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V.P.Gerdt, A.Yu.Zharkov*

COMPUTER CLASSIFICATION OF INTEGRABLE COUPLED KdV - LIKE SYSTEMS

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*Saratov State University, USSR

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1. Introduction

At present intensive work on testing of integrability and classification of integrable nonlinear evolution systems

$$U_t=F(x,U,U_1,\ldots U_N),\quad U=U(x,t)=(U^1,\ldots U^M)$$
$$F=(F^1,F^2,\ldots F^M),\quad U_1=D^1(U),\ D=d/dx \qquad (1)$$

is carried out. The integrability means that the system (1) can either be reduced to a linear one by differential substitutions $U\to F(V,..V_k)$ or can be integrated by inverse spectral transform. Prototype equations of these two groups are the Burgers and the Korteveg-de Vries (KdV) equations, respectively. In both cases the initial nonlinear problem reduces to a linear one that can be investigated and solved. Note that evolution systems of the second type are especially interesting in physics due to their multi-soliton solutions. In the present paper we use computer algebra to find all integrable coupled KdV - like evolution systems.

2. Symmetry approach

The classification problem consists in obtaining a complete list of integrable systems (1) for some fixed N and to describe the most general transforms connecting these systems. It can be achieved effectively by using the symmetry approach (see reviews [1-3]). In the framework of the symmetry approach, the integrable criterium is based on the property of (1) to have an infinite algebra of higher-order symmetries, i.e. the evolution systems of the form

$$U_t=H(x,U,U_1,\ldots U_n),\ H=(H^1,\ldots H^M),\ n>N, \qquad (2)$$

compatible with (1). The compatibility condition can be written as

$$\frac{dH}{dt} = F_*(H),$$

where F_* is the matrix differential operator

$$F_* = \sum_{i=0}^{N} \mathbb{F}_i \mathbb{D}^i, \quad |\mathbb{F}_i|_{jk} = \frac{\partial F^i}{\partial U_j^k} .$$

It is shown in [1,2] that in the case where the evolution system (1) has an infinite algebra of symmetries, it has an infinite number of local conservation laws

$$\frac{dR_{ij}}{dt} = \frac{dQ_{ij}}{dx}, \quad i=1,2,\ldots, \quad j=1,\ldots M, \tag{3}$$

where the densities R_{ij} can be expressed in terms of F and Q_{ik} (k<j). The conditions (3) generate an overdetermined system of equations in F. By solving this system it is possible to find a list of concrete F's containing all the integrable cases (usually it is sufficient to use conditions (3) for i<3). This list is then checked for higher i against the condition (3) in order to remove non-integrable cases. The next step consists in finding higher-order symmetries and a Lax representation.

3.Role of Computer Algebra

The algorithms for the most tedious steps of classification,such as computing the densities R_{ij}, checking the conditions (3), derivation of the overdetermined systems in F and finding the symmetries, have been suggested in [4,5]. They have been implemented using the computer algebra system FORMAC for scalar equations (M=1 in (1)) [4] and for the following wide class of systems [5]

$$U_t = \Lambda U_N + f(x, U, U_1, \ldots U_{N-1}), \quad f = (f^1, \ldots f^M)$$
$$\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_M), \quad \lambda_i \neq 0, \quad \lambda_i \neq \lambda_j (i \neq j), \quad \lambda_i \in \mathbb{C} . \tag{4}$$

The step in our algorithm which remains to be done by hand is to solve the overdetermined system of differential equations in F. However,in a special case (which is very important for applications), the $F^1,\ldots F^M$ are polynomials in $U, U^1, \ldots U^N$ with certain homogeneity properties, and the overdetermined system in F reduces to an algebraic system for the coefficients of these polynomials. The general approach to solve such a system exactly is based on the well-known technique of Groebner basis construction [6] which is implemented, for example, in the last version of the computer algebra system REDUCE [7]. An example of a Groebner basis computation for one of the classification problems [8] is given in [9]. Thus the classification of the integrable evolution systems (4) with homogeneously-polynomial right hand sides can be completely automated by means of computer algebra.

4.Coupled KdV - Like Systems

In this paper we apply the above technique to the classification of the following systems from class (4) (coupled KdV-like systems)

$$u_t = a_0 u_3 + a_1 u u_1 + a_2 v v_1 + a_3 u v_1 + a_4 v u_1, \quad a_0 \neq b_0, \; a_0 \neq 0, \; b_0 \neq 0,$$
$$v_t = b_0 v_3 + b_1 v v_1 + b_2 u u_1 + b_3 v u_1 + b_4 u v_1, \quad a_i, b_i \in \mathbb{C} \; (i=0 \div 5) , \tag{5}$$

which have an infinite algebra of symmetries. Using the FORMAC program described in [5] we obtain a system of equations for the parameters a_i, b_j consisting of twelve equations of sixth degree in ten unkowns obtained from (3) for i=1÷4 and in part for i=5

$$e_k = \hat{e}_k = 0, \quad (k=1 \div 6) , \tag{6}$$

where $\hat{e}_k = e_k \big|_{a_i \leftrightarrow b_i}$ and

$e_1 = a_1(a_3 - a_4) - a_4(b_3 - b_4),$

$e_2 = (2a_3 - a_4) y_1 - b_2 y_2, \qquad y_1 = 6a_0 a_3 b_2 + (a_0 - b_0)(a_1^2 + a_4 b_2),$

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$$v_t=b_0v_3+b_1vv_1+b_2uu_1+b_3vu_1+b_4uv_1, \quad a_i, b_i\in\mathbb{C} \ (i=0\div 5), \tag{5}$$

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$e_3=a_2y_1-(2b_3-b_4)y_2, \qquad y_2=6a_0a_2b_3+(a_0-b_0)(a_1a_2+a_4b_1)$,

$e_4=3a_0(a_2b_2+a_3b_3)+(a_0-b_0)(a_1+b_3)a_4$,

$e_5=2(2a_0^2+8a_0b_0-b_0^2)a_3b_3+2(a_0-b_0)(4a_0-b_0)a_3b_4-6a_0(a_0+2b_0)a_2b_2$

$\quad +(a_0-b_0)^2(5a_1a_3-5a_1a_4+a_4b_4)-(a_0-b_0)(7a_0-b_0)a_4b_3$,

$e_6=3a_0[(a_0-b_0)^3-3a_0(a_0+2b_0)^2](a_2b_2+a_3b_3)+(a_0-b_0)^3[3a_0a_1a_3-2(2a_0+b_0)a_1a_4]$

$\quad +9a_0^2(a_0-b_0)[(a_0-b_0)a_4-(a_0+2b_0)a_3]b_4-(a_0-b_0)(2a_0^3-30a_0^2b_0+b_0^3)a_4b_3$.

5.Solving the System (6)

In order to solve system (6) one may use the technique of Groebner basis. However, we have used instead a much more effective algorithm [10] which exploits the special structure of the system (6). Its main idea is to consider several alternative cases:

$$1)\ a_1\neq 0, a_4\neq 0 \quad 2)\ a_1\neq 0, a_4=0 \quad 3)\ a_1=0, a_4\neq 0 \quad 4)\ a_1=a_4=0$$

and two subcases inside each case:

$$a)\ y_1\neq 0 \text{ or } y_2\neq 0 \quad b)\ y_1=y_2=0 .$$

One can find simple relations connecting a_i, b_i for each subcase and thus the system (6) can be considerably simplified. For example, in the case 1), the equations

$$e_1=\hat{e}_1=0$$

lead to two possibilities:

$$i)\ a_3=a_4, b_3=b_4 \quad ii)\ a_3\neq a_4 \text{ or } b_3\neq b_4, \ a_1b_1=a_4b_4.$$

In the case ii) we can set $a_1=a_4$ and $b_1=b_4$ by taking into account

$$e_3=a_2y_1-(2b_3-b_4)y_2, \qquad y_2=6a_0a_2b_3+(a_0-b_0)(a_1a_2+a_4b_1),$$

$$e_4=3a_0(a_2b_2+a_3b_3)+(a_0-b_0)(a_1+b_3)a_4,$$

$$e_5=2(2a_0^2+8a_0b_0-b_0^2)a_3b_3+2(a_0-b_0)(4a_0-b_0)a_3b_4-6a_0(a_0+2b_0)a_2b_2$$
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$$+9a_0^2(a_0-b_0)[(a_0-b_0)a_4-(a_0+2b_0)a_3]b_4-(a_0-b_0)(2a_0^3-30a_0^2b_0+b_0^3)a_4b_3 .$$

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the invariance of the ten-parametric family (5) under the scale transformations $u \Rightarrow \alpha u$, $v \Rightarrow \beta v$ $(\alpha,\beta \in \mathbb{C})$. On the other hand, the subcase a) implies $a_2b_2=(2a_3-a_4)(2b_3-b_4)$, so a_2, b_2 can immediately be eliminated from the equations (6) for $k=3\div 6$, etc. Applying this method and carring out all computations in the interactive mode of the computer algebra system REDUCE, we have found all the non-trivial solutions of (6) (see ref.[10] fore more details). It should be noted that in each alternative case the problem reduces to simple *gcd* and *resultant* computations which are built-in in REDUCE. It turns out that in the most tedious case 1) $a_1\neq 0, a_4\neq 0$ the system (6) has a single non-trivial solution (up to a scale transformation)

$$a_0=(3\pm\sqrt{5})/6,\ a_1\neq 0,\ a_2=a_1(9a_0-7)/(12a_0-1),\ a_3=a_1/(3a_0),\ a_4=a_1, \tag{7}$$
$$b_0=(-3\pm\sqrt{5})/6 \quad b_1=-a_1/(3a_0),\ b_2=a_1(3a_0+1)/(9a_0-7),\ b_3=-a_1,\ b_4=-a_1/(3a_0).$$

The evolution system (5) with the coefficients (7) can be transformed to the well-known integrable Drinfeld-Sokolov system [11] by appropriate linear transformation of the vector space (u,v). For the cases 2)÷4), we have obtained a list of four nonlinear coupled systems of the form (5) with coefficients a_i, b_i satisfying (6) and containing four to six arbitrary constants.

6.List of Integrable Systems.Conclusion

Using our FORMAC program [5] we have checked the list obtained whether or not conditions (3) for $i=5\div 8$ are satisfied. We have found that only three evolution systems satisfy these conditions, namely

$$u_t=u_3+uu_1+vv_1, \quad v_t=-2v_3-uv_1 \tag{8}$$

$$u_t=u_3+uu_1, \quad v_t=4v_3+uv_1+1/2\, u_1v \tag{9}$$

$$u_t=u_3+uu_1, \quad v_t=-2v_3-vu_1-v_1 . \tag{10}$$

The system (8) is the well-known Hirota-Satsuma system [12] with an the invariance of the ten-parametric family (5) under the scale transformations $u \Rightarrow \alpha u$, $v \Rightarrow \beta v$ ($\alpha,\beta \in \mathbb{C}$). On the other hand, the subcase a) implies $a_2 b_2=(2a_3-a_4)(2b_3-b_4)$, so a_2, b_2 can immediately be eliminated from the equations (6) for k=3÷6, etc. Applying this method and carring out all computations in the interactive mode of the computer algebra system REDUCE, we have found all the non-trivial solutions of (6) (see ref.[10] fore more details). It should be noted that in each alternative case the problem reduces to simple *gcd* and *resultant* computations which are built-in in REDUCE. It turns out that in the most tedious case 1) $a_1\neq 0, a_4\neq 0$ the system (6) has a single non-trivial solution (up to a scale transformation)

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The system (8) is the well-known Hirota-Satsuma system [12] with an

infinite algebra of symmetries. The system (9) was firstly considered in our paper [5]. Based on the generally accepted conjecture (see, for example, [3]) that the existance of higher-order symmetries implies integrability, we may conclude that the system (9) also has an infinite algebra of symmetries. In [5] we have found the following 5-order symmetry for the system (9)

$$u_t=u_5+5/3\ uu_3+10/3\ u_1u_2+5/6\ u^2u_1,$$

$$v_t=16v_5+20/3\ uv_3+5/2vu_3+10u_1v_2+25/3\ v_1u_2+5/6\ u^2v_1+5/6\ vuu_1\ .$$

We believe that the system (10) is also integrable. It has the 5-order symmetry with the same first equation in accordance with the structure of (9) and (10)

$$u_t=u_5+5/3\ uu_3+10/3\ u_1u_2+5/6\ u^2u_1,$$

$$v_t=-4v_5-10/3\ uv_3-5/3\ vu_3-20/3\ u_1v_2-5v_1u_2-5/18\ u^2v_1-5/9\ uu_1\ .$$

We may conclude from the above computations that computer algebra is a powerful tool for investigating nonlinear evolution equations. It allows to make a complete classification of the integrable coupled systems from the ten-parametric family (5). All integrable cases are exhausted by the four systems (7)÷(10).

Authors are thankful to K.S.Kölbig and S.I.Svinolupov for useful discussions.

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