(2,0)$ symmetry to the solution with $\mathcal{U}(0,2)$ symmetry occurs when the breather velocity equals to some critical velocity V_2, which is given by the expression

$$(V_3 - V_2)(V_2 - V_1) = \frac{4p\alpha^2}{(p+1)^2 a^2} + \frac{4p\beta_2^2}{a^2} . \qquad (18)$$

It should be noted that the above conditions can be satisfied if the poles $\varkappa_1, \varkappa_2, K_1$ and K_2 do not lie on one circle.

Computer experiments for some particular values of parameters show that the breather solution (12) can be considered to consist of two one-soliton states.

Finally, a couple words of possible physical interpretation of the solutions obtained. Repulsive S3 equation usually called the Ginzburg-Landau equation, is applied to describe phenomenologically superfluids. The plane wave solutions then describe the Bose-condensate, while the kinks describe hole-type excitations in one-dimensional models and vortices, in two-dimensional models respectively.

Strictly speaking, this picture is valid at zero temperature $(T=0)$ only. On the other hand a physical system consists of two components (superfluid normal) at $T \neq 0$ and we have to apply at least two-component vector Schrödinger equation to describe it. Besides these two components must posses different types of internal symmetry. As a result, we come to an equation with total symmetry e.g. $U(1,1)$. It means that a new breather type mode of excitation should arise when the superfluid component moves through the normal one with a velocity greater than some critical velocity (see above). This new branch of excitation with internal is absent in the pure superfluid component (equation S3 with $U(0,1)$ symmetry).

Such excitations may display themselves in, e.g. light scattering experiments. The point is that the holes will determine the central peak, while the breathers will lead to satellites to occur at average breather frequency. /3/

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Решения бризерного типа векторного нелинейного уравнения Шредингера с квазипостоянными граничными условиями

В работе рассматривается векторное нелинейное уравнение Шредингера с квазипотенциальными граничными условиями. Получены новые двухсолитонные решения с нетривиальной динамикой, которые можно назвать бризерными, причем такие решения существуют для всех типов внутренней симметрии исследованной модели. Для получения этих решений был использован вариант алгебро-геометрического подхода Новикова, Дубровина и Кричевера. Условия, определяющие симметрию решения, сформулированы в терминах скорости бризера и некоторых критических скоростей, выраженных посредством параметров задачи.

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Breather Type Solutions of the Vector Nonlinear Schrödinger Equation with Quasi-Constant Boundary Conditions

Vector nonlinear Schrödinger equation (VS3) is investigated under quasi-constant boundary conditions. New two-soliton solutions are obtained with such non-trivial dynamics that they may be called the breather solutions. A version of the basic Novikov - Dubrovin - Krichever algebro-geometrical approach is applied to obtain breather like solutions existing for all types of internal symmetry of the model under study. Conditions under which symmetry is specified are formulated in terms of the soliton velocity expressed via the parameters of the problem.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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