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ON THE CONNECTION BETWEEN THE ONE-DIMENSIONAL s = 1/2 HEISENBERG CHAIN AND HALDANE-SHASTRY MODEL

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О связи между спиновыми цепочками Гейзенберга и Холдейна-Шастри

Для модели взаимодействующих спинов с гамильнонианом

$$H = \frac{J}{2} \sum_{\substack{j,k=1,\\ j \neq k}}^{N} \wp(j-k)\vec{\sigma}_j\vec{\sigma}_k,$$

где один из периодов $\wp$-функции Вейерштрасса равен N, найдены дополнительные интегралы движения и представление Лакса. Цепочки Гейзенберга и Холдейна-Шастри являются предельными случаями этой модели, соответствующими некоторым значениям второго периода. Найдены собственные векторы гамильтониана, отвечающие состояниям рассеяния двух спиновых волн, как для конечных систем, так и для их бесконечномерного аналога.

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On the Connection between the One-Dimensional s = 1/2 Heisenberg Chain and Haldane-Shastry Model

Extra integrals of motion and the Lax representation are found for interacting spin systems with the Hamiltonian

$$H = \frac{J}{2} \sum_{\substack{j,k=1\\ j \neq k}}^{N} \wp(j-k)\vec{\sigma}_j\vec{\sigma}_k,$$

where one of the periods of the Weierstrass $\wp$-function is equal to N. The Heisenberg and Haldane-Shastry chains appear as limiting cases of these systems at some values of the second period. The simplest eigenvectors and eigenvalues of H corresponding to the scattering of two spin waves are presented explicitly for these finite-dimensional systems and for their infinite-dimensional version.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

This paper is devoted to the study of the problem on integrability of the one-dimensional S = 1/2 spin chains with the Hamiltonian

$$H = \frac{J}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} h(j-k)\, \vec{\sigma}_j \vec{\sigma}_k , \qquad h(x) = h(-x), \; x \in \mathbf{Z}, \tag{1}$$

which for a long time have been used as a model of ferromagnerism and antiferromagnetism. The simplest possible model of the type (1) is the famous periodic Heisenberg chain /1/ with the interaction only between nearest neighbours

$$h(x) = \delta_{1x} + \delta_{N-1,x} , \quad 0 < x < N . \tag{2}$$

It is well known that this model can be included in the Yang - Baxter scheme and has transfer matrix with the dependence on a complex parameter. All the local integrals of motion can be generated as derivatives of the logarithm of the transfer matrix on this parameter evaluated at a fixed point /2/. The spectrum is relatively complicated and can be obtained by solving the set of transcendental equations of the Bethe ansatz.

Recently Haldane /3/ and Shastry /4/ have constructed a number of eigenvectors of the Hamiltonian (1) with the "potential"

$$h(x) = \frac{\pi^2}{N^2 \sin^2\left(\frac{\pi x}{N}\right)} . \tag{3}$$

The spectrum of this model seems to be completely equidistant, the most of the energy levels are highly degenerated. There is no doubt about integrability of such a system, but the extra integrals of motion have not been found.

Nothing is known also about the analog of the transfer matrix and the connection of the model with the Yang-Baxter equations.

It is natural to suppose that the integrability of the spin 1/2 chains like (1) and of one-dimensional systems of interacting particles in classical mechanics is based on essentially the same Lie-algebraic ground. One can expect a deep analogy between them, as mentioned also in /4/. One of the purposes of this paper is to exploit the methods known in classical dynamics for investigations of the quantum systems (1). I shown that the "potentials" (2) and (3) are connected in a simple but unusual way. The systems (1) with them have some common properties. Some extra integrals of motion for the Haldane - Shastry model are also presented.

2. THE LAX REPRESENTATION

The integrability in classical dynamics in most cases is associated with the existence of the Lax representation of the equations of motion, i.e., the equivalence of these equations to the bilinear matrix relation

$$\frac{dL}{dt} = \{H_{Cl}, L\} = [L, M], \qquad (4)$$

where L and M are quadratic (possibly infinite) matrices depending on dynamical variables, [...] is the matrix commutator, H_{Cl} is the classical Hamiltonian, {...} is the Poisson bracket. As a consequence of (4), all the invariants of L, for example $I_k = \mathrm{tr}(L^k)$, $k \in \mathbf{Z}$, belong to the variety of classical integrals of motion.

For the systems of particles interacting with each other through the pair potentials, the structure of matrices L and M was established in /5/. In the case of interacting spins it is natural to construct the operator-valued matrices L and M obeying the quantum analog of equation (4)

$$[H, L] = [L, M], \qquad (5)$$

where the elements of the matrix [H,L] on the left-hand side are commutators of the Hamiltonian and matrix elements of L. The proper modification of the classical ansatz /5/ in this case is the following: the dimension of L and M is equal to the number of spins, N, and

$$L_{jk} = (1 - \delta_{jk}) f(j-k) (1 + \vec{\sigma}_j \vec{\sigma}_k),$$

$$M_{jk} = (1 + \vec{\sigma}_j \vec{\sigma}_k)(1 - \delta_{jk}) g(j-k) + \delta_{jk} \sum_{s \neq j}^{N} z(j-s)(1 + \vec{\sigma}_j \vec{\sigma}_s), \qquad (6)$$

where δ is the usual Kronecker symbol (all the diagonal elements of L are equal to zero), f, g and z are unknown functions of the argument $x \in \mathbf{Z}$. It is easy to show by direct substitution of (1) and (6) into (5) that the "quantum Lax representation" exists if the following conditions are satisfied for all nonzero $x, y \in \mathbf{Z}$:

$$z(x) = -h(x), \qquad (7a)$$

$$f(x)\, g(y) - f(y)\, g(x) = f(x+y)\,(h(y) - h(x)), \qquad (7b)$$

$$f(x)\, g(-x) - f(-x)\, g(x) = f(x+N)\, g(-x-N) - f(-x-N)\, g(x+N). \qquad (7c)$$

The first two conditions appear also in the classical theory, where the arguments x and y are arbitrary real or complex numbers. The last condition of periodicity, (7c), appears only for the spin chains and is completely absent for continuum systems.

The general solution to (7a-b) is well known /6/. Up to trivial exponential factor it is given by the formulas

$$h(x) = -z(x) = -f(x) f(-x) + \mathrm{const} = \wp(x) + \mathrm{const},$$

$$g(x) = -\frac{df(x)}{dx}, \qquad (8a)$$

$$f(x) = \frac{\sigma(x-\alpha)}{\sigma(x)\sigma(\alpha)} \exp(x\zeta(\alpha)), \qquad (8b)$$

where $\wp(x)$, $\zeta(x)$, $\sigma(x)$ are the usual Weierstrass elliptic functions

$$\wp(x) = \frac{1}{x^2} + \sum_{\gamma \in \Gamma}{}' \left[\frac{1}{(x-\gamma)^2} - \frac{1}{\gamma^2}\right], \quad \zeta'(x) = -\wp(x),\ \zeta(x) - \frac{1}{x} \to 0 \text{ at } x \to 0, \qquad (9)$$

$$\sigma'(x) = \zeta(x)\sigma(x),\ \frac{\sigma(x)}{x} \to 1 \quad \text{at } x \to 0.$$

The sum in (9) runs over all points of the lattice Γ on the complex plane, $\Gamma = \{ m_1\omega_1 + m_2\omega_2 \}$ ($m_1, m_2 \in \mathbf{Z}$, $\omega_1, \omega_2 \in \mathbf{C}$, $\mathrm{Im}\frac{\omega_2}{\omega_1} \neq 0$) except of the origin of the coordinate system $m_1 = m_2 = 0$. The "spectral parameter" α is defined on a complex torus $\mathbf{C}/\Gamma$.

The function (8b) obeys also the elliptic Lame equation [6]. The last condition (7c) is satisfied if and only of one of the periods of Weierstrass functions (for example, ω_1) is equal to N. The choice of the second period, ω_2, in the imaginary axis,

$$\omega_2 \equiv \omega = i\kappa, \quad \mathrm{Im}\,\kappa = 0 \tag{10}$$

guarantees that $\wp(x)$ is real at real x. For definiteness we shall choose κ to be positive. Finally, we have shown that the systems (1) with a real "potential" $h(x) = \wp(x)$ depending on an arbitrary real parameter κ have a "quantum Lax representation" of the type (5-8).

Let us consider some limiting situations. As the second period $\omega \to \infty$, the asymptotic behaviour of the $\wp$-function is

$$\wp(x)\big|_{\omega\to\infty} = \frac{\pi^2}{N^2}\left(\frac{1}{\sin^2\frac{\pi x}{N}} - \frac{1}{3}\right) \tag{11}$$

and we obtain, up to a trivial term proportional to the square of the total spin $\vec{S}$ commuting with all the Hamiltonians (1), the Haldane-Shastry chain. Another situation when the Weierstrass functions are degenerated into trigonometric ones, is the limit of a small second period. One finds

$$\wp(x)\big|_{\omega\to 0} = \frac{\pi^2}{\kappa^2}\Big(\frac{1}{3} + 4\big(e^{-\frac{2\pi}{\kappa}|x|} + e^{-\frac{2\pi}{\kappa}|N-x|} + e^{-\frac{2\pi}{\kappa}|N+x|}\big) +$$

$$+\, O\big(e^{-\frac{4\pi}{\kappa}|x|} + e^{-\frac{4\pi}{\kappa}|N-x|} + e^{-\frac{4\pi}{\kappa}|N+x|}\big)\Big), \quad |x| < N,\ x \in \mathbf{Z}$$

By adding to the Hamiltonian (1) with $h(x) = \wp(x)$ the term $-\frac{J\pi^2}{6\kappa^2}(4\vec{S}^2 - 3N)$, performing "renormalization" of the constant J in (1), $J \to \frac{J\kappa^2}{4\pi^2}\exp(\frac{2\pi}{\kappa})$, and taking the limit $\omega \to 0$, we obtain

$$h_0(x) = \lim_{\omega\to 0}\left(\frac{\kappa^2}{4\pi^2}\exp\frac{2\pi}{\kappa}\right)\left(\wp(x) - \frac{\pi^2}{3\kappa^2}\right) = \delta_{1x} + \delta_{N-1,x}, \quad x \in \mathbf{Z}, \quad 0 < x < N$$

that is the "potential" of the periodic Heisenberg chain. We see that both the models (2) and (3) can be obtained from (8a) as some limits and also have a Lax representation of the type (5-8). The situation is completely analogous to the case of continuum classical models where the periodic Toda and Sutherland particle systems can be treated as the limits of the systems with interaction through an elliptic potential [7].

Finally, when the namber of spins and, consequently, the real period of $\wp$ tend to infinity,

$$\wp(x)\big|_{N\to\infty} = \frac{\pi^2}{\kappa^2}\left(\frac{1}{\sinh^2(\frac{\pi x}{\kappa})} + \frac{1}{3}\right)$$

and we get a model for an infinite one-dimensional magnetic chain with short-range interaction depending on the parameter κ,

$$h_\infty(x) = \frac{\pi^2}{\kappa^2\sinh^2(\frac{\pi x}{\kappa})}. \tag{12}$$

Taking the limit $\kappa \to 0$ after a trivial renormalization of J in (1), $J \to \frac{J\kappa^2}{4\pi^2}e^{\frac{2\pi}{\kappa}}$, we obtain an infinite Heisenberg chain treated by Bethe [1].

3. THE EXTRA INTEGRALS OF MOTION

Contrary to the classical models, the existence of the Lax matrices does not guarantee that the invariants of L would be integrals of motion in the quantum case. For the matrix L of the form (6) the situation is even more pessimistic: it is easy to show that the first two invariants $\mathrm{tr}(L^k)$ are trivial c-numbers, i.e. they do not depend on spin operators $\{\vec{\sigma}_j\}$. One needs a new way to construct nontrivial integrals.

Let us consider the 2N x 2N operator-valued matrix

$$(\Lambda)_{jk,\alpha\beta} = (1-\delta_{jk}) f(j-k)\,(t_j + t_k)_{\alpha\beta}\,, \qquad (13)$$

where

$$t_j = \frac{1}{2}(1 + \vec{\sigma}_0 \vec{\sigma}_j)$$

and the Greek indices of Λ stand for elements of the extra Pauli matrices $\{\vec{\sigma}_0\}$. The matrices t_j have usual properties

$$t_j^2 = 1\,, \quad \mathop{\mathrm{tr}}_{(0)} t_j t_k = \frac{1}{2}(1 + \vec{\sigma}_j \vec{\sigma}_k)$$

($\mathop{\mathrm{tr}}_{(0)}$ denotes the trace over the indices of $\vec{\sigma}_0$, the multiplication of t's is performed so that $\{\vec{\sigma}_j\}$ are treated as operator coefficients of $\vec{\sigma}_0$). For the operator (13) there is no analog of the Lax equation (5). However, it is easy to show that, up to the total spin, $\vec{S}^2$,

$$\mathrm{Tr}\,\Lambda^2 \simeq H = \frac{1}{2} \sum_{j,k=1,j\neq k}^{N} h(j-k)\,\vec{\sigma}_j \vec{\sigma}_k\,, \quad h(j-k) = f(j-k)\,f(k-j)\,,$$

where Tr denotes the trace over both the Latin and Greek indices of Λ. The calculation of the next invariant of Λ, Tr Λ^3, gives, up to a constant additive term,

$$\mathrm{Tr}\,\Lambda^3 = -\frac{i}{2} \sum_{j\neq k\neq \ell}^{N} f(j-k)\,f(k-\ell)\,f(\ell-j)\,(\sigma_j \sigma_k \sigma_\ell) +$$
$$+ \frac{7}{4} \sum_{j\neq k}^{N} \vec{\sigma}_j \vec{\sigma}_k \sum_{\ell\neq j,k}^{N} [\,f(j-k)\,f(k-\ell)\,f(\ell-j) + f(k-j)\,f(\ell-k)\,f(j-\ell)]\,, \qquad (14)$$

where the operator $(\sigma_j \sigma_k \sigma_\ell) \equiv \vec{\sigma}_j \cdot (\vec{\sigma}_k \times \vec{\sigma}_\ell)$ is completely antisymmetric in the indices (jkl).

By the direct calculation of the commutator of these invariants I have obtained the following result: if f(x) and h(x) obey the conditions (7b-c) guaranteeing the existence of the Lax representation (5), then

$$[\,\mathrm{Tr}\,\Lambda^2, \mathrm{Tr}\,\Lambda^3] \equiv 0\,. \qquad (15)$$

The terms quartic in the spin operators disappear in the commutator if the functional equation (7b) is satisfied. The

terms of third and second orders in spin operators are absent if the periodicity (7c) also takes place.

The use of an explicit form of f(x) (8b) simplifies (14). With the help of addition theorems for $\wp$,ζ-funtions and the well-known in the theory of sigma-function formulas

$$\frac{\sigma(x-a)\sigma(x+a)}{\sigma^2(x)\,\sigma^2(a)} = \begin{vmatrix} 1 & \wp(x) \\ 1 & \wp(a) \end{vmatrix}, \quad \frac{\sigma(x-a)\sigma(y-a)\sigma(x-y)}{\sigma^3(x)\,\sigma^3(y)\,\sigma^3(a)} = \frac{1}{2} \begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(a) & \wp'(a) \end{vmatrix},$$

one can show that the second term in (14) is proportional to the square of S, and

$$\mathrm{Tr}\,\Lambda^3 \simeq -\frac{i}{2}(\hat{I}_1 \wp(a) + \hat{I}_2) + \mathrm{const}\cdot \vec{S}^2\,, \qquad (16)$$

where

$$\hat{I}_1 = \sum_{j\neq k\neq \ell}^{N} [\,\zeta(j-k) + \zeta(k-\ell) + \zeta(\ell-j)]\,(\sigma_j \sigma_k \sigma_\ell)\,,$$

$$\hat{I}_2 = \sum_{j\neq k\neq \ell}^{N} [\,2(\zeta(j-k) + \zeta(k-\ell) + \zeta(\ell-j))^3 + \wp'(j-k) + \wp'(k-\ell) + \wp'(\ell-j)] \times$$
$$\times (\sigma_j \sigma_k \sigma_\ell)\,.$$

Both the operators $\hat{I}_1$ and $\hat{I}_2$ commute with the Hamiltonian because of eqs.(15), (16) and arbitrariness of the "spectral parameter" a. They are functionally independent. So, the spin models are principally different at this point from classical particle systems where the trace of the (k+1)th degree of the L matrix comtains only one integral independent of the integrals in traces of the first k degrees of L.

For the trigonometric degeneration corresponding to the Haldane - Shastry model there are more simple combinations of the limits of $\hat{I}_1$, $\hat{I}_2$:

$$\hat{I}_s = \sum_{j\neq k\neq \ell}^{N} \phi_s(j-k)\,\phi_s(k-\ell)\,\phi_s(\ell-j)\,(\sigma_j \sigma_k \sigma_\ell)\,, \quad s = 1, 2\,,$$

$$\phi_1(x) = \coth\frac{\pi x}{N}\,, \qquad \phi_2(x) \doteq \left(\sinh\frac{\pi x}{N}\right)^{-1}.$$

So, the first four terms of a decomposition of the operator

$$r(\lambda, a) = \mathrm{Tr}[\exp(\lambda\Lambda(a))] \qquad (17)$$

in the parameter λ, give the integrals of motion for the model with the "potential" (8a). It is likely that (17) can be treated as the generating function of these integrals depending on two parameters, $\lambda \in \mathbf{C}$ and $a \in \mathbf{C}/\Gamma$. One may suppose that this operator is a formal analog of transfer matrix for this model (and the Haldane-Shastry model as a limiting case). The full proof of this hypothesis finally confirming the integrability of the model is yet absent.

4. THE SIMPLEST EIGENVECTORS

Here we shall consider only the ferromagnetic case and investigate the state vectors corresponding to one or two spin waves. Let us denote by $|0\rangle$ the state in which all spins have the same projection on the z-axis. Let the operator a_j^+ transform $|0\rangle$ to the state in which the sign of the projection of jth spin on the z-axis is opposite. It is convenient to begin the consideration with a slightly more simple case of the infinite chain (12).

We shall use the Hamiltonian differing from (1) and (12) by the constant term,

$$\tilde{H}_\infty = -\frac{1}{2}\sum_{\substack{j,k=-\infty \\ j\neq k}}^{\infty} \frac{\pi^2}{\kappa^2\sinh^2(\frac{\pi}{\kappa}(j-k))}\left(\frac{\vec{\sigma}_j\vec{\sigma}_k - 1}{2}\right), \quad \tilde{H}_\infty|0\rangle = 0. \qquad (18)$$

The calculation of the energy of a spin wave with the momentum p,

$$\psi_p = \sum_{k=-\infty}^{\infty} \exp(ipk)\, a_k^+|0\rangle \qquad (19)$$

is based on the formula

$$F(z) = \sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2[\sinh\frac{\pi}{\kappa}(k+z)]^2}\, e^{ikp} = -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)}\exp\left(\frac{pz}{\pi}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right) \times$$

$$\times\left[\tilde{\wp}(z) - \tilde{\wp}(r_p) + \left(\tilde{\zeta}(r_p) - \frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right)\left(\frac{\tilde{\wp}'(z) - \tilde{\wp}'(r_p)}{\tilde{\wp}(z) - \tilde{\wp}(r_p)} - \frac{\tilde{\wp}''(r_p)}{\tilde{\wp}'(r_p)}\right)\right]. \qquad (20)$$

Hereafter $\omega = i\kappa$, $r_p = -\frac{\omega p}{4\pi}$, $\tilde{\wp}, \tilde{\zeta}, \tilde{\sigma}$ are the Weierstrass functions with the periods $(1, \omega)$. The derivation of (20) is based on the quasiperiodicity of the sum in the left-hand side of (20),

$$F(z+\omega) = F(z), \quad F(z+1) = \exp(-ip)\,F(z),$$

on the structure of its only singularity at the point Z=0 on a torus obtained by a factorization of a complex Z-plane on the lattice of periods $(1, \omega)$, and on the Liouville theorem for elliptic functions. The substitution of (19) into the equation $\tilde{H}_\infty\psi_p = \epsilon_p^{(1)}\psi_p$ gives

$$\epsilon_p^{(1)} = \tilde{\wp}(r_p) + \frac{\wp''(r_p)}{\wp'(r_p)}\left(\tilde{\zeta}(r_p) - \frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right) + 2\left(\tilde{\zeta}(r_p) - \frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right)^2 + \frac{2}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right). \qquad (21)$$

Taking the limit $\kappa \to 0$ after the multiplication of (21) by $\frac{\kappa^2}{4\pi^2}\exp\left(\frac{2\pi}{\kappa}\right)$ we get the standard dispersion relation for the spin wave in an infinite Heisenberg chain.

Before constructing two-magnon states note that formula (20) admits the following evident generalization ($\ell \in \mathbf{Z}$)

$$\sum_{k=-\infty}^{\infty} \frac{\pi}{\kappa^2[\sinh\frac{\pi}{\kappa}(k+z)]^2}\, e^{ikp}\coth\frac{\pi}{\kappa}(k+\ell+z) =$$

$$= -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)}\coth\left(\frac{\pi\ell}{\kappa}\right)\exp\left(\frac{pz}{\pi}\tilde{\zeta}\left(\frac{\omega}{z}\right)\right) \times \qquad (22)$$

$$\times\left[\tilde{\wp}(z) - \tilde{\wp}(r_p) + \left(\tilde{\zeta}(r_p) - \frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right) + \frac{\pi}{\kappa\sinh\left(\frac{2\pi\ell}{\kappa}\right)}\left(1 - e^{-ip\ell}\right)\right) \times\right.$$

$$\left.\times\left(\frac{\tilde{\wp}'(z) - \tilde{\wp}'(r_p)}{\tilde{\wp}(z) - \tilde{\wp}(r_p)} - \frac{\tilde{\wp}'(r_p)}{\tilde{\wp}'(r_p)}\right)\right].$$

The scheme of the proof for (22) is the same as for (20). The structure of this formula shows that the two-magnon state is described by the vector

$$\psi^{(\infty)}_{p_1p_2} = \sum_{\substack{k_1,k_2=-\infty \\ k_1\neq k_2}}^{\infty} [e^{i(p_1k_1+p_2k_2)}\sinh\frac{\pi}{\kappa}(k_1-k_2+\frac{\kappa}{\pi}\gamma) + e^{i(p_2k_1+p_1k_2)}\sinh\frac{\pi}{\kappa}(k_1-k_2-\frac{\kappa}{\pi}\gamma)](\sinh\frac{\pi}{\kappa}(k_1-k_2))^{-1} a^+_{k_1}a^+_{k_2}|0\rangle \qquad (23)$$

substitution of which to $\tilde{H}_\infty\psi^{(\infty)}_{p_1p_2} = \epsilon^{(2)}_{p_1p_2}\psi^{(\infty)}_{p_1p_2}$ gives

$$\epsilon^{(2)}_{p_1p_2} = \epsilon^{(1)}_{p_1} + \epsilon^{(1)}_{p_2}, \qquad (24)$$

where $\epsilon^{(1)}_{p_s}$ are calculated accoprding to (21), and the phase γ is connected with impulses p_1 and p_2 by the relation

$$\coth\gamma = \frac{1}{2}[\tilde{\zeta}(\frac{p_2\omega}{2\pi}) - \tilde{\zeta}(\frac{p_1\omega}{2\pi}) + \frac{p_1-p_2}{\pi}\tilde{\zeta}(\frac{\omega}{2})].$$

In the limit $\omega\to 0$ it is just the expression for the Bethe phase in the Orbach parametrization. As for the infinite Heisenberg ferromagnet, according to (24), the additivity of magnon energies takes place.

In the case of finite spin systems consider the Hamiltonian

$$\tilde{H} = -\frac{1}{2}\sum_{\substack{j\neq k \\ j,k=1}}^{N} \wp(j-k)(\frac{\vec{\sigma}_j\vec{\sigma}_k - 1}{2}), \quad \tilde{H}|0\rangle = 0.$$

It the same way as for (20), (22) one can obtain the formulas for the sums of Weierstrass functons,

$$\sum_{k=0}^{N-1}\exp(\frac{2\pi i}{N}mk)\,\tilde{\wp}(k+z) = -\frac{\tilde{\sigma}(z+r_m)}{\tilde{\sigma}(z-r_m)}\exp(\frac{2\tilde{\zeta}(\frac{\omega}{2})mz}{N})[\tilde{\wp}(z) - \tilde{\wp}(r_m) + (\tilde{\zeta}(r_m) - \frac{2r_m}{\omega}\tilde{\zeta}(\frac{\omega}{2}))(\frac{\tilde{\wp}'(z)-\tilde{\wp}'(r_m)}{\tilde{\wp}(z)-\tilde{\wp}(r_m)} - \frac{\tilde{\wp}''(r_m)}{\tilde{\wp}'(r_m)})], \qquad (25a)$$

where $m\in\mathbf{Z}$, $m<N$; $r_m = -\frac{\omega m}{2N}$,

$$\sum_{k=0}^{N-1}\wp(k+z)\frac{\sigma(k-\ell+\gamma+z)}{\sigma(k-\ell+z)}\exp(i\alpha k) = -\frac{\sigma(\ell-\gamma)}{\sigma(\ell)}\,(\frac{\tilde{\sigma}(z+r_{\alpha\gamma})}{\tilde{\sigma}(z-r_{\alpha\gamma})})\times$$

$$\times\exp\{\frac{z}{2\pi i}[\zeta(\frac{N}{2})\tilde{\zeta}(\frac{\omega}{2})\gamma + i\zeta(\frac{\omega}{2})\alpha]\}[\tilde{\wp}(z) - \tilde{\wp}(r_{\alpha\gamma}) + \qquad (25b)$$

$$+(\frac{\tilde{\wp}'(z)-\tilde{\wp}'(r_{\alpha\gamma})}{\tilde{\wp}(z)-\tilde{\wp}(r_{\alpha\gamma})} - \frac{\tilde{\wp}''(r_{\alpha\gamma})}{\tilde{\wp}'(r_{\alpha\gamma})})(\tilde{\zeta}(r_{\alpha\gamma}) + \frac{\zeta(\ell-\gamma)-\zeta(\ell)}{2} -$$

$$-\frac{e^{i\alpha\ell}\sigma(\gamma)\sigma(\ell)}{2\sigma(\ell-\gamma)}\wp(\ell) + \frac{\zeta(\frac{N}{2})\tilde{\zeta}(\frac{\omega}{2})\gamma + i\zeta(\frac{\omega}{2})\alpha}{4\pi i})],$$

where α and γ are connected by the relation

$$\exp(i\alpha N + \gamma\zeta(\frac{N}{2})) = 1, \quad \ell\in\mathbf{Z}; \quad r_{\alpha\gamma} = -(4\pi)^{-1}(\alpha\omega + i^{-1}\gamma\zeta(\frac{\omega}{2})).$$

The expression for the energy of the spin wave (19) for which the quasimomenta $\{p\}$ are quantized according to the periodicity condition, $p_m = \frac{2\pi}{N}m$, $0\le m\le N-1$, $m\in\mathbf{Z}$, can be easily found from (25a),

$$\epsilon^{(1)}(p_m) = \phi(r_m), \quad r_m = -\frac{\omega m}{2N},$$

$$\phi(r) = \tilde{\wp}(r) + \frac{\tilde{\wp}''(r)}{\tilde{\wp}'(r)}(\tilde{\zeta}(r) - \frac{2r}{\omega}\zeta(\frac{\omega}{2})) + 2(\zeta(r) - \zeta(\frac{\omega}{2})\frac{2r}{\omega}) + \frac{2}{\omega}[\tilde{\zeta}(\frac{\omega}{2}) - N\zeta(\frac{\omega}{2})].$$

Let us search the vectors of two-magnon states in the form analogous to (23),

$$\psi^{(N)}_{p_1p_2} = \sum_{\substack{k_1,k_2=1 \\ k_1\neq k_2}}^{N}[e^{i(p_1k_1+p_2k_2)}\frac{\sigma(k_1-k_2+\gamma)}{\sigma(k_1-k_2)} + e^{i(p_2k_1+p_1k_2)}\frac{\sigma(k_1-k_2-\gamma)}{\sigma(k_1-k_2)}]a^+_{k_1}a^+_{k_2}|0\rangle. \qquad (26)$$

The quasimomenta p_1, p_2 and the phase γ must be determined from the periodicity conditions and $\tilde{H}\psi^{(N)}_{p_1p_2} = \epsilon^{(2)}_{p_1p_2}\psi^{(N)}_{p_1p_2}$.

By using eq.(25b) one makes sure that (26) is just the eigenvector of $\tilde{H}$ with the eigenvalue

$$\epsilon^{(2)}_{p_1p_2} = \phi(r_{p_1\gamma}) + \phi(r_{p_2\gamma}) + \wp(\gamma) + \zeta^2(\gamma),$$

where

$$r_{p_1\gamma} = -(4\pi)^{-1}(p_1\omega + i^{-1}\gamma\zeta(\frac{\omega}{2})),\quad r_{p_2\gamma} = -(4\pi)^{-1}(p_2\omega + i^{-1}\gamma\zeta(\frac{\omega}{2})),$$

and (p_1, p_2, γ) is an arbitrary solution of the system of transcendental equations

$$\exp(ip_1 N + 2\gamma\zeta(\frac{N}{2})) = 1,\quad \exp(ip_2 N - 2\gamma\zeta(\frac{N}{2})) = 1,$$

$$\tilde{\zeta}(2r_{p_1\gamma}) - \tilde{\zeta}(2r_{p_2\gamma}) + \frac{4\tilde{\zeta}(\frac{\omega}{2})}{\omega}(r_{p_2\gamma} - r_{p_1\gamma}) + \frac{4\zeta(\frac{\omega}{2})\gamma}{\omega} - 2\zeta(\gamma) = 0.$$

In the limit $\omega \to 0$ these equations coincide with the equations of the Bethe ansatz for the quasimomenta of two-magnon states in periodic Heisenberg chain.

The investigation of the states with a larger number of magnons can be performed in the case of the infinite chain on the basis of the summation formula for trigonometric series generalizing (22),

$$\sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \frac{e^{ikp}}{(\sinh\frac{\pi}{\kappa}(k+z))^2} \prod_{\lambda=1}^{n} \coth\frac{\pi}{\kappa}(k+z+\ell_\lambda) = -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)}\exp(\frac{pz}{\pi}\tilde{\zeta}(\frac{\omega}{2}))\times$$

$$\times(\prod_{\lambda=1}^{n}\coth\frac{\pi\ell_\lambda}{\kappa})\{\tilde{\mathcal{P}}(z) - \tilde{\mathcal{P}}(r_p) + (\frac{\tilde{\mathcal{P}}'(z) - \tilde{\mathcal{P}}'(r_p)}{\tilde{\mathcal{P}}(z) - \tilde{\mathcal{P}}(r_p)} - \frac{\tilde{\mathcal{P}}''(r_p)}{\tilde{\mathcal{P}}'(r_p)})\times \qquad (27)$$

$$\times[\tilde{\zeta}(r_p) - \frac{2r_p}{\omega}\tilde{\zeta}(\frac{\omega}{2}) + \frac{\pi}{\kappa}(\sum_{\nu=1}^{n}(\sinh\frac{2\pi\ell_\nu}{\kappa})^{-1} -$$

$$-\sum_{\nu=1}^{n}\frac{\exp(-ip\ell_\nu)}{2[\sinh\frac{\pi\ell_\nu}{\kappa}]^2}(\prod_{\epsilon\neq\nu}\coth\frac{\pi}{\kappa}(\ell_\epsilon - \ell_\nu))(\prod_{\lambda=1}^{n}\coth\frac{\pi\ell_\lambda}{\kappa})^{-1})]\},$$

where $\{\ell_\nu\}$ are nonzero integers, $\prod_{\lambda>\mu}^{n}(\ell_\lambda - \ell_\mu) \neq 0$. I know also an analogous formula for the summation of a finite series containing like (25) the elliptic functions.

But, contrary to the infinite chain, this formula is not useful for the construction of the eigenvectors of $\tilde{H}$. The situation bears a strong resemblance to the quantum systems of particles on a line. In this case the wave functions can be easily found for the trigonometric Sutherland systems, but for elliptic potentials of pair interactions the single known result is a solution of the Lame equation for two-particle systems. At the classical level, the trajectories of particle systems in an elliptic case were found by Krichever [6] by the methods of algebraic geometry and the solution containing the multidimensional Riemann theta functions. As for my knowledge, nobody for this time indicated the way of solution to the corresponding quantum problem.

5. SUMMARY

In this paper the simplest properties of the spin model generalizing the Heisenberg and Haldane-Shastry chain were found. The most important problem in further investigation is, in my opinion, the proof of the hypothesis on the existence of the generating function of integrals of motion (17) and the finding of its connection with Yang-Baxter equations. As for purely calculation schemes, it would be interesting to indicate a simple way of constructing the states with an arbitrary number of magnons, especially for the periodic chain.

It would be interesting also to investigate the possibility of the destruction of SU(2)-symmetry of the Hamiltonian. In particular, one can expect in the XXZ case, as in the Haldane-Shastry model, the conservation of integrability for values of the anisotropy parameter $\Delta = \frac{m(m+1)}{2}$, $m \in \mathbf{Z}$, $m > 1$. These numbers appear in the equations determining the Legendre polynomials and Lame functions as parameters at which these equations have solutions without any branch points. For the problems of finding the eigenvectors of the Hamiltonian like (1) this fact corresponds to the possibility of analytical summation of the series of the type

$$\sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \frac{\exp(ikp)}{[\sinh\frac{\pi}{\kappa}(k+z)]^2} P(\coth\frac{\pi}{\kappa}(k+z))$$

and their generalizations like (25) and (27) (P denotes an arbitrary polynomial). It is likely that the corresponding formulas would be lengthy and complicated.

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