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# A CONVERSE OF THE KATO-ROSENBLUM THEOREM

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## 1. Introduction

Let $\mathfrak{h}$ be a separable Hilbert space. We denote by $\mathcal{S}(\mathfrak{h})$, $\mathcal{B}_{self}(\mathfrak{h})$ and $\mathcal{L}_1(\mathfrak{h})$ the set of self-adjoint operators, the set of bounded operators and the set of trace class operators on $\mathfrak{h}$, respectively. Further $P^{ac}(.)$ denotes the orthogonal projection onto the absolutely continuous subspace of a self-adjoint operator (see e.g. [1]). Then the famous trace class existence theorem of Kato and Rosenblum ([3],[9]) states that for $H_o \in \mathcal{S}(\mathfrak{h})$ and $V \in \mathcal{L}_1(\mathfrak{h}) \cap \mathcal{B}_{self}(\mathfrak{h})$ the wave operators $W_\pm(H_o + V, H_o)$,

$$W_\pm(H_o + V,H_o) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it(H_o+V)}e^{-itH_o} P^{ac}(H_o), \tag{1.1}$$

exist and give a unitary equivalence between the absolutely continuous parts of $H_o + V$ and $H_o$. Note that the operator $H_o + V$ defined by

$$(H_o + V)f = H_of + Vf, \quad f \in \mathrm{dom}(H_o + V) = \mathrm{dom}(H_o) \tag{1.2}$$

is self-adjoint (see e.g. [4]).

Naturally, the question arises whether there is a wider class of operators from $\mathcal{B}_{self}(\mathfrak{h})$ such that for any $V$ of this class and $H_o \in \mathcal{S}(\mathfrak{h})$ the wave operators $W_\pm(H_o + V,H_o)$ exist. More precisely, we can introduce the sets $\mathcal{W}_\pm(\mathfrak{h}) \subseteq \mathcal{B}_{self}(\mathfrak{h})$ defined by

$$\mathcal{W}_\pm(\mathfrak{h}) = \{V \in \mathcal{B}_{self}(\mathfrak{h}) : \forall\, H_o \in \mathcal{S}(\mathfrak{h})\ \exists\, W_\pm(H_o + V,H_o)\}. \tag{1.3}$$

On account of the mentioned Kato-Rosenblum theorem we have

$$\mathcal{L}_1(\mathfrak{h}) \cap \mathcal{B}_{self}(\mathfrak{h}) \subseteq \mathcal{W}_\pm(\mathfrak{h}). \tag{1.4}$$

Now the question is whether the equality



$$\mathcal{L}_1(\mathfrak{h}) \cap \mathcal{B}_{self}(\mathfrak{h}) = \mathcal{W}_\pm(\mathfrak{h}) \tag{1.5}$$

holds. In other words (using some results of Section 3), does there exist an operator $V \in \mathcal{B}_{self}(\mathfrak{h})$, $V \notin \mathcal{L}_1(\mathfrak{h})$, such that for all $H_o \in \mathcal{S}(\mathfrak{h})$ the absolutely continuous parts of $H_o + V$ and $H_o$ are unitary equivalent via the wave operators? The problem of the validity of (1.5) has been posed by M.M.Skriganov from the Leningrad State University during the stay of the authors at this university.

It is expected that equality (1.5) is valid. The expectation is based on the famous Weyl-von Neumann theorem and its generalization by Kuroda (see [4],[5]). In the generalized version this theorem says that for any self-adjoint operator $H_o$, any $\epsilon > 0$, and any cross norm $\gamma$ not equivalent to the trace norm there is a self-adjoint compact perturbation $V$ such that $H_o + V$ has pure point spectrum and $\gamma(V) < \epsilon$. However, the quoted theorem does not prove (1.5). For this, it would be necessary to have a converse of the generalized Weyl-von Neumann theorem. Namely, for any compact self-adjoint operator $V$, $V \notin \mathcal{L}_1(\mathfrak{h})$, there is an absolutely continuous self-adjoint operator $H_o$ such that $H_o + V$ has a pure point spectrum. But such a theorem is unknown for the authors. In this paper we show that the equality (1.5) is valid. This means that the class of self-adjoint trace class operators is the largest class of self-adjoint perturbations giving the existence of the wave operators independent of the chosen unperturbed self-adjoint operators. More precisely, we prove a stronger result, namely that (1.5) is also true if we use a more weaker definition of the wave operators.

The proof of (1.5) is done as follows. First we shortly introduce the generalized wave operators and some notions in the next section. In Section 3 we prove some consequences of the assumption that (1.5) is not true. In particular, it follows that for the multiplication operator $H_o$ by the independent variable $\lambda$ on $L^2([-\pi,\pi])$ there is a special self-adjoint operator V such that the generalized wave operators exist but $V \notin \mathcal{L}_1(\mathfrak{h})$. Using the special form of the operators V and $H_o$ from Section 3, we prove in Section 4 that the wave operators $W_\pm(H_o + V,H_o)$ cannot exist. This gives the desired contradiction for the assumption that (1.5) is not true.

## 2. Generalized wave operators

Generalized wave operators were systematically investigated in [6] and [7]. They are defined with the help of more general limiting processes. We only present generalized wave operators defined with the help of the Cesaro mean. We say, the generalized wave operators $\tilde{W}_\pm(H_o + V,H_o)$ exist if the limits

$$\tilde{W}_\pm(H_o + V,H_o) = \operatorname*{s-lim}_{T\to+\infty} \frac{1}{T}\int_0^T dt\, e^{\pm it(H_o + V)}e^{\mp itH_o} P^{ac}(H_o) \quad (2.1\ )$$

exist and are partial isometries satisfying

$$\tilde{W}_\pm(H_o + V,H_o)^* \tilde{W}_\pm(H_o + V,H_o) = P^{ac}(H_o). \quad (2.2\ )$$

For brevity and in accordance with [1] we will use the notation

$$\tilde{W}_\pm(H_o + V,H_o) = |A|\text{-}\lim_{t\to\pm\infty} e^{it(H_o+V)}e^{-itH_o} P^{ac}(H_o) \quad (2.3\ )$$

if (2.1) and (2.2) are fulfilled. Note that the existence of the wave operators in the Cesaro mean is equivalent for the existence



of the wave operators in a whole class of limiting processes (for details see [7]). Thus, the Cesaro mean is only a concrete description for a more general definition of wave operators. Further note that the existence of $W_\pm(H_o + V,H_o)$ implies the existence of $\tilde{W}_\pm(H_o + V,H_o)$ but the converse is not true in general.

We say, the generalized wave operators (or wave operators) are complete if their final projections coincide with $P^{ac}(H_o + V)$, i.e.

$$\tilde{W}_\pm(H_o + V,H_o)\, \tilde{W}_\pm(H_o + V,H_o)^* = P^{ac}(H_o + V). \quad (2.4\ )$$

A necessary and sufficient condition for the completeness is that the wave operators $\tilde{W}_\pm(H_o,H_o + V)$ exist (the same is true for the usual wave operators $W_\pm$).

Now we introduce the sets $\tilde{\mathcal{V}}_\pm(\mathfrak{h}) \subseteq \mathcal{B}_{self}(\mathfrak{h})$ by

$$\tilde{\mathcal{V}}_\pm(\mathfrak{h}) = \{V \in \mathcal{B}_{self}(\mathfrak{h}):\ \tilde{W}_\pm(H_o + V,H_o) \text{ exist for all } H_o \in \mathcal{S}(\mathfrak{h})\} \quad (2.5\ )$$

which obviously obey the relations

$$\mathcal{V}_\pm(\mathfrak{h}) \subseteq \tilde{\mathcal{V}}_\pm(\mathfrak{h}). \quad (2.6\ )$$

In the following sections we show that

$$\tilde{\mathcal{V}}_\pm(\mathfrak{h}) = \mathcal{L}_1(\mathfrak{h}) \cap \mathcal{B}_{self}(\mathfrak{h}). \quad (2.7\ )$$

First, we remark that the introduction of the generalized wave operators has clarified some problems, e.g. the invariance principle in a quite satisfactory manner (see [11], [12]) (The invariance principle in its strong form is true for generalized wave operators but not for the usual wave operators). Thus, it is not clear whether the validity of (1.5) implies the validity of (2.7). This is the reason why we have solved a converse of the Kato-Rosenblum theorem in a more general setting.

Let us finish this section with another remark. The proposed problem is meaningful only if $\dim(\mathfrak{h}) = +\infty$. In a finite dimensional Hilbert space $P^{ac}(.)$ is always the null operator. Thus we obtain $P^{ac}(H_o) = P^{ac}(H_o + V) = 0$ and $\tilde{\mathcal{V}}_{\pm}(\mathfrak{h}) = \mathcal{B}_{self}(\mathfrak{h})$. Since every linear operator on a finite dimensional Hilbert space is a trace class operator we get $\mathcal{L}_1(\mathfrak{h}) = \mathcal{B}(\mathfrak{h})$. This implies that equality (2.7) is true for every finite dimensional Hilbert space.

## 3. Properties of $\tilde{\mathcal{V}}_{\pm}(\mathfrak{h})$

In this section we establish some properties of the sets $\tilde{\mathcal{V}}_{\pm}(\mathfrak{h})$ which will be useful in the next one. We agree to call $\hat{A}$ a part of a self-adjoint $A$ on $\mathfrak{h}$ if there is a reducing subspace $\hat{\mathfrak{h}}$ of $A$ such that $\hat{A} = A|\hat{\mathfrak{h}} \cap \mathrm{dom}(A)$. In particular, $A$ can be regarded as a part of itself.

<u>Lemma 3.1.</u>

( i) $\tilde{\mathcal{V}}_+(\mathfrak{h}) = \tilde{\mathcal{V}}_-(\mathfrak{h}) \overset{def}{=} \tilde{\mathcal{V}}(\mathfrak{h})$.

( ii) *If* $V \in \tilde{\mathcal{V}}(\mathfrak{h})$, *then* $-V \in \tilde{\mathcal{V}}(\mathfrak{h})$.

(iii) *If* $\hat{\mathfrak{h}}$ *is a reducing subspace of* $V \in \tilde{\mathcal{V}}(\mathfrak{h})$, *then* $\hat{V} = V|\hat{\mathfrak{h}} \in \tilde{\mathcal{V}}(\hat{\mathfrak{h}})$.

( iv) *If* $V \in \mathcal{B}_{self}(\mathfrak{h})$ *is unitarily equivalent to some* $V' \in \tilde{\mathcal{V}}(\mathfrak{h}')$, *then* $V \in \tilde{\mathcal{V}}(\mathfrak{h})$.

( v) *If* $K \in \mathcal{L}_1(\mathfrak{h}) \cap \mathcal{B}_{self}(\mathfrak{h})$ *and* $V \in \tilde{\mathcal{V}}(\mathfrak{h})$, *then* $K + V \in \tilde{\mathcal{V}}(\mathfrak{h})$.



<u>Proof.</u> (i) Let $J$ be a conjugation on $\mathfrak{h}$ commuting with $V \in \tilde{\mathcal{V}}_+(\mathfrak{h})$ (such a conjugation always exists, see [10,p.223]). If $H_o \in \mathcal{S}(\mathfrak{h})$, then

$$J\, e^{-it(H_o + V)}\, J = e^{it(JH_oJ + V)}, \quad t \in \mathbb{R}^1, \qquad (3.1)$$

where the operator $JH_oJ$ defined on $\mathrm{dom}(JH_oJ) = J\mathrm{dom}(H_o)$ is obviously self-adjoint, i.e. $JH_oJ \in \mathcal{S}(\mathfrak{h})$. Since $V \in \tilde{\mathcal{V}}_+(\mathfrak{h})$ the wave operator $\tilde{W}_+(JH_oJ + V, JH_oJ)$, $H_o \in \mathcal{S}(\mathfrak{h})$, exists. Furthermore, $P^{ac}(.)$ is a spectral projection [1]. Thus, $P^{ac}(JH_oJ) = JP^{ac}(H_o)J$. Using (3.1) we get

$$J\, \tilde{W}_+(JH_oJ + V, JH_oJ)\, J =$$

$$|A|\text{-}\lim_{t\to+\infty} J e^{it(JH_oJ + V)} e^{-itJH_oJ} P^{ac}(JH_oJ)\, J = \qquad (3.2)$$

$$|A|\text{-}\lim_{t\to+\infty} e^{-it(H_o + V)} e^{itH_o} P^{ac}(H_o) = \tilde{W}_-(H_o + V, H_o),$$

$H_o \in \mathcal{S}(\mathfrak{h})$. Hence, $\tilde{\mathcal{V}}_+(\mathfrak{h}) \subseteq \tilde{\mathcal{V}}_-(\mathfrak{h})$. Similarly, we find $\tilde{\mathcal{V}}_-(\mathfrak{h}) \subseteq \tilde{\mathcal{V}}_+(\mathfrak{h})$ giving the desired relation.

(ii) We note that $V \in \tilde{\mathcal{V}}(\mathfrak{h})$ implies the existence of $\tilde{W}_-(-H_o+V, -H_o)$ for every $H_o \in \mathcal{S}(\mathfrak{h})$. But the existence of $\tilde{W}_-(-H_o+V, -H_o)$ is equivalent to the existence of $\tilde{W}_+(H_o-V, H_o)$, $H_o \in \mathcal{S}(\mathfrak{h})$. Hence $-V \in \tilde{\mathcal{V}}(\mathfrak{h})$.

(iii) Every $\hat{H}_o \in \mathcal{S}(\hat{\mathfrak{h}})$ can be extended to some operator $H_o \in \mathcal{S}(\mathfrak{h})$ such that $\hat{H}_o$ is a part of $H_o$, i.e. $H_o|\hat{\mathfrak{h}} \cap \mathrm{dom}(H_o) = \hat{H}_o$. Obviously, we have $(H_o + V)|(\hat{\mathfrak{h}} \cap \mathrm{dom}(H_o + V)) = \hat{H}_o + \hat{V}$ and $P^{ac}(H_o)|\hat{\mathfrak{h}} = P^{ac}(\hat{H}_o)$. The last property is again based on the fact that $P^{ac}(.)$ is a spectral projection. Since $V \in \tilde{\mathcal{V}}(\mathfrak{h})$, the wave operator $\tilde{W}_+(H_o + V, H_o)$ exists and we have

$$\tilde{W}_+(H_o+V,H_o)|\hat{\mathfrak{h}} = |A|\text{-}\lim_{t\to+\infty} e^{it(H_o+V)}e^{-itH_o}P^{ac}(H_o)|\hat{\mathfrak{h}} = \tag{3.3}$$

$$|A|\text{-}\lim_{t\to+\infty} e^{it(\hat{H}_o+\hat{V})}e^{-it\hat{H}_o}P^{ac}(\hat{H}_o) = \tilde{W}_+(\hat{H}_o+\hat{V},\hat{H}_o)$$

for every $\hat{H}_o \in \mathscr{S}(\hat{\mathfrak{h}})$. Hence $\hat{V} \in \tilde{\mathscr{V}}(\hat{\mathfrak{h}})$.

(iv) Since $V' \in \tilde{\mathscr{V}}(\mathfrak{h}')$ and $V \in \mathscr{B}_{self}(\mathfrak{h})$ are unitarily equivalent, there is an isometry U from $\mathfrak{h}$ onto $\mathfrak{h}'$ such that $V = U^{-1}V'U$. If $H_o \in \mathscr{S}(\mathfrak{h})$, then $UH_oU^{-1} \in \mathscr{S}(\mathfrak{h}')$. By $V' \in \tilde{\mathscr{V}}(\mathfrak{h}')$ the wave operator $\tilde{W}_+(UH_oU^{-1}+V',UH_oU^{-1})$ exists for every $H_o \in \mathscr{S}(\mathfrak{h})$. Hence, we get

$$U^{-1}\tilde{W}_+(UH_oU^{-1}+V',UH_oU^{-1})U =$$

$$|A|\text{-}\lim_{t\to+\infty} U^{-1}e^{it(UH_oU^{-1}+V')}e^{-itUH_oU^{-1}}P^{ac}(UH_oU^{-1})\,U = \tag{3.4}$$

$$|A|\text{-}\lim_{t\to+\infty} e^{it(H_o+U^{-1}V'U)}e^{-itH_o}P^{ac}(H_o) = \tilde{W}_+(H_o+V,H_o)$$

for every $H_o \in \mathscr{S}(\mathfrak{h})$. Thus, we have proved (iv).

(v) Since $V \in \tilde{\mathscr{V}}(\mathfrak{h})$ $\tilde{W}_+(H_o+K+V,H_o+K)$ exists for every $K \in \mathscr{L}_1(\mathfrak{h}) \cap \mathscr{B}_{self}(\mathfrak{h})$ and $H_o \in \mathscr{S}(\mathfrak{h})$. The trace class existence theorem implies the existence of $W_+(H_o+K,H_o)$ for every $K \in \mathscr{L}_1(\mathfrak{h}) \cap \mathscr{B}_{self}(\mathfrak{h})$ and every $H_o \in \mathscr{S}(\mathfrak{h})$. Using the chain rule [1] we get the existence of $\tilde{W}_+(H_o+K+V,H_o)$ for every $K \in \mathscr{L}_1(\mathfrak{h}) \cap \mathscr{B}_{self}(\mathfrak{h})$ and $H_o \in \mathscr{S}(\mathfrak{h})$. Hence, $K+V \in \tilde{\mathscr{V}}(\mathfrak{h})$.∎

<u>Lemma 3.2.</u> *If $V \in \tilde{\mathscr{V}}(\mathfrak{h})$, then for every $H_o \in \mathscr{S}(\mathfrak{h})$ the wave operators $\tilde{W}_\pm(H_o+V,H_o)$ are complete.*

<u>Proof.</u> By virtue of (ii) of Lemma 3.1 the wave operators $\tilde{W}_\pm(H-V,H)$ exist for every $H \in \mathscr{S}(\mathfrak{h})$. Choosing $H = H_o+V$, $H_o \in \mathscr{S}(\mathfrak{h})$, we obtain the existence of $\tilde{W}_\pm(H_o,H_o+V)$ for every $H_o \in \mathscr{S}(\mathfrak{h})$. But $\tilde{W}_\pm(H_o+V,H_o)$ is complete if $\tilde{W}_\pm(H_o,H_o+V)$ exist.∎

Denoting by $\mathscr{L}_\infty(\mathfrak{h})$ the class of compact operators on $\mathfrak{h}$ the following important lemma can be proved.

<u>Lemma 3.3.</u> $\tilde{\mathscr{V}}(\mathfrak{h}) \subseteq \mathscr{L}_\infty(\mathfrak{h})$.

<u>Proof.</u> By the intertwining property [1] of the wave operators we get

$$(H_o+V)\tilde{W}_+(H_o+V,H_o)e^{-itH_o} = \tilde{W}_+(H_o+V,H_o)H_oe^{-itH_o}, \tag{3.5}$$

$V \in \mathscr{V}(\mathfrak{h})$, $H_o \in \mathscr{B}_{self}(\mathfrak{h})$, $t \in \mathbb{R}^1$. But (3.5) yields

$$\tilde{W}_+(H_o+V,H_o)H_o\,e^{-itH_o} - H_o\tilde{W}_+(H_o+V,H_o)e^{-itH_o} = \tag{3.6}$$

$$V\,\tilde{W}_+(H_o+V,H_o)e^{-itH_o},$$

$t \in \mathbb{R}^1$. Since $|A|\text{-}\lim_{t\to+\infty}(\tilde{W}_+(H_o+V,H_o) - I)e^{-itH_o}P^{ac}(H_o) = 0$ (see [1]) we get

$$|A|\text{-}\lim_{t\to+\infty}\{\tilde{W}_+(H_o+V,H_o)H_o - H_o\tilde{W}_+(H_o+V,H_o)\}e^{-itH_o}P^{ac}(H_o) = 0 \tag{3.7}$$

for any $H_o \in \mathscr{B}_{self}(\mathfrak{h})$. Hence, we obtain

$$|A|\text{-}\lim_{t\to+\infty} V\tilde{W}_+(H_o+V,H_o)e^{-itH_o}P^{ac}(H_o) = 0. \tag{3.8}$$

Taking into account the intertwining property of the wave operators and Lemma 3.2 we find

$$|A|\text{-}\lim_{t\to+\infty} V\,e^{-it(H_o+V)}P^{ac}(H_o+V) = 0, \tag{3.9}$$

$H_o \in \mathscr{B}_{self}(\mathfrak{h})$. Proposition 6.48 of [1] completes the proof.∎

**Lemma 3.4.** *Let* $\dim(\mathfrak{h}) = +\infty$. *If* $\tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h}) \neq \emptyset$, *then there is a* $V \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$ *such that*

( i) $V \in \mathscr{L}_\infty(\mathfrak{h})$,

( ii) $V \geq 0$,

(iii) $V|(\mathrm{ima}(V))^-$ *is simple,*

( iv) $\dim(\ker(V)) = +\infty$.

*Proof.* The property (i) is a consequence of Lemma 3.3. To prove (ii) let $L \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$. Since L is self-adjoint there exist two reducing subspaces $\mathfrak{h}_\pm$ such that $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, $L_+ = L|\mathfrak{h}_+ \geq 0$ and $L_- = L|\mathfrak{h}_- \leq 0$. Hence, we obtain either $L_+ \notin \mathscr{L}_1(\mathfrak{h}_+)$ or $L_- \notin \mathscr{L}_1(\mathfrak{h}_-)$. If $L_+ \notin \mathscr{L}_1(\mathfrak{h}_+)$, then $\dim(\mathfrak{h}_+) = +\infty$. Moreover, by Lemma 3.1(iii) we find $L_+ \in \tilde{\mathscr{W}}(\mathfrak{h}_+)\setminus\mathscr{L}_1(\mathfrak{h}_+)$. If $L_- \notin \mathscr{L}_1(\mathfrak{h}_-)$, then $\dim(\mathfrak{h}_-) = +\infty$ and $L_- \in \tilde{\mathscr{W}}(\mathfrak{h}_-)\setminus\mathscr{L}_1(\mathfrak{h}_-)$. Applying Lemma 3.1(ii) we get $0 \leq -L_- \in \tilde{\mathscr{W}}(\mathfrak{h}_-)\setminus\mathscr{L}_1(\mathfrak{h}_-)$. Thus, in every case there is a non-negative operator $V'$ on some infinite dimensional Hilbert space $\mathfrak{h}'$ belonging to $\tilde{\mathscr{W}}(\mathfrak{h}')\setminus\mathscr{L}_1(\mathfrak{h}')$. Since $\dim(\mathfrak{h}') = +\infty$ there exists an isometry U from $\mathfrak{h}$ onto $\mathfrak{h}'$. Setting $V = U^{-1}V'U$ and applying Lemma 3.1(iv) we find $V \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$ and $V \geq 0$.

In order to prove (iii) let $0 \leq L \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$. Since L is self-adjoint and compact, L admits the spectral representation

$$L = \sum_{l=0}^{m} \lambda_l P_l, \tag{3.10}$$

where $\{\lambda_l\}_{l=0}^m$ and $\{P_l\}_{l=0}^m$ denote the nonzero eigenvalues of L and the corresponding sequence of eigenprojections, respectively. Obviously, we have $\dim(P_l\mathfrak{h}) < +\infty$, $l = 0,1,2,\ldots,m$. Since $L \notin \mathscr{L}_\infty(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$ we get $m = +\infty$. Moreover, by $L \geq 0$ we can suppose that $\{\lambda_l\}_{l=0}^\infty$ is a strongly decreasing sequence of positive numbers converging to zero as $l \to +\infty$. Introducing the subspaces $P_l\mathfrak{h} = \mathfrak{h}_l$,



$l = 0,1,2,\ldots,$ we obtain the decomposition

$$(\mathrm{ima}(L))^- = \bigoplus_{l=0}^{+\infty} \mathfrak{h}_l. \tag{3.11}$$

We choose a sequence of non-negative simple self-adjoint operators $\{K_l\}_{l=0}^{+\infty}$ defined on $\mathfrak{h}_l$ so that $\sum_{l=0}^{+\infty} \mathrm{tr}(K_l) < +\infty$ and $\mathrm{spec}(K_l) \subseteq [0,\lambda_{l-1} - \lambda_l)$, $\lambda_{-1} \stackrel{\mathrm{def}}{=} \lambda_0 + 1$, $l = 0,1,2,\ldots$ In particular, we can choose $K_l = 0$ if $\dim(\mathfrak{h}_l) = 1$. Obviously, such a sequence $\{K_l\}_{l=0}^{+\infty}$ always exists. Setting

$$K = 0 \oplus \{\bigoplus_{l=0}^{+\infty} K_l\} \tag{3.12}$$

with respect to the decomposition

$$\mathfrak{h} = \ker(L) \oplus \{\bigoplus_{l=0}^{+\infty} \mathfrak{h}_l\} \tag{3.13}$$

we obtain a nuclear operator K on $\mathfrak{h}$. Defining V by

$$V = L + K \tag{3.14}$$

it is easy to see that V is simple. Moreover, by Lemma 3.1(v) we find $V \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$.

It remains to prove (iv). To this end we assume that $L \in \tilde{\mathscr{W}}(\mathfrak{h})\setminus\mathscr{L}_1(\mathfrak{h})$ satisfies the conditions (i), (ii) and (iii). Introducing again the spectral decomposition (3.11), we can now assume $\dim(\mathfrak{h}_l) = 1$, $l = 0,1,2,\ldots$ We choose a monotonously decreasing infinite subsequence $\{\lambda_{l_j}\}_{j=0}^{+\infty}$ of $\{\lambda_l\}_{l=0}^{+\infty}$ obeying the condition

$$\sum_{j=0}^{+\infty} \lambda_{l_j} < +\infty. \tag{3.15}$$

Since $\lambda_l \to 0$ as $l \to +\infty$ such a subsequence always exists. We set

$$K = \sum_{j=0}^{+\infty} \lambda_{1_j} P_{1_j}. \tag{3.16}$$

Due to (3.16) K is a nuclear operator. Moreover, the subspace

$$\hat{\mathfrak{H}} = (\mathrm{ima}(K))^{-} = \bigoplus_{j=0}^{+\infty} \mathfrak{H}_{1_j} \tag{3.17}$$

is infinite dimensional and reducing for L. Obviously, we have

$$Lf = Kf, \ f \in \hat{\mathfrak{H}}. \tag{3.18}$$

Thus setting

$$V = L - K \tag{3.19}$$

we obtain a simple non-negative operator V such that $\ker(V) \geq \hat{\mathfrak{H}}$. Hence, $\dim(\ker(V)) = +\infty$. Since $L \in \tilde{\mathscr{V}}(\mathfrak{H})\backslash\mathscr{L}_1(\mathfrak{H})$ we find $V \in \tilde{\mathscr{V}}(\mathfrak{H})\backslash\mathscr{L}_1(\mathfrak{H})$ applying Lemma 3.1(v).■

In the following we denote by $\{\varphi_1\}_{1=-\infty}^{+\infty}$ the orthonormal system $\varphi_1(x) = \frac{1}{\sqrt{2\pi}} e^{-i1x}$, $x \in [-\pi,\pi]$, in $L^2([-\pi,\pi])$.

Lemma 3.5. *If for some infinite dimensional separable Hilbert space $\mathfrak{H}$ the set $\tilde{\mathscr{V}}(\mathfrak{H})\backslash\mathscr{L}_1(\mathfrak{H})$ is not empty, then there is a strongly decreasing infinite sequence of positive numbers $\{\lambda_1\}_{1=0}^{+\infty}$ tending to zero as $1 \to +\infty$ and satisfying*

$$\sum_{1=0}^{+\infty} \lambda_1 = +\infty \tag{3.20}$$

*such that*

$$V = \sum_{1=0}^{+\infty} \lambda_1 (\cdot,\varphi_1)\varphi_1 \tag{3.21}$$

*belongs to $\tilde{\mathscr{V}}(L^2([-\pi,\pi]))\backslash\mathscr{L}_1(L^2([-\pi,\pi]))$.*

Proof. Let $L \in \tilde{\mathscr{V}}(\mathfrak{H})\backslash\mathscr{L}_1(\mathfrak{H})$. On account of Lemma 3.4 we can assume that in addition L satisfies the conditions (i) - (iv) of Lemma



3.4. Since $L \in \mathscr{L}_\infty(\mathfrak{H})\backslash\mathscr{L}_1(\mathfrak{H})$ and $L \geq 0$, the spectrum of L consists of $\{0\}$ and an infinite strongly decreasing sequence of positive numbers $\{\lambda_1\}_{1=0}^{+\infty}$ tending to zero as $1 \to +\infty$. Moreover, by (ii) of Lemma 3.4 the sequence of eigenprojections $\{P_1\}_{1=0}^{+\infty}$ of L consists of one dimensional projections. Denoting by $\{\psi_1\}_{1=0}^{+\infty}$ the corresponding sequence of eigenfunctions of L we obtain the representation

$$L = \sum_{1=0}^{+\infty} \lambda_1 (\cdot,\psi_1)\psi_1. \tag{3.22}$$

By Lemma 3.4(iv) there is an isometry U from $\mathfrak{H}$ onto $L^2([-\pi,\pi])$ such that $U\ker(L) = \mathrm{clo\,spa}\{\varphi_1 : 1 = -1,-2,\ldots\}$ and $U\psi_1 = \varphi_1$, $1 = 0,1,2,\ldots$ Setting $V = ULU^{-1}$ by (3.22) V admits the representation (3.21). Applying Lemma 3.1(iv) we find $V \in \tilde{\mathscr{V}}(L^2([-\pi,\pi]))\backslash\mathscr{L}_1(L^2([-\pi,\pi]))$.■

## 4. Main theorem

For the proof of our main theorem we need two technical lemmas.

Lemma 4.1. *If $V \in \tilde{\mathscr{V}}(\mathfrak{H})$, then*

$$\tilde{W}_+(H_0 + V, H_0) = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{inH} e^{-inH_0} P^{ac}(H_0). \tag{4.1}$$

Proof. Put $H = H_0 + V$ and define a bounded operator G by

$$G = \int_{-1}^{0} e^{itH} e^{-itH_0}\, dt. \tag{4.2}$$

From (2.1) we get the representation

$$\tilde{W}_+(H,H_0) = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{inH}\, G\, e^{-inH_0} P^{ac}(H_0). \tag{4.3}$$

Using the formula

$$e^{itH} e^{-itH_o} = I + i\int_0^t ds\, e^{isH}\, V\, e^{-isH_o},$$

$t \in \mathbb{R}^1$, we obtain $G = I + C$ with

$$C = i\int_{-1}^{0} dt \int_0^t ds\, e^{isH}\, V\, e^{-isH_o}\ .$$

By Lemma 3.3 we have $V \in \mathcal{L}_\infty(\mathfrak{h})$. Hence, we find $C \in \mathcal{L}_\infty(\mathfrak{h})$ as an easy calculation shows. This implies the relation $\text{s-}\lim_{t\to\infty} Ce^{-itH_o} P^{ac}(H_o) = 0$ (see e.g. [1], Theorem 6.45). Thus,

$$\text{s-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{inH}\, C\, e^{-inH_o}\, P^{ac}(H_o) = 0.$$

This relation together with $G = I + C$ and (4.3) gives (4.1).■

The next lemma is essentially an application of a result from the theory of divergent series to our objects.

Lemma 4.2. *Let $\{f_m\}_{m\geq 1}$ be a sequence of elements of $\mathfrak{h}$ converging in the Cesaro mean to $f \in \mathfrak{h}$, i.e.*

$$\lim_{N\to+\infty} \frac{1}{N+1} \sum_{m=0}^{N} f_m = f. \tag{4.4}$$

*Let $\{\lambda_m\}_{m=0}^{\infty}$ be a strongly decreasing sequence Of positive numbers converging to zero as $m \to \infty$ and obeying*

$$\lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n = +\infty. \tag{4.5}$$

*Set $0 = \lambda_{-1} = \lambda_{-2} = \ldots$ and assume $\|f_m\| \leq 1$, $m \in \mathbb{N}$. Then,*

$$\Big\| \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m \Big\| \leq 1,\ k \in \mathbb{Z}, \tag{4.6}$$

*and*



$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m = f,\ k \in \mathbb{Z}. \tag{4.7}$$

Proof. 1. We write the expression in (4.6) in another form. Let $k$ be fixed. We put $\nu_m = \lambda_{m+k}$ for $m \in \mathbb{Z}$. Then $\nu_m \geq 0$, $\sum_{m=-k}^{\infty} \nu_m = +\infty$, $\lim_{m\to\infty} \nu_m = 0$. Further, we put

$$g_n = \frac{1}{n+1} \sum_{m=0}^{n} f_m. \tag{4.8}$$

This gives

$$\sum_{m=0}^{n-1} \nu_m f_m = n\nu_{n-1} g_{n-1} + \sum_{m=-1}^{n-2} (m+1)(\nu_m - \nu_{m+1}) g_m$$

with $g_{-1} = 0$. Using this connection between $f_m$ and $g_m$ we can write

$$\sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m = \sum_{n=1}^{N} \tilde{C}_{N,n}\, g_{n-1}, \tag{4.9}$$

where

$$\tilde{C}_{N,n} = n(N-n+1)\nu_{n-1} - n(N-n)\nu_n. \tag{4.10}$$

The term $\sigma_N = \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_m$ can be written by

$$\sigma_N = \sum_{n=1}^{N}\sum_{m=0}^{n-1} \nu_{m-k} = \sum_{n=1}^{N}\sum_{j=-k}^{n-1-k} \nu_j =$$

$$\sum_{n=1}^{N} \tilde{C}_{N,n} + \begin{cases} \sum_{n=1}^{N} \{ \sum_{j=-k}^{-1} \nu_j - \sum_{j=n-k}^{n-1} \nu_j \} & (k \geq 0) \\ \sum_{n=1}^{N} \{ -\sum_{j=0}^{-k-1} \nu_j + \sum_{j=n}^{n-k-1} \nu_j \} & (k < 0) \end{cases}.$$

Since $\nu_j = \lambda_{j+k} = 0$ for $j < -k$ we note that $\sum_{j=0}^{-k-1} \nu_j = 0$. Setting $C_{N,n} = \tilde{C}_{N,n}\, \sigma_N^{-1}$ we obtain

$$\sum_{n=1}^{N} C_{N,n}\, g_{n-1} = \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k}\, f_m. \qquad (4.11)$$

Thus, we have to prove that $\lim_{n\to\infty} g_n = f$ and $\|g_n\| \le 1$ (this follows immediately from $\|f_n\| \le 1$) imply

$$\Big\| \sum_{n=1}^{N} C_{N,n}\, g_n \Big\| \le 1 \qquad (4.12)$$

and

$$\lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n}\, g_n = f. \qquad (4.13)$$

2. We note some properties of the "transformation" $C_{N,n}$. It is a straightforward calculation to show that

$$\sum_{n=1}^{N} C_{N,n} \le 1 \text{ and } \lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n} = 1.$$

Further, we also obtain that $\lim_{N\to\infty} C_{N,n} = \lim_{N\to\infty} \tilde{C}_{N,n}\, \sigma_N^{-1} = 0$ for each $n$. This is an easy consequence of $\nu_m \ge 0$, $\lim_{m\to\infty} \nu_m = 0$ and $\lim_{N\to\infty} \sum_{m=0}^{N} \nu_m = +\infty$. These properties of the transformation $C_{N,n}$ imply that $C_{N,n}$ is a so-called regular transformation. This means that if $S_n$ is a number sequence tending to $S$, then the sequence $\sum_{n=1}^{N} C_{N,n} S_n$ converges to $S$ too (see [2,Theorem 1]).

3. The last step in our proof is now to extend the assertions about number sequences to sequences of Hilbert space vectors $g_n$. We have

$$\Big\| \sum_{n=1}^{N} C_{N,n}\, g_n \Big\| \le \sum_{n=1}^{N} C_{N,n} \|g_n\| \le \sum_{n=1}^{N} C_{N,n} \le 1 \qquad (4.14)$$

because of $\|g_n\| \le 1$. This proves (4.12) and thus (4.6). Let $u \in \mathfrak{h}$ and set $a_n = (u, g_n)$. Obviously, $a_n$ converges by assumption to



$a = (u,f)$. Thus,

$$\lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n}\, a_n = a,$$

because of the considerations in 2. Thus $\sum_{n=1}^{N} C_{N,n}\, g_n = w_n$ converges weakly to $f$. On the other hand, since $\|g_n\|$ converges to $\|f\|$ we also get

$$\lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n} \|g_n\| = \|f\|.$$

Using (4.14) we find that

$$\limsup_{N\to\infty} \|w_n\| \le \|f\|. \qquad (4.15)$$

Now, it is a standard conclusion to verify the strong convergence of $w_n$ to $f$ from its weak convergence to $f$ and (4.15). This proves (4.13) and therefore (4.7).∎

<u>Theorem 4.3.</u> *For every separable Hilbert space* $\mathfrak{h}$ *we have* $\tilde{\mathcal{V}}(\mathfrak{h}) = \mathcal{L}_1(\mathfrak{h})$.

<u>Proof.</u> As it was pointed out in Section 2, the problem is trivial if $\dim(\mathfrak{h}) < +\infty$. Thus, we assume $\dim(\mathfrak{h}) = +\infty$ in addition. Let $\tilde{\mathcal{V}}(\mathfrak{h}) \setminus \mathcal{L}_1(\mathfrak{h}) \ne \emptyset$. By $V \in \mathcal{B}_{self}(L^2([-\pi,\pi]))$ we denote the operator characterized by Lemma 3.5. In the following, we will see that $V \in \tilde{\mathcal{V}}(L^2([-\pi,\pi]))$ is incompatible with all the other properties indicated in Lemma 3.5, in particular, with (3.20).

Let us introduce the self-adjoint operator $H_o$ on $L^2([-\pi,\pi])$ defined by $(H_o f)(x) = x f(x)$, $f \in L^2([-\pi,\pi])$. Notice that

$$e^{-inH_o}\varphi_l = \varphi_{l+n}, \quad n,l \in \mathbb{Z}, \qquad (4.16)$$

where $\{\varphi_l\}_{l=-\infty}^{+\infty}$ denotes the special orthonormal system used in Lemma 3.5. Let $h(.) \ne 0$ be a $C^\infty$-function defined on $[-\pi,\pi]$ which is zero in a neighbourhood of $-\pi$ and $\pi$. Setting $H = H_o + V \in \mathcal{B}_{self}(L^2([-\pi,\pi])$ we intend to show that

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, e^{-i\tau H_o}\, h(H_o)\, \varphi_o = \tag{4.17}$$

$$\tilde{W}_+(H,H_o)\, e^{-i\tau H_o}\, h(H_o)\, \varphi_o,$$

$|\tau| \le 1$. We remark that the existence of $\tilde{W}_+(H,H_o)$ follows from $V \in \tilde{\mathcal{V}}(L^2([-\pi,\pi]))$.

First of all, we note that $g(\tau) = e^{-i\tau H_o}\, h(H_o)\, \varphi_o$ admits a Fourier series given by

$$(g(\tau))(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k(\tau)\, e^{-ikx} = \sum_{k=-\infty}^{+\infty} c_k(\tau)\, \varphi_k(x), \tag{4.18}$$

$x \in [-\pi,\pi]$, for every $\tau \in [-\pi,\pi]$. A straightforward calculation shows that

$$\sup_{|\tau|\le 1} \sum_{k=-\infty}^{+\infty} |c_k(\tau)| < +\infty. \tag{4.19}$$

We find

$$\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, g(\tau) = \sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o} \sum_{k=-\infty}^{+\infty} c_k(\tau)\, \varphi_k, \tag{4.20}$$

$n \ge 1$, $\tau \in [-1,1]$. Due to (4.19) the sums can be interchanged. Moreover, taking into account (3.21) and (4.16) we get

$$\sum_{m=0}^{n-1} e^{imH}\, V e^{-imH_o}\, g(\tau) = \sum_{k=-\infty}^{+\infty} c_k(\tau) \sum_{m=0}^{n-1} \lambda_{m+k}\, e^{imH}\, e^{-imH_o}\, \varphi_k,$$

$n \ge 1$, $\tau \in [-1,1]$. Hence, we have to calculate

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, g(\tau) = \tag{4.21}$$

$$\lim_{N\to+\infty} \sum_{k=-\infty}^{+\infty} c_k(\tau) \left\{ \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k}\, e^{imH}\, e^{-imH_o}\, \varphi_k \right\},$$

where we have interchanged $\sum_{n=1}^{N}$ and $\sum_{k=-\infty}^{+\infty}$ on account of (4.19). Introducing the sequence $\{f_m^{(k)}\}_{m=0}^{+\infty}$, $k \in \mathbb{Z}$, $f_m^{(k)} = e^{imH}\, e^{-imH_o}\, \varphi_k$, and applying Lemma 4.2 we see that the expression in the curved brackets is norm bounded by one. Consequently, because of (4.19) $\lim_{N\to+\infty}$ and $\sum_{k=-\infty}^{+\infty}$ can be interchanged. Thus, we arrive at

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, g(\tau) =$$

$$\sum_{k=-\infty}^{+\infty} c_k(\tau) \lim_{N\to+\infty} \left\{ \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k}\, f_m^{(k)} \right\},$$

$\tau \in [-1,1]$. The sequence $\{f_m^{(k)}\}_{m=0}^{+\infty}$ converges in the Cesaro mean to $\tilde{W}_+(H,H_o)\varphi_k$ as $m \to +\infty$ for every $k \in \mathbb{Z}$. Applying Lemma 4.2 we obtain

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, e^{-i\tau H_o}\, h(H_o)\, \varphi_o =$$

$$\sum_{k=-\infty}^{+\infty} c_k(\tau)\, \tilde{W}_+(H,H_o)\varphi_k.$$

But taking into account the Fourier series (4.18) we find (4.17).

Using the representation

$$e^{inH}\, e^{-inH_o}\, f = f + i\int_0^1 d\tau\, e^{i\tau H} \sum_{m=0}^{n-1} e^{imH}\, V e^{-imH_o}\, e^{-i\tau H_o}\, f,$$

$n \ge 1$, $f \in L^2([-\pi,\pi])$, we get

$$\left(\frac{1}{N}\sum_{n=1}^{N} e^{inH}\, e^{-inH_o} f, f'\right) = (f, f') +$$

$$i\int_0^1 d\tau \left(\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V\, e^{-imH_o}\, e^{-i\tau H_o} f,\ e^{-i\tau H} f'\right),$$

$f,\ f' \in L^2([-\pi,\pi])$. Dividing by $\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m$ and setting $f = h(H_o)\varphi_o$ we find

$$\left(\frac{1}{\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\ \frac{1}{N}\sum_{n=1}^{N} e^{inH}\, e^{-inH_o}\, h(H_o) f,\ f'\right) =$$

$$\frac{1}{\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\,(f,\ f') + \tag{4.22}$$

$$i\int_0^1 d\tau \left(\frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V e^{-imH_o}\, e^{-i\tau H_o}\, h(H_o)\varphi_o, e^{-i\tau H} f'\right),$$

$f' \in L^2([-\pi,\pi])$. Now, the left-hand side and the first term of the right-hand side tend to zero as $N \to +\infty$ since $\sum_{m=0}^{\infty}\lambda_m = \infty$ implies $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m = \infty$ (Cesaro mean is a totally regular transformation. See [2]). Moreover, by the representation (4.20) and Lemma 4.2 we have

$$\left|\left(\frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH}\, V e^{-imH_o}\, e^{-i\tau H_o}\, h(H_o)\varphi_o,\ e^{-i\tau H} f'\right)\right| \le$$

$$\|f'\|\ \left\|\sum_{k=-\infty}^{+\infty} c_k(\tau)\ \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_{m+k}\, e^{imH}\, e^{-imH_o}\,\varphi_k\right\| \le$$

$$\|f'\|\ \sum_{k=-\infty}^{+\infty}|c_k(\tau)| \le \|f'\| \sup_{\tau\in[-1,1]}\ \sum_{k=-\infty}^{+\infty}|c_k(\tau)| < +\infty,$$

$f' \in L^2([-\pi,\pi])$. Thus, applying the Lebesgue dominated convergence



theorem to the second term of the right-hand side of (4.22) and taking into account (4.17) we obtain

$$0 = i(\tilde{W}_+(H,H_o)h(H_o)\varphi_o, f'), \tag{4.23}$$

$f' \in L^2([-\pi,\pi])$. Since $f' \in L^2([-\pi,\pi])$ is an arbitrary element, (4.23) yields $\tilde{W}_+(H,H_o)h(H_o)\varphi_o = 0$. But $\tilde{W}_+(H,H_o)$ is an isometry. Hence, $h(H_o)\varphi_o = 0$ which implies $h(.) \equiv 0$. But this fact contradicts $h(.) \not\equiv 0$. Consequently, $\tilde{\mathscr{V}}(\mathfrak{H})\setminus\mathscr{L}_1(\mathfrak{H}) \neq \emptyset$ is impossible for any infinite dimensional separable Hilbert space.∎

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Обратное утверждение к теореме Като-Розенблюма

Хорошо известная теорема Като-Розенблюма утверждает, что если мы возмущаем произвольный самосопряженный оператор ядерным оператором, тогда волновые операторы $W_{\pm}(H_0 + V, H_0)$ всегда существуют. В данной работе мы покажем, что в определенном смысле обратное утверждение к этой теореме имеет место. А именно: если для заданного ограниченного самосопряженного оператора V волновые операторы $W_{\pm}(H_0 + V, H_0)$ существуют для всех самосопряженных операторов $H_0$, тогда V - ядерный оператор. Результат остается тоже верным, если мы воспользуемся более слабым определением волновых операторов.

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A Converse of the Kato-Rosenblum Theorem

The well-known Kato-Rosenblum theorem says that if we perturb an arbitrary self-adjoint operator $H_0$ by a self-adjoint trace class operator, then the wave operators $W_{\pm}(H_0+V,H_0)$ always exist. In this paper we show that a kind of converse of this theorem holds. Namely, if we have that for a given bounded self-adjoint operator V the wave operators $W_{\pm}(H_0+V,H_0)$ for all self-adjoint operators $H_0$ exist, then V is a trace class operator. This result is also true if we use weaker notion of wave operators.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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