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E5-89-258

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ON STATIONARY TOPOLOGICAL SOLITONS IN TWO-DIMENSIONAL ANISOTROPIC HEISENBERG MODEL

Submitted to journal "Letters in Mathematical Physics"

1989

The importance of studying conditions for localized solutions (solitons and instantons) to exist in non-one-dimensional Heisenberg models (three-component σ-models) is determined by wide using of these models in the condensed matter physics (see, e.g. [1,2]) and contemporary field theory [3]. In papers [4,5] Heisenberg models and their localized solutions are considered in order to explore possible mechanizms of high-temperature (HT) superconductivity. Note that due to the anisotropy and layer structure of the HT-superconductors (see, e.g. [6]) the investigations of two-dimensional (2D) anisotropic Heisenberg magnets seem to be of special interest. Unlike the case of 2D isotropic model [7] in anisotropic 2D magnet with the Hamiltonian density

$$\mathcal{H} = \alpha^2 (\partial_i s^a)^2 + \mathcal{H}_2 (s^a) \tag{1}$$

$$s^a s^a = 1 \quad , \quad i=1,2 \quad , a=1,2,3 \quad , \mathcal{H}_2 \geq 0 \quad , \mathcal{H}_2 \not\equiv 0 \; ,$$

stationary two-dimensional solitons cannot exist. This can be shown by Derrick's scaling transformation method [8] (see below). For the stable stationary solitons to exist in 2D anisotropic and 3D isotropic Heisenberg models they should be completed by stabilizing terms with higher degrees (a fourth ones, for example) of space derivatives in the Hamiltonian density (see [9,10,11] and references cited therein). Such terms appear when the short-wave Pomeranchuk fluctuations are taken into account [4] or if one considers the lattice models with not only nearest but with next-to-nearest neighbours interactions too [9] . In this Letter we study stationary solitons in the 2D anisotropic magnet with the simplest form of the fourth order term, $\mathcal{H}_3 = \gamma^2 (\partial_i s^a \partial_i s^a)^2$, and the "easy-axis" anisotropy (see [1]).

In this case the Hamiltonian density has the form :

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 = \alpha^2 (\partial_i s^a)^2 + \beta^2 \sin^2\theta + \gamma^2 (\partial_i s^a \partial_i s^a)^2 \tag{2}$$

here α , β , γ are constants, θ is an angle between the positive direction of the "easy" axis and the unit vector $\vec{s}(x_i)$, $\vec{s} = (s^1, s^2, s^3)$, $\partial_i = \frac{\partial}{\partial x_i}$, i =1,2. The system (2) has two vacuum states, $\vec{s}(x_i) = \vec{s}_0 = \pm \vec{e}_3$, with $\mathcal{H}_k \equiv 0$ for them, k =1,2,3, here $\vec{e}_3$ is the unit "easy" axis vector. We call a soliton such a continuous localized excitation of the vacuum state (choose $\vec{s}_0 = \vec{e}_3$ for definiteness sake) which is a solution to equations of motion and has a finite energy $H=\int \mathcal{H}\, d^2x$. Stationary solitons satisfy the equation $\frac{\delta H}{\delta s^a}=0$. Stable solitons should correspond to functional H minima.

Continuous localized distributions $s^a(x_i)$, $s^a(\infty)=s^a_0$, are divided into classes with different topological charge Q values, where Q is the degree of mapping $s^a(x_i)$ of R^2_{comp}, compactified by the boundary condition $s^a(\infty)=s^a_0$, to unit sphere S^2 [1,3,12],

$$Q=\frac{1}{8\pi}\int \varepsilon_{ik}\, \varepsilon^{abc} s^a \partial_i s^b \partial_k s^c\, d^2x, \tag{3}$$

$i, k=1,2,\ a, b, c=1,2,3.$

For distributions $s^a(x_i)$ with $|Q|\geqslant 1$ the following inequalities are valid :

$$H_k>0,\quad k=1,2,3\ ,\quad H_k=\int \mathcal{H}_k\, d^2x$$
$$H=H_1+H_2+H_3>0. \tag{4}$$

Perform the scaling transformation $s^a(x_i)\rightarrow s^a(\lambda x_i)$. One can see that then $H_1(1)\rightarrow H_1(1)$, $H_2(1)\rightarrow \lambda^{-2} H_2(1)$, $H_3(1)\rightarrow \lambda^2 H_3(1)$, where $H_k(\lambda)=\int \mathcal{H}_k[s^a(\lambda x_i)]d^2x$. It is necessary for stationary soliton existence that $\frac{dH(\lambda)}{d\lambda}=0$ at $\lambda=1$, as for $|\lambda-1|\ll 1$ scaling transformation give rise to small variations of distributions $s^a(x_i)$. Hence we get for stationary solitons :

$$H_2=H_3. \tag{5}$$

It follows from (5) that the 2D stationary topological solitons cannot exist in : i) isotropic magnet ($\mathcal{H}_2\equiv 0$) with nonzero $\mathcal{H}_3$ term ($\mathcal{H}_3\not\equiv 0$) and ii) anisotropic magnet with $\mathcal{H}_2\geqslant 0$ (independently of anisotropic type) and $\mathcal{H}_3\equiv 0$, i.e., without stabilizing terms.

Introduce dimensionless variables $\vec{\tau}=(x_d, y_d)=\alpha^{-1}\beta\vec{R}$, $\vec{R}=(x,y)$, then Eq.(2) takes the form :

$$\mathcal{H}=\beta^2[(\partial_i s^a)^2+\sin^2\theta+p(\partial_i s^a\partial_i s^a)^2], \tag{6}$$

where $\partial_1=\frac{\partial}{\partial x_d}$, $\partial_2=\frac{\partial}{\partial y_d}$, $p=\beta^2\gamma^2\alpha^{-4}$.

In the model (6) the existence of nontopological ($Q=0$) stationary solitons of the form

$$s^1=\sin\theta\ ,\quad s^2=0\ ,\quad s^3=\cos\theta\ ,\quad \theta=\theta(\tau),\quad \tau^2=x_d^2+y_d^2, \tag{7}$$

is not excluded a priori. For distributions (7)

$$\mathcal{H}(x_d,y_d)=\mathcal{H}(\tau)=\beta^2\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta+p\left(\frac{d\theta}{d\tau}\right)^4\right]. \tag{8}$$

Varying (8) and setting $\delta H=0$, we get the equation for soliton function $\theta_s(\tau)$

$$\frac{d^2\theta}{d\tau^2}\left[1+6p\left(\frac{d\theta}{d\tau}\right)^2\right]+\frac{1}{\tau}\frac{d\theta}{d\tau}\left[1+2p\left(\frac{d\theta}{d\tau}\right)^2\right]-\sin\theta\cos\theta=0 \tag{9}$$

which being completed by boundary conditions $\frac{d\theta}{d\tau}(0)=0$, $\theta(\infty)=0$, determines the boundary problem. Using the expansion of $\theta_s(\tau)$ for small τ

$$\theta_s(\tau)=C_0+C_2\tau^2+o(\tau^2) \tag{10}$$

we get from Eq.(11)

$$C_2=\frac{1}{4}\sin C_0\cos C_0\ . \tag{11}$$

The search for solitons was accomplished by "shooting" method, using (10) and (11). The value of the parameter p was varied at the whole interval $(0,\pi)$, but curves $\theta(\tau)$ which have been obtained by numerical solving of Eq.(9) do not satisfy the condition $\theta(\infty)=0$, if $0<C_0<\pi$. Thus, in the model (6) as well as in the previously considered [11] model with $\mathcal{H}_2=\beta^2\sin^2\theta$, $\mathcal{H}_3=\gamma^2[(\partial_i s^a\partial_i s^a)^2-(\partial_i s^a\partial_j s^a)^2]$, non-topological solitons do not exist.

In order to obtain topological ($|Q|\geqslant 1$) localized solutions $s^a(x_i)$ we use the well-known ansatz [1,12]

$$s^1=\sin\theta\cos\chi,\quad s^2=\sin\theta\sin\chi,\quad s^3=\cos\theta,$$
$$\chi=m\varphi, \tag{12}$$
$$\theta=\theta(\tau),\quad \theta(0)=\pi,\quad \theta(\infty)=0,$$

where $x_d=\tau\cos\varphi$, $y_d=\tau\sin\varphi$.

Topological charge of distributions (12) is expressed by the formula [1] :

$$Q=\frac{m}{2}[\cos\theta(\infty)-\cos\theta(0)]=m. \tag{13}$$

The ansatz (12) with $m=1$ makes it possible to find out the functional H minimum in the class of localized distributions $s^a(x_i)$ with $Q=1$. Note that the Hamiltonian H of the model (6) is invariant with respect to the group $G=SO(2)_S\otimes SO(2)_I$ (indices S and I mean the space (x_i) and isotopic (s^a) variables). Now we can separate the subgroup $G_1\subset G$, $G_1=diag[SO(2)_S\otimes SO(2)_I]$, with the coinciding transformation parameters of $SO(2)_S$ and $SO(2)_I$ groups. In other words, G_1 may be defined as a group of transformations [12]

$$\vec{s}(\vec{x}) \rightarrow R^3(\alpha)\, \vec{s}\,[\rho^{-1}(\alpha)\vec{x}\,], \tag{14}$$

where $\vec{x} = (x_1, y_1) = (\tau, \varphi)$, $\rho(\alpha)\vec{x} = (\tau, \varphi + \alpha)$, and $R^3(\alpha)$ is an operator of rotating through the angle α about the third isotopic axis. The fields $s^a(x_i)$ of the form (12) with $m = 1$ form a set M_1 of the fields invariant under the subgroup G_1, the topological charge Q of these fields is equal to $m = 1$. According to Coleman-Palais theorem [12-14] the extremum of the functional H over the set M_1 of the invariant fields with $Q = 1$ is the extremum over the whole set of the fields $s^a(x_i)$ with $Q = 1$. Thus, the use of the ansatz (12) with $m = 1$ permits one to find solitons with $Q = 1$ in the model (6). Unfortunately, nowadays there is no effective well-grounded ansatz for solitons with $|Q| > 1$. But note that the ansatz (12) for all $m = Q$ describes distributions of the energy density $\mathcal{H}(x_i)$ which are independent of the angle φ:

$$\mathcal{H}(\tau) = \beta^2\left\{\left(\frac{d\theta}{d\tau}\right)^2 + \frac{m^2}{\tau^2}\sin^2\theta + \sin^2\theta + p\left[\left(\frac{d\theta}{d\tau}\right)^2 + \frac{m^2}{\tau^2}\sin^2\theta\right]^2\right\} \tag{15}$$

Varying (15) and setting $\delta H = 0$, we get that in the case $|Q| > 0$ the functions $\theta(\tau)$ is a solution to the boundary value problem ($Q = m$)

$$\frac{d^2\theta}{d\tau^2}\left[1 + 6p\left(\frac{d\theta}{d\tau}\right)^2 + 2pm^2\frac{\sin^2\theta}{\tau^2}\right] + \frac{d\theta}{d\tau}\left[\frac{1}{\tau} + \frac{2p}{\tau}\left(\frac{d\theta}{d\tau}\right)^2 - 2pm^2\frac{\sin^2\theta}{\tau^3} + \right.$$
$$\left. + 2pm^2\frac{\sin\theta\cos\theta}{\tau^2}\,\frac{d\theta}{d\tau}\right] - \sin\theta\cos\theta\left(\frac{m^2}{\tau^2} + 1\right) - 2pm^2\frac{\sin^3\theta\cos\theta}{\tau^4} = 0, \tag{16}$$

$$\theta(0) = \pi, \qquad \theta(\infty) = 0. \tag{17}$$

For small τ the solutions $\theta_m(\tau)$ to the Eq.(18) at a given m are described by the expansion

$$\theta_m(\tau) = \pi - C_m\tau^m + o(\tau^m) \tag{18}$$

which was used for a numerical study of the problem (16),(17) solutions by the shooting method. Functions $\theta_1(\tau)$ of solitons with $Q = 1$ ($m = 1$) and functions $\theta_2(\tau)$ of localized distributions with $Q = 2$ ($m = 2$) have been found by computer for various p values (see Table).

T a b l e

p	C_1	h_1	C_2	h_2
10^{-6}	62.64	25.27	3570	50.38
10^{-5}	32.56	25.54	110.3	50.61
10^{-4}	16.53	26.25	34.23	51.35
10^{-3}	8.116	28.12	10.52	53.61
10^{-2}	3.879	32.93	3.225	60.37
10^{-1}	1.853	45.69	1.0145	80.35
1	0.9205	82.21	0.3459	140.8
20	0.4026	261.2	0.1034	443.2
500	0.1758	1178	0.03466	1998

The table contains values $C_m(p)$, $m = 1,2$ which characterize the solutions to the problem (16),(17), and dimensionless energy $h_m = \alpha^{-2}H[\theta_m(\tau)]$ values of corresponding localized distributions $s^a(x_i)$.

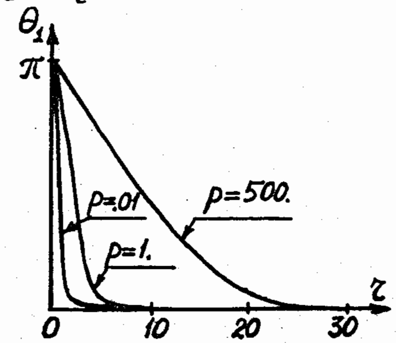

Figure 1a

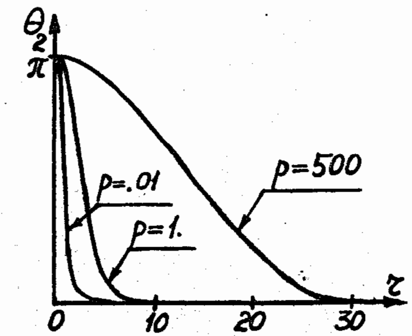

Figure 1b

Functions $\theta_1(\tau)$ and $\theta_2(\tau)$ are presented in Figure 1a and Figure 1b; the characteristic radius of these solutions increases monotonously with the growth of p, $0 < p < \infty$. The investigation of numerical results for small p shows that the half-width R_m of these solutions defined by equality $\theta_m(R_m) = \frac{\pi}{2}$ is proportional to p^{γ_m}, $\gamma_1 \approx 0.27$ ($R_1 \approx 3.8 \cdot 10^{-2}$ for $p = 10^{-6}$, $R_2 \approx 0.46$ for $p = 10^{-2}$), $\gamma_2 \approx 0.25$. The dependence $h_1(p)$ for solitons with $Q = 1$ is pre-

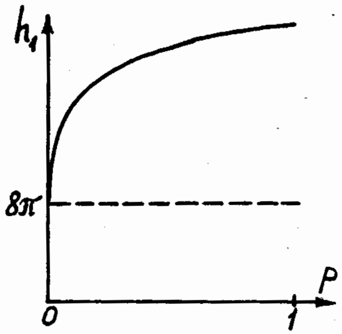

Figure 2

sented in Figure 2. For small p it may be described by the formula

$$h_1(p) \approx 8\pi + Bp^{\varkappa}, \quad \varkappa \approx 0.42, \quad B = const. \qquad (19)$$

Using (19) and the relationship $R_1 \propto p^{\gamma_1}$, we get for solitons of small radius, $R_1 \ll 1$,

$$h_1 \approx 8\pi + \mathcal{D} R_1^{\delta}, \quad \delta \approx 1.5, \quad \mathcal{D} = const. \qquad (20)$$

Note that for all p values (see the Table) the inequality $h_2 < 2h_1$ is valid, that is for all p considered the existence of localized distributions $s^a(x_i)$ with Q=2, possessing energy less than the double energy of solitons with Q =1, is shown. Hence, the formation of the bound state of two solitons with Q =1 is energetically favourable. We hope that the solitons with minimum energy in the Q =2 sector will be obtained in the course of further investigations.

In conclusion we would like to discuss the relation of the solutions obtained for the anisotropic magnet with Belavin-Polyakov solitons in the isotropic magnet [7] . Note that for solitons of small radii, $R_{sol} \ll 1$, that is for $p \ll 1$ the terms $\mathcal{H}_2$ and $\mathcal{H}_3$ are small as compared with $\mathcal{H}_1$ [1, 10,11] (in fact $\mathcal{H}_1 \sim p^{-2\gamma_m}$, $\mathcal{H}_2 \sim p^0$, $\mathcal{H}_3 \sim p^{1-4\gamma_m} \approx p^0$), so aspiration to develop for the model (6) the perturbation theory in small p with Belavin-Polyakov solitons being the solutions of the zeroth approximation is quite natural [10,11] . But this program meets essential difficulties related to the divergence of the "easy-axis" anisotropy energy integral on Belavin-Polyakov solitons with Q =1 [15] .

The authors are grateful to B.A.Ivanov, B.G.Konopelchenko, V.E.Korepin, A.M.Kosevich, V.G.Makhankov, A.S.Schwartz, V.E.Zakharov, E.P.Zhidkov for useful discussions.

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Received by Publishing Department
on April 13, 1989.