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NONLINEAR EVOLUTION EQUATIONS AND SOLVING ALGEBRAIC SYSTEMS: THE IMPORTANCE OF COMPUTER ALGEBRA

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1. INTRODUCTION

A wide-spread problem in the theory of nonlinear evolution equations (NEE) is to find the exact solutions of complicated algebraic systems. For example, let us consider the following evolution systems

$$U_t = \Lambda U_N + F(U, U_1, ..U_{N-1}), \quad U=U(x,t)=(U^1, ..U^M), \quad U_i = \mathbb{D}^i(U), \quad \mathbb{D}=d/dx$$
$$F=(F^1, ...F^M), \quad \Lambda=\mathrm{diag}(\lambda_1, ...\lambda_M), \quad \lambda_i \neq 0, \quad \lambda_i \neq \lambda_j \ (i \neq j). \qquad (1)$$

The general symmetry approach to checking up the integrability and classification of integrable NEE (see the reviews [1,2] and references therein) allows to obtain the integrability conditions which are related to existence of higher order symmetries and conservation laws in a fully algorithmic way.

The implementation of these algorithms in a form of FORMAC program FORMINT have been given for scalar equations ($M=1$ in (1)) in [3] and for the general case ($M>1$) in [4]. Using this program one can automatically obtain the equations in right hand side of (1) which follow from the necessary integrability conditions. The integrability conditions for the right hand side with arbitrary functions have the form of systems of differential equations, the procedure of solving these equations is not generally algorithmic. But in the important particular cases when the right hand side F^i are polynomials and the integrability conditions are reduced to a system of nonlinear algebraic equations in coefficients of the polynomials. Recent intensive investigations in computer algebra [5] and, particularly, in algorithmization of analysis and construction of exact solutions of nonlinear algebraic systems [6] allow to solve the necessary conditions of integrability for polynomial nonlinear evolution equations (1) by a computer.

The fundamental algorithms are based [6] on the explicit construction of the so-called standard or Groebner basis (GB) of a polynomial ideal, which is generated by the initial polynomial system. The GB which contains the full information of ideal under consideration, including the one of the roots of the polynomial system, leads to essential reduction of complexity when solving many computational problems in commutative polynomial algebra [6,7]. In the case of null dimensional ideal, i.e. when we have a finite number of solutions the knowledge of GB in lexicographic ordering of variables makes the procedure of finding solutions very simple. In the case of "separable" GB [6] the problem is reduced to determination of the roots of the polynomial in a single variable.

Let us introduce the following notations: $f_i(\{x\})$, $i=1,\dots N$ is the set of polynomials and $\{x\}=(x_1,\dots x_N)$ is the set of variables. Let $g_j(\{x\})$, $j=1,\dots K$ be the GB of the corresponding system f_i. In this case we can write

$$GB\{(f_1,\dots f_N),(x_1,\dots x_N)\} = \{g_1,\dots g_K\}\ .$$

Note that the choice of variable ordering has an essential influence on practical computations. It defines a particular form of GB and also determines its structure [6,15]. Therefore, in arranging the variables we shall use the chosen ordering i.e. $x_1>x_2>\dots>x_N$.

A more complicated problem is the solution of nonlinear polynomial systems with an infinite number of solutions, i.e. when we have a positive dimension. In this case we select some of variables as parameters. We note that the construction of GB in rest of variables is very cumbersome procedure. It is related to appearing complicated rational coefficients in parameters.

As a useful example let us consider the algebraic system which appears in the process of finding the algebraic curve (Riemann surface) for finite-gap elliptic potentials of Lame type and its generalizations [8]. Let introduce Lame potential

$$u(x) = n(n+1)\wp(x) \quad c \quad n=4, \tag{2}$$

where $\wp(x)$ is the $\wp$-function of Weierstrass.

This paper is organized as following. In sect.2 we give the basic

equations of type (1) and in sect.3 we demonstrate the usefulness of Groebner basis method and computer algebra for solving the systems of equations, which appear in the classification problem of polynomial non-linear equations (1) of uniform rank. In sect.4 we study the algebraic system with parameters, which corresponds to potential (2) and discuss some computational aspects of its solving with the help of computer algebra systems. Some conclusions are given in sect.5.

2. THE BASIC EQUATIONS

In this paper we shall consider the important subclass of equations of type (1) with polynomial nonlinear right hand sides with uniform rank [19]. Hence we suppose that $F^i(U,..)$ in (1) are sums of monomials of the following type:

$$\lambda\ (U^{i_1})^{d_1}(U_1^{i_2})^{d_2}\dots(U_{N-1}^{i_{N-1}})^{d_{N-1}},\quad d_k\geq 0,\ 1\leq i_k\leq M,\ (k=1,..N-1),\ \lambda\in\mathbb{C},$$

such that the ranks of all monomials are equal (including linear terms in the right hand side (1)). The rank R of the differential monomial $U^{d_0}U_1^{d_1}..U_k^{d_k}$ (for simplicity we miss the vector indexes i_k of functions $U(x,t)$) is called the following number

$$R = W_D D + W_I I,$$

where W_D и W_I are the weights of the corresponding vector function U (which are equal for all components) and the spatial differentiation operator d/dx. The degree D of the monomial and the full number of differentiations I are given by

$$D = \sum_{l=0}^{k} d_l,\quad I = \sum_{l=0}^{k} l d_l.$$

Further we call the polynomial - homogeneous equations the nonlinear evolution equations with uniform rank.

Let us consider some multiparametric polynomial - homogeneous equations. The integrability test of such equations is reduced to solving a nonlinear algebraic systems.

1) The scalar NEE's of degree 7 of Korteweg-de Vries (KdV) type. The classification problem for these equations

$$u_t=u_7+\lambda_1 uu_5+\lambda_2 u_1u_4+\lambda_3 u_2u_3+\lambda_4 u^2u_3+\lambda_5 uu_1u_2+\lambda_6 u_1^3+\lambda_7 u^3u_1 \tag{3}$$

and for equations degree 9 have been solved in [10].

2)The scalar NEEs of degree 7 of modified KdV (MKdV) type

$$\begin{aligned} u_t=u_7+\lambda_1(u_1u_5+3u_2u_4+2u_3^2)+\lambda_2(u^2u_5+6uu_1u_4+6u_1^2u_3+7u_1u_2^2)+\lambda_3(2u_1u_2^2+u_1^2u_3)+\\ \lambda_4(3u^2u_1u_3+12uu_1^2+3u^2u_2^2+2u_1^4)+\lambda_5(u^4u_3+8u^3u_1u_2+6u^2u_1^3)+\lambda_6 u^6u_1 \end{aligned} \tag{4}$$

are investigated in [11].

3)The system of two coupled nonlinear equations of KdV type [12,13]

$$\begin{aligned} &u_t=a_0u_3+a_1uu_1+a_2vv_1+a_3uv_1+a_4vu_1,\ a_0\neq b_0,\ a_0\neq 0,\ b_0\neq 0,\\ &v_t=b_0v_3+b_1vv_1+b_2uu_1+b_3vu_1+b_4uv_1,\ a_i,b_i\in\mathbb{C}\ (i=0\div 4). \end{aligned} \tag{5}$$

The nonlinear algebraic equations in the next section follow from some first necessary conditions of integrability. These conditions are related to the existence of the nontrivial local conservation laws (within the canonical series [1,2]) of the following type

$$\frac{dR_{ij}}{dt}=\frac{dQ_{ij}}{dx},\ i=1,2....,\ j=1,...M. \tag{6}$$

The polynomials R_{ij} can be computed in terms of polynomials in the right-hand side of (1) using the algorithmic procedure [4] implemented in computer algebra system FORMAC. In the examples given above (see examples 1)-3)) R_{ij} are differential polynomials in the components of the vector-function U whose coefficients are polynomials in numeric parameters of the NEE. Due to (6), the time derivatives of conservation laws can be the total derivatives with respect to spatial

variables of a local function. Hence, we obtain nonlinear algebraic equations in parameters of NEE. Note that the process of generation of these equations is fully automated using the FORMINT program.[3,4].

The problems closely related to this classification analysis of NEE (3)-(5) arise in the theory of NEE with finite-gap elliptic potential. For example, let us consider the following spectral problem for Hill's equation with Lame finite-gap potential

$$\left[\frac{d^2}{dx^2}-n(n+1)\wp(x)\right]\Psi=\lambda\Psi. \tag{7}$$

The construction of the so called Baker-Akhiezer function [8] $\Psi(x,\lambda)$ is based on the slightly improved Hermite ansatz

$$\Psi(x,\lambda)=\exp(kx)\left[a_0(\lambda,k,\alpha)\Phi(x,\alpha)+\sum_{j=1}^{n-1}a_j(\lambda,k,\alpha)D^j\Phi(x,\alpha)\right], \tag{8}$$

where

$$\Phi(x,\alpha)=\sigma(\alpha-x)/(\sigma(\alpha)\sigma(x))\exp(\zeta(\alpha)x),$$

and σ,ζ are the Weierstrass elliptic functions. The function $\Phi(x,\alpha)$ is a solution of equation (7) when n=2. Using the Laurent expansion of $\Phi(x,\alpha)$ at x=0, we can obtain [8] the overdetermined linear system of algebraic equations in a_i from (8). Solving this system we obtain the new system of two nonlinear polynomial equations of the following type

$$F_1(k,\lambda,\wp(\alpha),\wp'(\alpha))=F_2(k,\lambda,\wp(\alpha),\wp'(\alpha))=0, \tag{9}$$

where F_1 and F_2 are polynomials in their arguments, and $\wp(\alpha)$ and $\wp'(\alpha)$ are connected with relation [14]

$$[\wp'(\alpha)]^2=4[\wp(\alpha)]^3-g_2\wp(\alpha)-g_3, \tag{10}$$

where g_2,g_3 are elliptic constants [14]. Taking k and λ as a variables in (9) we shall consider the rest variables as parameters. Solving (9) we can obtain the following algebraic curve [8]

$$C_g: \; R(k,\alpha)=k^L+\sum_{j=1}^{L-1} r_j(\alpha)k^{L-j}=0, \qquad (11)$$

where $r_j(\alpha)$ are meromorphic functions on the elliptic curve (torus) C_1: $(\wp'(\alpha),\wp(\alpha))$, which is defined by (10). The problem is to find the curve (11) which is the L-fold covering of torus (10).

3. ALGEBRAIC SYSTEMS IN THE CLASSIFICATION PROBLEMS

In this section we study some nonlinear algebraic systems which arise from the several first conditions of integrability (6) in classification problem for NEE (3)-(5). As we noted above, the program FORMINT [3,4] is used for automatic calculations of the conservation law densities in terms of right-hand sides in equations (3)-(5) and for checking the conditions (6) after that, i.e. the generation of the corresponding algebraic equations.

Below we consider systematically the examples 1)-3) of sect.2. For more details see refs. [10,13].

Example 1. Let us consider the explicit form of R_1 in (6) for equations with numbers i=1,3,5,7 (all the conditions (6) with even numbers are satisfied identically, and we omit the vector index j=1)

$$R_1=\lambda_1 u,\; R_3=(2/7\;\lambda_1^2-\lambda_4),\; R_5=a_1u_1^2+a_2u^3,\; R_7=b_1u_2^2+b_2uu_1^2+b_3u^4, \qquad (12)$$

where

$$a_1=-2\lambda_1^2+\lambda_1\lambda_2+2\lambda_1\lambda_3-\lambda_2^2-7\lambda_5+21\lambda_6,\; a_2=7\lambda_7-2\lambda_1\lambda_4+3/7\;\lambda_1^3,$$

$$b_1=\lambda_1(5\lambda_1-3\lambda_2+\lambda_3),\; b_2=\lambda_1(2\lambda_6-4\lambda_4),\; b_3=\lambda_1\lambda_3.$$

Substituting (12) in (6), we obtain the following system of 13 equations

$$\begin{gathered}
\lambda_1(\lambda_4-\lambda_5/2+\lambda_6)=(2/7\;\lambda_1^2-\lambda_4)(-10\lambda_1+5\lambda_2-\lambda_3)=(2/7\;\lambda_1^2-\lambda_4)(3\lambda_4-\lambda_5+\lambda_6)=0,\\
a_1(-3\lambda_1+2\lambda_2)+21a_2=a_1(2\lambda_4-2\lambda_5)+a_2(-45\lambda_1+15\lambda_2-3\lambda_3)=0,\\
2a_1\lambda_7+a_2(12\lambda_4-3\lambda_5+2\lambda_6)=b_1(2\lambda_2-\lambda_1)+7b_2=b_1\lambda_3+7b_2=0,\\
b_1(-2\lambda_4-2\lambda_5)+b_2(2\lambda_2-8\lambda_1)+84b_3=0,\\
b_1(8/3\;\lambda_5+6\lambda_6)+b_2(11\lambda_1-17/3\;\lambda_2-5/3\;\lambda_3)-168b_3=0,\\
15b_1\lambda_7+b_2(5\lambda_4-2\lambda_5)+b_3(-120\lambda_1+30\lambda_2-6\lambda_3)=0,\\
-3b_1\lambda_7+b_2(-\lambda_4/2+\lambda_5/4-\lambda_6/2)+b_3(24\lambda_1-6\lambda_2)=3b_2\lambda_7+b_3(40\lambda_4-8\lambda_5+4\lambda_6)=0.
\end{gathered} \qquad (13)$$

The system (13) is one of nonlinear algebraic systems arising in the problem of NEE classification, which have been firstly solved in a fully automatic way using computer (IBM 3081D) by means of Buchberger's method of constructing GB using SAC-2 [15]. The optimal ordering for example 2) as well as for example 1) is the lexicographic one i.e. $\lambda_i>\lambda_j$ (i<j). This is in a good agreement with empirical strategy (see [15]). Being rather cumbersome, the system (13) have moreover an infinite number of solutions. It is easy to establish that, using the well known invariance property $u\Rightarrow\beta u$. After fixing the arbitrary parameters and finding GB, the solutions of the system are exactly the same as in [10]. Further we follow the papers [11-13], where various alternative possibilities to reduce the initial system to simpler subsystems are given .

In the present paper we use the computer algebra system REDUCE 3.2 on the IBM PC. Due to that we have no possibility to solve the general system (13) because of memory restrictions. We note that in [15] this example is already have been computed for 55 sec. at IBM 3081D.

Example 2. The first three odd densities (all even densities are not informative) for equation (4) are given by

$$R_1=\lambda_2u^2,\; R_3=a_1u_1^2+a_2u^4,\; R_5=b_1u_2^2+b_2u_1^3+b_3u^2u_1^2+b_4u^6, \qquad (14)$$

where

$$a_1=-4\lambda_2+\lambda_3-2/7\ \lambda_1^2,\quad a_2=\lambda_5-2/7\ \lambda_2^2,\quad b_1=7\lambda_2,\quad b_2=6\lambda_1\lambda_2-2\lambda_1\lambda_3-7\lambda_4+3/7\ \lambda_1^3,$$
$$b_3=-42\lambda_5-6\lambda_1\lambda_4-2\lambda_1\lambda_3+9/7\ \lambda_1^2\lambda_2+16\lambda_2^2,\quad b_4=-2\lambda_2\lambda_5+7\lambda_6+3/7\ \lambda_2^3. \tag{15}$$

The application of FORMINT program in [11] shows that the condition (6) for density R_1 is fulfilled identically and reduced to the following system of equations for R_3 and R_5

$$a_1\lambda_1=a_1\lambda_2+14a_2=a_1\lambda_4=a_1(6\lambda_2+2\lambda_3+3\lambda_4)+168a_2=a_1\lambda_5+5a_2\lambda_2=0,$$
$$5b_1\lambda_1+21b_2=10b_1\lambda_2+14b_3=105b_4-5b_1\lambda_5-b_3\lambda_2=5b_1\lambda_4+2b_2\lambda_2=0. \tag{16}$$

It is easy to see that system (15)-(16) have in general an infinite number of solutions, due to invariance of 6-parametric family (4) under the scale transformation $u\to\beta u$. Below we shall check this by direct calculation of GB for the cases when all the densities (14) are not zero.

To simplify computation of solutions of (15)-(16) we consider the following alternative cases [11].

2.1) $R_1=R_3=R_5=0$. It follows from (14)-(16) that $\lambda_2=\lambda_5=0$. In this case system (15)-(16) can be rewritten as $a_1=b_2=b_3=b_4=0$. Now we give the result of GB computations, using the notation of sect.1

$$GB\{(a_1,b_2,b_3,b_4),(\lambda_1,\lambda_3,\lambda_4,\lambda_6)\} =$$
$$\{\lambda_1^2-7/2\ \lambda_3,\lambda_1\lambda_3+14\lambda_4,\lambda_1\lambda_4,\lambda_3^2,\lambda_3\lambda_4,\lambda_4^2,\lambda_6\}\ .$$

Hence we obtain immediately that $\lambda_i=0$ ($i=1\div6$) is the single solution of the system (15)-(16) and corresponds to the trivial case $u_t=u_7$ in equation (4).

2.2) $R_i\neq0$ ($i=1,3,5$). From equations (14)-(16) we obtain that $\lambda_1=\lambda_4=0$.

Keeping non-identical equations and using computer algebra calculations we obtain the explicit form of the initial system and the corresponding GB

$$GB\{(8\lambda_2^2-\lambda_2\lambda_3-14\lambda_5,36\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2-84\lambda_5,10\lambda_2^3-7\lambda_2\lambda_5-7\lambda_3\lambda_5,21\lambda_2^2-2\lambda_2\lambda_3-42\lambda_5,$$
$$29\lambda_2^3+2\lambda_2^2\lambda_3-203\lambda_2\lambda_5+735\lambda_6),(\lambda_2,\lambda_3,\lambda_5,\lambda_6)\} =$$
$$\{\lambda_2^2-14/5\ \lambda_5,\lambda_2\lambda_3-42/5\ \lambda_5,\lambda_2\lambda_5-7\lambda_6,\lambda_2\lambda_6-2/5\ \lambda_5^2,\lambda_3^2-126/5\ \lambda_5,$$
$$\lambda_3\lambda_5-21\lambda_6,\lambda_3\lambda_6-6/5\ \lambda_5^2,\lambda_5^3-35/2\ \lambda_6^2\}\ .$$

The variables in this GB are not separated, hence it is the case with an infinite number of solutions [6]. Let us fix the value of a variable, say, λ_2. If we put $\lambda_2=0$ then the GB structure leads to zero values of the rest variables. It means that all densities (14) vanish in contradiction with the initial assumption. Therefore, to cancel the numeric denominators in (15) (note that the concrete choise of $\lambda_2\neq0$ is not essential) let $\lambda_2=7$ and then we can again compute the GB

$$GB\{(-7\lambda_3-14\lambda_5+392,-\lambda_3^2+7\lambda_3-84\lambda_5+1764,-7\lambda_3\lambda_5-49\lambda_5+3430,-14\lambda_3-42\lambda_5+1029,$$
$$98\lambda_3-1421\lambda_5+735\lambda_6+9947),(\lambda_3,\lambda_5,\lambda_6)\} =$$
$$\{\lambda_3-21,\lambda_5-35/2,\lambda_6-35/2\}\ .$$

So we obtain the known solution $\lambda_1=0$, $\lambda_2=7$, $\lambda_3=21$, $\lambda_4=0$, $\lambda_5=35/2$, $\lambda_6=35/2$ see [11].

2.3) $R_1\neq0,R_3=0,R_5\neq0$. Let us fix one of the variables. When $\lambda_1=0$, we obtain $b_2=0$ from (16) and $\lambda_4=0$ from (15). Hence we have one of the above cases which leads to the contradiction. So we can set, for example, $\lambda_1=7$, as in [11]. Then we have

$$GB\{(4\lambda_2-\lambda_3+14,2\lambda_2^2-7\lambda_5,23\lambda_2-6\lambda_3-3\lambda_4+63,21\lambda_2^2-2\lambda_2\lambda_3+63\lambda_2-42\lambda_4-42\lambda_5,$$

$$29\lambda_2^3+2\lambda_2^2\lambda_3-63\lambda_2^2+42\lambda_2\lambda_4-203\lambda_2\lambda_5+735\lambda_6,12\lambda_2^2-4\lambda_2\lambda_3+3\lambda_2\lambda_4+42\lambda_2)$$

$$(\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)\} = \{\lambda_2+7,\lambda_3+14,\lambda_4+14/3,\lambda_5-14,\lambda_6+28/3\} \;,$$

It gives the solution $\lambda_1=7,\lambda_2=-7,\lambda_3=-14,\lambda_4=-14/3,\lambda_5=14,\lambda_6=-28/3$ which is the same as in [11]. Those solutions are obtained by different methods.

The cases 2.1)-2.3) considered above exhaust all solutions of system (15)-(16) [11]. To verify that it is sufficient to construct GB for other alternative possibilities when some of densities R_1,R_3,R_5 vanish. Let, for instance, $R_1=R_3=0,R_5\neq 0$. Then the construction of GB gives immediately $\lambda_i=0$ (i=1-6) which contradicts the assumption $R_5\neq 0$.

Example 3. In [13] we already gave a detailed analysis and solved the nonlinear algebraic system on the parameters a_i,b_i ($i=0\div 4$) in (5), which follows from the first 8 conditions (6), i.e. $i=1\div 8$ ($j=1,2$). This result was obtained in [12] using FORMINT program [4] at ES-1061 computer. Further we can solve the same problem in two steps.

At the first step we consider the system which is a result of the first four and partially fifth conditions. We can study the alternative cases and subcases, searching equivalent set of subsystems, which is simple to solve on the personal computer using the method of computation of the corresponding resultants and the great common divisors. To do this we use the REDUCE 3.2 system (interactive mode) at the ES-1061. As a result we obtained four classes of solutions

$$a_1=a_3=a_4=b_1=b_2=0,\; a_2\neq 0,\; b_4=2b_3, \tag{17.1}$$

$$a_3=a_4=b_1=b_2=0,\; a_1\neq 0, a_2\neq 0,\; b_4=-(b_0/a_0)a_1, \tag{17.2}$$

$$(a_0-b_0)^2a_1^2-12a_0b_3^2-2a_0^2(a_0+2b_0)a_1b_3=0,$$

$$a_3=a_4=b_1=b_2=b_3=0,\; a_1\neq 0,\; a_2\neq 0,\; b_4=-a_1, \tag{17.3}$$

$$a_2=a_3=a_4=b_1=0. \tag{17.4}$$

We note that the direct construction of GB for initial system is not possible because of very cumbersome calculations.

At the second step we consider the auxiliary algebraic equations which follows from the basic conditions of integrability of (6). The final conclusion about solutions of (17.1)-(17.3) was possible when taking the fifth (i=5) condition of integrability. The most cumbersome calculations are related to checking solutions of (17.4) for the basic conditions of integrability up to i=8. Next we write the explicit form of the auxiliary system of equations on the rest parameters of the solutions (17.4) when $a_1=0$ and $b_0=a_0-1$, to exclude the arbitrary rule related to the invariance of the family (5) with respect to the scale transformations $t\Rightarrow\alpha t,x\Rightarrow\beta x,u\Rightarrow\mu u,v\Rightarrow\nu v$:

$$E_1\equiv-2a_0b_4^3+3a_0b_4^2-2b_4^2-6a_0b_3b_4+6b_3b_4+6a_0b_3^2\; -6b_3^2-a_0b_4=0,$$

$$\begin{aligned}E_2\equiv\,&18a_0^2b_4^3-9a_0b_4^3-18a_0^2b_3b_4^2+18a_0b_3b_4^2-18a_0^2b_3^2b_4-18a_0b_3^2b_4-27a_0^2b_4^2+24a_0b_4^2\\&-5b_4^2+63a_0^2b_3b_4-78a_0b_3b_4+15b_3b_4+9a_0^2b_4-63a_0^2b_3^2+78a_0b_3^2-15b_3^2=0,\end{aligned} \tag{18}$$

$$\begin{aligned}E_3\equiv\,&-8a_0b_4^4+12b_3b_4^2-12a_0b_3b_4^2-12b_3^2b_4+12a_0b_3^2b_4-b_4^3+6a_0b_4^3+18b_3b_4-18a_0b_3b_4\\&-18b_3^2+18a_0b_3^2-4b_4^2+5a_0b_4^2-3a_0b_4=0,\end{aligned}$$

$$E_4\equiv b_3b_4+3a_0b_3^2-3b_3^2+3a_0b_3b_4^2-5b_3b_4^2-a_0b_3+15b_3^2b_4-15a_0b_3^2b_4+10a_0b_3^3-10b_3^3=0.$$

In [13] the system (18) was solved by the standard method of elimination of variables using resultants (certainly with the help of the computer algebra system REDUCE). But it is required to verify whether the roots were really solutions. The construction of GB allows to eliminate that cumbersome step. In this case we have

$$GB\{(E_1,E_2,E_3,E_4)(b_3,a_0,b_4)\} =$$

$$\{b_3^2-b_3b_4+1/8\; b_4^3+1/8\; b_4^2,a_0b_3+5/3\; b_3b_4^2+2b_3b_4-5/6\; b_4^3-5/6\; b_4^2,$$

$$b_3b_4^3+b_3b_4^2-1/2\; b_4^3-1/2\; b_4^2,b_3b_4+1/3\; b_4^2,b_4^4-b_4^2\} \;.$$

Hence we obtain all roots previously found in [13]: $(a_0,b_3,b_4) = \{(a_0\in\mathbb{C},0,0),\ (-1/3,1/2,1),(1/3,-1,-1)\}$. Note that the calculation of GB in examples 2), 3), including the case 2.3) which we do not consider here, takes only 5 minutes on the IBM PC AT (10 Mgz).

4. AN EXAMPLE WITH PARAMETERS

In the cases above there are no free parameters in the nonlinear algebraic systems. When we have such parameters the problem of solving these systems is much more complicated. Constructing the GB we can transform not numerical polynomial coefficients but complicated algebraic expressions. It needs incomparably more computer resources. Such a situation takes place in example 4) of Sect.2. Let us consider the explicit form of the algebraic system (9) for one-dimensional Schrödinger equation with Lame potential (7) (n=4 in [8]).

Example 4).

$$\begin{aligned} &F_1=35k^4-k^2(30\lambda+210\wp)+140\wp' k+3\lambda^2-105\wp^2-21g_2+30\wp\lambda=0 \\ &F_2=(5\lambda-140\wp)k^3+210\wp' k^3+(-3\lambda^2+45\wp\lambda+126g_2-420\wp^3)k+70\wp\wp'-25\lambda\wp'=0. \end{aligned} \tag{19}$$

We recall that the variables here are k and λ and the rest ones $\wp\equiv\wp(\alpha)$, $\wp'\equiv\wp'(\alpha)$, g_2 are parameters ,besides, $\wp$ и $\wp'$ are related with algebraic equation (10) which contains one more parameter g_3. The computation of GB of the system (19) on IBM PC is not possible because of memory restrictions (640 KB for the computer algebra system REDUCE at IBM PC and the compatible computers [16]. It is impossible to use larger memory even on the 32-byte computers). To solve this problem we have been used the most powerful computer algebra systems SCRATCHPAD II which has the developed facilities to treat complicated algebraic expressions (see review [16]). In addition, using the system SCRATCHPAD II we can take in account the relations like (10) at each step of computations and thereby to simplify intermediate expressions.

As a result we can obtain

$$\begin{aligned} &GB\{(F_1,F_2)(\lambda,k)\} = \\ &\{\lambda+G(k,\wp,\wp',g_2,g_3),\ k^{10}-45\wp k^8+120\wp' k^7+(-630\wp^2+399/4\ \ g_2)k^6\ \ +504\wp\wp' k^5 \\ &+(-1050\wp^3+1725/4\ g_3+735/4\ \wp g_2)k^4+(360\wp^2\wp'-165\wp' g_2)k^3+(-189/4\ g_2^2 \\ &-315\wp^4+2205/4\ \wp^2 g_2-855/2\ \wp g_3)k^2+(-163\wp\wp' g_2+125\wp' g_3+40\wp^3\wp')k \\ &-9\wp^5-75/4\ \wp g_2^2\ -75/4\ g_2 g_3+9/4\ \wp^2 g_3+309/4\ g_2\wp^3\}\ , \end{aligned} \tag{20}$$

where $G(k,\wp,\wp',g_2,g_3)$ is the polynomial in k of degree 9, coefficients of which are the complicated rational expressions in parameters. These calculations took 260 sec. of CPU time at IBM RT PC.

The second term of GB (20) is the desired algebraic curve (11). Indeed, the structure of GB (20) shows "variable separation". It means not only that system (19) has a finite number of solutions, namely ten (counting multiple roots), but the fact that the first term of the GB corresponds to elimination of λ from (19) without appearing superfluous roots. In the paper [8] algebraic curve (20) has been found using the Moses elimination method (see HH-algorithm in [8] implemented on the computer algebra system REDUCE as well). The method generalizes the standard technique of sequential calculating resultants to the case of arbitrary polynomial systems and so, generally, leads to superfluous solutions. As we mentioned above, the solution test and the elimination of superfluous roots is rather cumbersome procedure even for free-parameter case. For systems with parameters this procedure is impracticable.

5. FINAL REMARKS

The above analysis shows that the GB technique is very useful to automate the process of solving nonlinear algebraic systems which appear in study of NEE. This approach admits not only full algorithmization and realization of GB construction on computer algebra systems, but also investigation of other principle problems. In particular, the presence of a constant term (not depending on the variables) in GB list indicates the incompatibility of an algebraic system [6]. Moreover one can find the dimension of an algebraic variety [18], i.e. the dimension of the root space, for a given set of polynomials. In classification problems it allows to determine the number of variables which can be considered as arbitrary parameters and to express the others in terms of them.

Certainly, in general the solution process is not possible in the full analytical way. However, the structure of GB shows that in the case of finite number of solutions the problem is reduced to finding roots of polynomials in a single variable. To solve this problem there are effective numeric methods. For polynomials with the rational coefficients one can efficiently obtain rational boundaries of real roots by computer. In other words, one can compute a sequence of disjoint intervals with rational end points, each containing exactly one real root [19]. Besides, the recent great progress in algorithmization of polynomial factorizing [20] and in its implementation in highly developed computer algebra systems [16] simplifies the problem.

It is remarkable, that in the cases considered above which are related to the classification analysis of integrable polynomial - homogeneous NEE, the problem of finding solutions of the corresponding algebraic equations are completely solved. This is in accordance with the fact that GB has sufficiently simple structure (see the examples in Sect.3). There can be no doubt that it is closely connected with the property of integrability.

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REFERENCES

1. Sokolov V.V.,Shabat A.B. Classification of integrable evolution equations.Math.Phys.Rev.4,221-280,New York,1984.
2. Mikhailov A.V.,Shabat A.B.,Yamilov R.I. Symmetry approach to classification of nonlinear equations.Complete list of integrable systems.Usp.Mat.Nauk.42,3-53,1987 (in Russian).
3. Gerdt V.P.,Shvachka A.B.,Zharkov A.Yu. FORMINT - a program for classification of integrable nonlinear evolution equations. Comp.Phys.Comm.34,303-311,1985.
4. Gerdt V.P.,Shabat A.B.,Svinolupov S.I.,Zharkov A.Yu. Computer algebra application for investigating integrability of nonlinear evolution systems. Preprint JINR E5-87-40,Dubna,1987; J.Comp.Math.&Math.Phys.28,1674-1684,1988 (in Russian).
5. Davenport J.H.,Siret Y.,Tournier G. Computer Algebra, Academic Press,London,1988.
6. Buchberger B. Groebner basis: a method in symbolic mathematics. In: (Bose N.K.,ed.) Multidimensional Systems Theory, 184-232, Dordrecht: Reidel,1985.
7. Proceedings of the First International Meeting "Computer and Commutative Algebra" (Genoa, May 20-23,1986).J.Symb.Comp.V.6, No.2&3,1988.
8. Gerdt V.P.,Kostov N.A. Computer algebra in the theory of ordinary differential equations of Halphen type. In: (Kaltofen E.,Watt S.M., eds.) Computers and Mathematics, 279-288,Springer-Verlag,New York, 1989.
9. Ito M. A REDUCE program for finding symmetries of nonlinear evolution equations with uniform rank.Comp.Phys.Comm.42,351-375, 1986.
10. Gerdt V.P.,Schvacka A.B.,Zharkov A.Yu. Classification of integrable high-order KdV-like equation. Preprint JINR R5-84-489,Dubna,1984 (in Russian).
11. Gerdt V.P.,Zharkov A.Yu. Computer classification of integrable seventh order MKdV-like equations. Preprint JINR R5-86-371,Dubna, 1986 (in Russian).
12. Gerdt V.P.,Zharkov A.Yu. Computer classification of integrable coupled KdV - like systems. Preprint JINR E5-89-232,Dubna,1989.
13. Gerdt V.P.,Zharkov A.Yu. Solving the polynomial system arizing in classification of integrable coupled KdV - like systems. Preprint JINR R5-89-231,Dubna,1989 (in Russian).
14. Whittaker E.T.,Watson G.N. A course of modern analysis. Cambridge University Press, Cambridge,1973.
15. Boege W.,Gebauer R.,Kredel H. Some examples for solving systems of algebraic equations by calculating Groebner bases. J.Symb.Comp., V.2, No.1, 83-98, 1986.
16. Gerdt V.P.,Tarasov O.V.,Shirkov D.V. Computer algebra systems and their application in high energy physics. In: Proc. of Intern. school on the problems of use of computer in physical research. (Dubna, 28 November-3 December, 1988).JINR D10-89-70,134-178,Dubna, 1989 (in Russian).
17. Moses J. Solution of a system of polynomial equations by elimination. Comm.ACM.9,634-637,1966.

18. Galligo A., Traverso K. Practical determination of the dimension of an algebraic variety. In: (Kaltofen E., Watt S.M., eds.) Computers and Mathematics, 46-52, Springer-Verlag, New York, 1989.
19. Collins G.E., Loos R. Real zeros of polynomials. In: Computer algebra. Symbolic and algebraic computation. 2-nd edition. Computing Suppl. 4, 83-94. Vienna: Springer-Verlag, 1983.
20. Kaltofen E. Polynomial factorization. In: Computer algebra. Symbolic and algebraic computation. 2-nd edition. Computing Suppl. 4, 95-113, Vienna: Springer-Verlag, 1983.

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Нелинейные эволюционные уравнения и решение алгебраических систем: роль компьютерной алгебры

В работе описано применение методов компьютерной алгебры для решения нелинейных полиномиальных систем, возникающих при исследовании нелинейных эволюционных уравнений. Рассмотрен ряд систем, которые получаются при классификации интегрируемых нелинейных эволюционных уравнений однородного ранга. Полиномиальные системы другой природы связаны с нахождением алгебраических кривых для конечно-зонных эллиптических потенциалов типа потенциала Ламе и его обобщений. Все рассматриваемые в настоящей работе полиномиальные системы решены методом, основанным на построении базисов Гребнера для полиномиальных идеалов, генерируемых исходной системой и реализованных в виде программ, написанных на языках аналитических вычислений.

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Nonlinear Evolution Equations and Solving Algebraic Systems: the Importance of Computer Algebra

In the present paper we study the application of computer algebra to solve the nonlinear polynomial systems which arise in investigation of nonlinear evolution equations. We consider several systems which are obtained in classification of integrable nonlinear evolution equations with uniform rank. Other polynomial systems are related with the finding of algebraic curves for finite-gap elliptic potentials of Lame type and generalizations. All systems under consideration are solved using the method based on construction of the Groebner basis for corresponding polynomial ideals. The computations have been carried out using computer algebra systems.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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