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THE TROTTER-KATO PRODUCT FORMULA FOR GIBBS SEMIGROUPS

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1. Introduction

Let A and B be linear operators in a separable complex Hilbert space $\mathscr{H}$. Then under suitable conditions concerning {A,B} the strong limit

$$\underset{n\to\infty}{\text{s-lim}}\ (\exp(-\frac{t}{n}A)\ \exp(-\frac{t}{n}B))^n = \exp(-tC) \qquad (1.1)$$

exists for $t \geq 0$, where the operator C can be constructed by means of A and B. This is the well-known Trotter-Lie product formula for strongly continuous (C_o-) semigroups [1]. (For finite matrices it has been established by Sophus Lie about 1875). Since the discovery of the product formula it has permeated through mathematics and mathematical physics challenging the problem of relaxation and generalization of the hypotheses under which the formula holds, see [2-10].

A solution of this problem implies that one has to do the following:

(i) to find the set of pairs {A,B} for which the limit (1.1) exists;

(ii) to identify the operator C and to describe the mapping {A,B}: $\rightarrow$ C;

(iii) to generalize (if possible) the exponential functions involved in the left-hand side of (1.1) to a class of real-valued, Borel measurable functions f(.), g(.) such that in some operator topology τ

$$\underset{n\to\infty}{\tau\text{-lim}}(f(\frac{t}{n}A)\ g(\frac{t}{n}B))^n = \exp(-tC)\Pi \qquad (1.2)$$

for $t \in \mathbb{R}^1_+ = \{x \in \mathbb{R}^1 : x \geq 0\}$ (or its continuation into the right complex half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) \geq 0\}$) where Π is the orthogonal projection of $\mathscr{H}$ onto the closed subspace in which operator C is defined;

(iv) to indicate a natural topology τ in which the convergence in (1.2) will take place.

A lot of papers has been devoted to the points (i) and (ii) of the above program when (A,B) is a pair of self-adjoint operators. It was Trotter [1] who for the first time has proved (1.1) for C_o-semigroups whenever operators A,B are semibounded from below and the algebraic sum $A + B$ is essentially self-adjoint on a common dense domain $\mathcal{D} = \mathcal{D}(A) \cap \mathcal{D}(B)$, i.e., the operator $A + B$ has a unique self-adjoint extension defined by the closure $(A + B)^{-} = C$. For unitary groups $(t \to it)$ the semiboundedness can be canceled.

The proviso about the domain $\mathcal{D}$ is important because there exist examples of non-negative self-adjoint operators $\{A,B\}$ such that $\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}$ and $(\mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2}))^{-} = \mathcal{H}$. Therefore, Chernoff [2] (see also Faris [3] and Simon [4]) has extensively studied (1.1) for C_o-contraction semigroups to define a generalized sum of two unbounded non-negative self-adjoint operators A and B whenever their common form domain $Q = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2})$ is dense in $\mathcal{H}$ and the quadratic form $t_B[u] = \| B^{1/2} u \|^2$, $u \in Q(B) = \mathcal{D}(B^{1/2})$ is bounded relative to $t_A[u]$: $Q(A) \subset Q(B)$ and $t_B[u] \le a\|u\|^2 + bt_A[u]$, $u \in Q(A)$, for some $a,b \ge 0$. Then $C = A \dotplus B$ is the form sum of A and B [5], i.e., a unique non-negative self-adjoint operator associated with non-negative closed quadratic form

$$h[u] = t_A[u] + t_B[u], \; u \in Q(A) \cap Q(B). \qquad (1.3)$$

The essential contribution to the theory at the point (iii) has been made by Kato [6,7]. In two subsequent papers he has proved the product formula (1.2) in the strong operator topology $\tau = s$ for

a very general (but a natural) class of real-valued functions f,g: $\mathbb{R}^1_+ \to [0,1]$ and an *arbitrary* pair (A,B) of non-negative unbounded self-adjoint operators in the Hilbert space $\mathcal{H}$. Then Π is the orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{H}'$ spanned by $Q = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2})$ and $C = A \dotplus B$ is self-adjoint operator in $\mathcal{H}'$ associated with the non-negative closed quadratic form $h[u] = \| A^{1/2} u\|^2 + \| B^{1/2} u\|^2$, $u \in Q$, which is densely defined in $\mathcal{H}'$. In the first paper [6] (Kato I) he has proved the product formula (1.2) for the pairs of functions $\{f(x),g(x)\}$ including $\{e^{-x},(1+x)^{-1}\},\{(1+x)^{-1},e^{-x}\}$ and $\{(1+x)^{-1}, (1+x)^{-1}\}$, while in the second one [7] (Kato II) a completely different proof which allows one to include the important case of $\{e^{-x}, e^{-x}\}$ has been proposed. Below we shall refer to (1.2) for $\tau = s$ as the *Trotter-Kato product formula*. The problem to prove the Trotter-Kato product formula for unitary groups and imaginary resolvents $\{f,g\} = \{(1+ix)^{-1},(1+ix)^{-1}\}$ has been considered by Lapidus [8,9]. He has proved that in the latter case the conditions on the pair $\{A,B\}$ can be relaxed. If the self-adjoint operator A is assumed to be non-negative, then the positive part B_+ of the self-adjoint operator $B = B_+ - B_-$ can be arbitrary while its negative part B_- has to be small with respect to A in the sense of quadratic forms with the relative bound $b < 1$: $Q(A) \subset Q(B_-)$ and $t_{B_-}[u] \le a\|u\|^2 + bt_A[u]$, $u \in Q(A)$. Then again $C = A \dotplus B$ and $\mathcal{H}' = \Pi\mathcal{H} = (Q(A) \cap Q(B))^{-}$, $Q(B) = \mathcal{D}(|B|^{1/2})$, where $|B| = (B^*B)^{1/2}$ denotes the absolute value of the operator B.

Recently, one of the authors of the present paper has made an attempt to connect the topology τ in the product formula with that in which semigroups involved in (1.1) are continuous for $t \in \mathbb{R}^1_+\backslash\{0\}$ [10]. This question has been inspired by the point (iv) of the above program for Gibbs semigroups, see [11,12]. If at least one of the operators A or B generates a self-adjoint Gibbs semigroup and Trotter's conditions on the pair (A,B) are satisfied, then the strong operator convergence in (1.1) can be *lifted* for $t > 0$ to $\tau = \|.\|_1$-topology (trace-norm convergence).

The purpose of the present paper is to prove the Trotter-Kato product formula (1.2) for $\tau = \|.\|_1$ when at least one operator of the pair (A,B) is a generator of a self-adjoint Gibbs semigroup. We also discuss relaxation of the conditions on (A,B) imposed by Kato [6,7] which are relevant in applications to quantum statistical mechanics.

To formulate the problem more precisely we recall some notation and definitions, see e.g. [13]. If $\mathscr{H}$ is a separable Hilbert space, then $\mathscr{C}_p(\mathscr{H})$ is the Banach space of compact operators on $\mathscr{H}$ with finite $\|.\|_p$-norm:

$$\| X \|_p = \left\{ \sum_{k=1}^{\infty} (\lambda_k(X))^p \right\}^{1/p}, \quad 1 \le p < \infty. \qquad (1.4)$$

Here $\{\lambda_k(X)\}_{k=1}^{p}$ are the singular values of the operator $X \in \mathscr{C}_p(\mathscr{H})$, i.e. eigenvalues of the operator $|X| = (X^*X)^{1/2}$, e.g. the trace class $\mathscr{C}_1(\mathscr{H})$ and the Hilbert-Schmidt operators $\mathscr{C}_2(\mathscr{H})$ are defined by the trace-norm $\|X\|_1 = \mathrm{Tr}|X|$ and the Hilbert-Schmidt norm $\|X\|_2 = (\mathrm{Tr}(X^*X))^{1/2}$, respectively. The Banach spaces $(\mathscr{C}_p(\mathscr{H}))_{1\le p<\infty}$ are $*$-ideals in the Banach space of compact operators $\mathscr{C}om(\mathscr{H}) = \mathscr{C}_\infty(\mathscr{H})$ and bounded operators $\mathscr{B}(\mathscr{H})$ in $\mathscr{H}$ ordered by

$$\mathscr{C}_1(\mathscr{H}) \subset \mathscr{C}_2(\mathscr{H}) \subset \ldots \subset \mathscr{C}_p(\mathscr{H}) \subset \ldots \subset \mathscr{C}om(\mathscr{H}) \subset \mathscr{B}(\mathscr{H}). \qquad (1.5)$$

Definition 1.1 [11]. A C_o-semigroup $\{G(t)\}_{t\ge 0}$ in a separable Hilbert space $\mathscr{H}$ is called a Gibbs semigroup if $G(t)\colon (0,\infty) \to \mathscr{C}_1(\mathscr{H})$.

Remark 1.2. From the continuity of multiplication (Grümm [14]):

$$X_nY_n \xrightarrow{\|.\|_p} XY \quad \text{if} \quad X_n \xrightarrow{s} X,\ Y_n \xrightarrow{\|.\|_p} Y \quad \text{as } n \to \infty \qquad (1.6)$$

for $\{X_n\}_{n\ge1} \in \mathscr{B}(\mathscr{H})$, $\{Y_n\}_{n\ge1} \in \mathscr{C}_p(\mathscr{H})$, $1 \le p < \infty$, it follows that Gibbs semigroups are $\|.\|_1$-continuous for $t > 0$.

The Gibbs semigroups naturally arise in quantum statistical mechanics (QSM) as one-parameter self-adjoint C_o-semigroups generated by a Hamiltonian H: $G_H(\beta) = \exp(-\beta H)$. Here a parameter $\beta > 0$ is nothing but the inverse temperature of the system described by the operator H. For continuous systems of QSM H is a sum of two parts: an ideal (kinetic-energy operator T) and a nonideal (interaction operator U). It is known [4] that for singular two-body potentials the operator U is not being small with respect to the kinetic-energy operator T in the usual operator sense [5]. Therefore, in this case the definition of the Hamiltonian of the system is not very obvious. Moreover, as far as in QSM the main object of investigations is the partition function $Z(\beta) = \mathrm{Tr}(G_H(\beta))$, regularizations or limit procedures defining the Hamiltonian H have to be such that the corresponding families of operators approximating Gibbs semigroup $G_H(\beta)$ should be $\|.\|_1$-convergent [11,15]. The same arguments are applied to the Trotter product formula which is often used (under the Tr) for constructing a sum of T and U, trace Feynman-Kac formula and other calculations, see e.g. [16].

The outline of the paper is as follows. In section 2 we accumulate technical preliminaries which in our opinion have also their own interest for the theory of the $*$-ideals $\mathscr{C}_p(\mathscr{H})$, $1 \le p \le \infty$. In section 3 we prove the Trotter-Kato product formula (1.2) in $\|.\|_1$- topology. This is done in two steps. First, we consider a special case when Kato I conditions plus requirement that $f(tA) \in \mathscr{C}_p(\mathscr{H})$ for $t > 0$ and $1 \le p < \infty$ are fulfilled. In contrast with the case of C_o-semigroups and $\tau = s$ (see [7]) we cannot avoid this intermediate step exploiting monotony properties of auxiliary operator families. The vindication of this line of reasoning becomes clear when one follows the proof of the product formula (1.2) in the general case of conditions à la Kato II. The last section 4 is devoted to concluding remarks and possible applications.

2. Technical preliminaries

In the following we prove some generalizations of the existing convergence theorems in trace ideals which will be useful in the sequel. The generalizations are mainly connected with the uniformity of certain convergences in $\mathscr{C}_p$-ideals and, therefore, they are of independent interest.

Proposition 2.1. *Let* $\{X_n\}_{n\ge 1}$ *and* $\{Y_n\}_{n\ge 1}$ *be operator sequences from* $\mathscr{B}(\mathscr{H})$ *and* $\mathscr{C}_p(\mathscr{H})$, $1 \le p < \infty$, *respectively, and let* $X \in \mathscr{B}(\mathscr{H})$ *and* $Y \in \mathscr{C}_p(\mathscr{H})$ *be operators such that*

$$\|.\|_p\text{-}\lim_{n\to\infty} Y_n = Y. \tag{2.1}$$

(i) *If* $s\text{-}\lim_{n\to\infty} X_n = X$, *then* $\|.\|_p\text{-}\lim_{n\to\infty} X_n Y_n = XY$.

(ii) *If* $s\text{-}\lim_{n\to\infty} X_n^* = X^*$, *then* $\|.\|_p\text{-}\lim_{n\to\infty} Y_n X_n = YX$.

Proof. The first conjecture is due to Grümm [14], see Remark 1.2. To prove the second conjecture we note that $\mathscr{C}_p(\mathscr{H})$ is $*$-ideal in $\mathscr{B}(\mathscr{H})$, i.e., the norm is invariant under the involution $Z \to Z^*$. Hence $Y_n \xrightarrow{\|.\|_p} Y$ yields $Y_n^* \xrightarrow{\|.\|_p} Y^*$. Applying (i) to $\{X_n^* Y_n^*\}_{n\ge 1}$ we obtain $X_n^* Y_n^* \xrightarrow{\|.\|_p} X^* Y^*$. Using again the invariance of $\|.\|_p$ under the involution $Z \to Z^*$ we prove (ii).■

Corollary 2.2. *Let* $\{X_n(.)\}_{n\ge 1}$ *and* $\{Y_n(.)\}_{n\ge 1}$ *be sequences of operator-valued functions defined on* $\mathscr{K}$ *with values in* $\mathscr{B}(\mathscr{H})$ *and* $\mathscr{C}_p(\mathscr{H})$, *respectively, such that*

$$\sup_{\substack{t\in\mathscr{K}\\ n\ge 1}} \|X_n(t)\| < +\infty. \tag{2.2}$$

Let $X(.): \mathscr{K} \longrightarrow \mathscr{B}(\mathscr{H})$ *and* $Y(.): \mathscr{K} \longrightarrow \mathscr{C}_p(\mathscr{H})$ *be operator-valued functions such that*

$$\sup_{t\in\mathscr{K}} \|Y(t)\| < +\infty \tag{2.3}$$

and

$$\|.\|_p\text{-}\lim_{n\to\infty} Y_n(t) = Y(t) \tag{2.4}$$

uniformly in $t \in \mathscr{K}$.

(i) *If* $s\text{-}\lim_{n\to\infty} X_n(t) = X(t)$ *uniformly in* $t \in \mathscr{K}$ *and if for some sequence of finite dimensional orthogonal projections* $\{P_l\}_{l\ge 1}$ *obeying* $s\text{-}\lim_{l\to\infty} P_l = I$ *we have*

$$\lim_{l\to\infty} \sup_{t\in\mathscr{K}} \|(I - P_l)Y(t)\|_p = 0, \tag{2.5}$$

then $\|.\|_p\text{-}\lim_{n\to\infty} X_n(t)Y_n(t) = X(t)Y(t)$ *uniformly in* $t \in \mathscr{K}$.

(ii) *If* $s\text{-}\lim_{n\to\infty} X_n(t)^* = X(t)^*$ *uniformly in* $t \in \mathscr{K}$ *and if for some sequence of finite dimensional orthogonal projections* $\{Q_l\}_{l\ge 1}$ *obeying* $s\text{-}\lim_{l\to\infty} Q_l = I$ *we have*

$$\lim_{l\to\infty} \sup_{t\in\mathscr{K}} \| Y(t)(I - Q_l)\|_p = 0, \tag{2.6}$$

then $\|.\|_p\text{-}\lim_{n\to\infty} Y_n(t)X_n(t)$ *uniformly in* $t \in \mathscr{K}$.

Proof. We note that $X_n(t) \xrightarrow[n\to\infty]{s} X(t)$ or $X_n(t)^* \xrightarrow[n\to\infty]{s} X(t)^*$ and (2.2) imply

$$\sup_{t\in\mathcal{X}} \| X(t) \| < +\infty. \tag{2.7}$$

To prove (i) we use the estimate

$$\|X_n(t)Y_n(t) - X(t)Y(t)\| \le \|X_n(t)\| \, \|Y_n(t) - Y(t)\|_p +$$
$$+ \|X_n(t) - X(t)\| \, \|(I - P_l)Y(t)\|_p + \|(X_n(t) - X(t))P_l\|_p \, \|Y(t)\|.$$

The first term tends to zero uniformly in $t \in \mathcal{X}$ as $n \to \infty$ by (2.2) and (2.4). The second term goes to zero uniformly in $t \in \mathcal{X}$ and $n \ge 1$ as $l \to \infty$ by (2.2), (2.7) and (2.5). Choosing a suitable integer l and fixing it we obtain that on account of the estimate

$$\|(X_n(t) - X(t))P_l\|_p \le l^{1/p} \, \|(X_n(t) - X(t))P_l\|$$

and the uniformity of convergence $X_n(t) \xrightarrow[n\to\infty]{s} X(t)$ in $t \in \mathcal{X}$ the expression $\|(X_n(t) - X(t))P_l\|$ converges to zero uniformly in $t \in \mathcal{X}$ as $n \to \infty$. Hence, by (2.3) the third term tends to zero uniformly in $t \in \mathcal{X}$ as $n \to \infty$. Summing up we prove part (i).

In order to prove part (ii) we have only to use the results of (i) and the invariance of the operator norm $\|.\|$ and the ideal norm $\|.\|_p$ under the involution $Z \to Z^*$.■

The next lemma will be necessary to establish a certain generalization of the Grümm convergence theorem [13,14].

Lemma 2.3. *Let* $(X_n(.))_{n\ge 1}$ *be a sequence of operator-valued functions defined on* $\mathcal{X}$ *with values in* $\mathcal{B}(\mathcal{X})$ *such that* (2.2) *is valid. Let* $X(.): \mathcal{X} \longrightarrow \mathcal{B}(\mathcal{X})$ *be an operator-valued function such that for some sequences of finite dimensional projections* $(P_l)_{l\ge 1}$ *and* $(Q_l)_{l\ge 1}$ *obeying* $\operatorname{s-lim}_{l\to\infty} P_l = \operatorname{s-lim}_{l\to\infty} Q_l = I$ *we have*

$$\lim_{l\to\infty} \sup_{t\in\mathcal{X}} \|(I - P_l)X(t)\| = \lim_{l\to\infty} \sup_{t\in\mathcal{X}} \|X(t)(I - Q_l)\| = 0. \tag{2.8}$$

If $\operatorname{s-lim}_{n\to\infty} X_n(t) = X(t)$ *and* $\operatorname{s-lim}_{n\to\infty} X_n(t)^* = X(t)^*$ *uniformly in* $t \in \mathcal{X}$, *then* $\operatorname{s-lim}_{n\to\infty} X_n(t)^m = X(t)^m$, $\operatorname{s-lim}_{n\to\infty} (X_n(t)^*)^m = (X(t)^*)^m$, $m \in \mathbb{N}$, *and* $\operatorname{s-lim}_{n\to\infty} |X_n(t)|^\theta = |X(t)|^\theta$, $\operatorname{s-lim}_{n\to\infty} |X_n(t)^*|^\theta = |X(t)^*|^\theta$, $0 < \theta < +\infty$, *uniformly in* $t \in \mathcal{X}$.

Proof. The first two assertions can be proven by induction and the proof is following the line of reasoning of the previous Corollary 2.2. Similarly we show the validity of the last two assertions for $\theta = 2,4,6,\ldots$. To handle the case $0 < \theta < 2$ we exploit the representation

$$|X_n(t)|^{2\upsilon} = C_\upsilon \int_0^{+\infty} \frac{1}{\lambda^{1-\upsilon}} \frac{X_n(t)^*X_n(t)}{\lambda + X_n(t)^*X_n(t)} \, d\lambda . \tag{2.9}$$

$0 < \upsilon < 1$. We remark that for $\delta > 0$ we get the estimate

$$\left\| C_\upsilon \int_0^{\delta} \frac{1}{\lambda^{1-\upsilon}} \frac{X_n(t)^*X_n(t)}{\lambda + X_n(t)^*X_n(t)} \, d\lambda \right\| \le C_\upsilon \, \delta^\upsilon \tag{2.10}$$

which is uniform in $t \in \mathcal{X}$ and $n \in \mathbb{N}$. Further, the identity

$$\lambda \left\{ \frac{X_n(t)^*X_n(t)}{\lambda + X_n(t)^*X_n(t)} - \frac{X(t)^*X(t)}{\lambda + X(t)^*X(t)} \right\} =$$

$$= \frac{\lambda}{\lambda + X_n(t)^*X_n(t)} (X_n(t)^*X_n(t) - X(t)^*X(t)) -$$

$$\frac{\lambda}{\lambda + X_n(t)^*X_n(t)} (X_n(t)^*X_n(t) - X(t)^*X(t)) \, Q_l \, \frac{X(t)^*X(t)}{\lambda + X(t)^*X(t)} -$$

$$\frac{\lambda}{\lambda + X_n(t)^*X_n(t)} (X_n(t)^*X_n(t) - X(t)^*X(t))(I - Q_l) \, \frac{X(t)^*X(t)}{\lambda + X(t)^*X(t)}$$

and (2.8) imply the uniformity of

$$\text{s-}\lim_{n\to\infty} \lambda \left\{ \frac{X_n(t)^* X_n(t)}{\lambda + X_n(t)^* X_n(t)} - \frac{X(t)^* X(t)}{\lambda + X(t)^* X(t)} \right\} = 0 \qquad (2.11)$$

in $t \in \mathcal{X}$ and $\lambda \geq \delta$. Now the decomposition

$$|X_n(t)|^{2\upsilon} - |X(t)|^{2\upsilon} = C_\upsilon \int_0^\delta \frac{d\lambda}{\lambda^{1-\upsilon}} \left\{ \frac{X_n(t)^* X_n(t)}{\lambda + X_n(t)^* X_n(t)} - \frac{X(t)^* X(t)}{\lambda + X(t)^* X(t)} \right\}$$

$$+ C_\upsilon \int_\delta^{+\infty} \frac{d\lambda}{\lambda^{2-\upsilon}} \lambda \left\{ \frac{X_n(t)^* X_n(t)}{\lambda + X_n(t)^* X_n(t)} - \frac{X(t)^* X(t)}{\lambda + X(t)^* X(t)} \right\},$$

(2.10) and (2.11) immediately prove $\text{s-}\lim_{n\to\infty} |X_n(t)|^{2\upsilon} = |X(t)|^{2\upsilon}$ uniformly in $t \in \mathcal{X}$ for $0 < \upsilon < 1$. Combining the result for $\theta = 2,4,6,\ldots$ and $\theta = 2\upsilon$, $0 < \upsilon < 1$, and taking into consideration (2.8) we easily show that $\text{s-}\lim_{n\to\infty} |X_n(t)|^\theta = |X(t)|^\theta$ for $0 < \theta < +\infty$ uniformly in $t \in \mathcal{X}$ if one follows the proof line of Corollary 2.2. Similarly we prove the uniformity of $\text{s-}\lim_{n\to\infty} |X_n(t)^*|^\theta = |X(t)^*|^\theta$, $0 < \theta < +\infty$, in $t \in \mathcal{X}$.∎

Proposition 2.4. (uniform Grümm convergence theorem)

Let $\{X_n(\cdot)\}_{n\geq 1}$ *be a sequence of operator-valued functions defined on* $\mathcal{X}$ *with values in* $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < +\infty$, *such that* (2.2) *is satisfied. Let* $X(\cdot)$: $\mathcal{X} \longrightarrow \mathcal{C}_p(\mathcal{H})$ *be an operator-valued function such that for some sequences of finite dimensional projections* $\{P_l\}_{l\geq 1}$ *and* $\{Q_l\}_{l\geq 1}$ *obeying* $\text{s-}\lim_{l\to\infty} P_l = \lim_{l\to\infty} Q_l = I$ *the condition* (2.8) *holds and in addition*

$$\left.\begin{aligned} &\text{either } \lim_{l\to\infty} \sup_{t\in\mathcal{X}} \| (I - P_l) X(t) \|_p = 0 \\ &\text{or } \lim_{l\to\infty} \sup_{t\in\mathcal{X}} \| X(t)(I - Q_l) \|_p = 0 \end{aligned}\right\} \qquad (2.12)$$

is valid. If $\text{s-}\lim_{n\to\infty} X_n(t) = X(t)$, $\text{s-}\lim_{n\to\infty} X_n(t)^* = X(t)^*$ *and*

$$\lim_{n\to\infty} \| X_n(t) \|_p = \| X(t) \|_p \quad , \; p = 1,2,\ldots, \qquad (2.13)$$

uniformly in $t \in \mathcal{X}$, *then* $\|.\|_p\text{-}\lim_{n\to\infty} X_n(t) = X(t)$ *uniformly in* $t \in \mathcal{X}$ *for every* $p = 1,2,3,\ldots$.

Proof. Due to Lemma 2.1 it is clear that either $\{|X_n(\cdot)|\}_{n\geq 1}$ and $|X(\cdot)|$ or $\{|X_n(\cdot)^*|\}_{n\geq 1}$ and $|X(\cdot)^*|$ satisfy the assumptions of Proposition 2.4 too. Since the second case can be tried analogously to the first one we consider only this case. First of all let us show that for any fixed $l = 1,2,\ldots$ we have

$$\lim_{n\to\infty} \| (I-Q_l)|X_n(t)|(I-Q_l) \|_p = \| (I-Q_l)|X(t)|(I-Q_l) \|_p \qquad (2.14)$$

uniformly in $t \in \mathcal{X}$. By a simple calculation we prove that

$$\| (I - Q_l)|X_n(t)|(I - Q_l) \|_p^p - \| |X_n(t)| \|_p^p =$$

$$\text{Tr} \left\{ [(I - Q_l)|X_n(t)|(I - Q_l)]^p - |X_n(t)|^p \right\} =$$

$$\sum_{j=0}^{p-1} \text{Tr} \left\{ [(I - Q_l)|X_n(t)|(I - Q_l)]^j \, (-Q_l|X_n(t)| - |X_n(t)|Q_l + Q_l|X_n(t)|Q_l) \; |X_n(t)|^{p-1-j} \right\}.$$

On account of Proposition 2.1 and Lemma 2.3 we find that for any fixed $l = 1,2,\ldots$ the expression under the trace tends in the $\|.\|_1$-norm to $[(I-Q_l)|X(t)|(I-Q_l)]^j(-Q_l|X(t)|-|X(t)|Q_l+Q_l|X(t)|Q_l) \times |X(t)|^{p-1-j}$ as $n \to \infty$ uniformly in $t \in \mathcal{X}$. But this implies that

$$\lim_{n\to\infty} \left\{ \| (I - Q_l)|X_n(t)|(I - Q_l) \|_p^p - \| |X_n(t)| \|_p^p \right\} = \left\{ \| (I - Q_l)|X(t)|(I - Q_l) \|_p^p - \| |X(t)| \|_p^p \right\} \qquad (2.15)$$

uniformly in $t \in \mathcal{X}$. On account of (2.12) and (2.2) which imply (2.7) we obtain $\sup_{t\in\mathcal{X}} \| |X(t)| \|_p < +\infty$. Since (2.13) there is n_0 such that $\{\| |X_n(t)| \|_p\}_{n\geq 1, t\in\mathcal{X}}$ is uniformly bounded in $t \in \mathcal{X}$ and $n \geq n_0$.

Hence, the uniformity of (2.13) yields the uniformity of convergence $\text{s-}\lim_{n\to\infty} \| |X_n(t)| \|_p^p = \| |X(t)| \|_p^p$ in $t \in \mathcal{X}$. Thus, from (2.15) we conclude that

$$\lim_{n\to\infty} \|(I-Q_l)|X_n(t)|(I-Q_l)\|_p^p = \|(I-Q_l)|X(t)|(I-Q_l)\|_p^p \qquad (2.16)$$

uniformly in $t \in \mathcal{X}$ for every fixed $l = 1,2,\ldots$. Finally, using the inequality $|a - b| \le |a^p - b^p|^{1/p}$, $a \ge 0$, $b \ge 0$, $p \ge 1$, we obtain (2.14) from (2.16).

Further, a straightforward calculation gives the estimate

$$\begin{aligned}
&\|X_n(t) - X(t)\|_p \le \\
&\|(X_n(t) - X(t))Q_l\|_p + \||X_n(t)|(I - Q_l)\|_p + \||X(t)|(I - Q_l)\|_p \le \\
&\|(X_n(t)-X(t))Q_l\|_p + |\|Q_l|X_n(t)|(I-Q_l)\|_p - \|Q_l|X(t)|(I-Q_l)\|_p| + \\
&|\|(I-Q_l)|X_n(t)|(I-Q_l)\|_p - \|(I-Q_l)|X(t)|(I-Q_l)\|_p| + \\
&+ 2\||X(t)|(I - Q_l)\|_p .
\end{aligned}$$

By (2.12) there is a suitable integer l such that the last term is sufficiently small uniformly in $t \in \mathcal{X}$. Let us fix this l. Now the first term goes to zero uniformly in $t \in \mathcal{X}$ as $n \to +\infty$ by Proposition 2.1. On account of Proposition 2.1 and Lemma 2.3 we find that $\|.\|_p\text{-}\lim_{n\to\infty} Q_l|X_n(t)|(I - Q_l) = Q_l|X(t)|(I - Q_l)$ uniformly in $t \in \mathcal{X}$. Hence, the second term goes to zero uniformly in $t \in \mathcal{X}$ as $n \to +\infty$. The same property of the third term can be derived from (2.14).∎

Remark 2.5. The previous proposition was proved for $p = 1,2,\ldots$. But using (2.9) the proof can be extended to every $p \in [1,+\infty)$.

Proposition 2.4 admits a certain modification which we will need in the following.

Corollary 2.6. *Let* $(X_n(.))_{n\ge1}$ *and* $(Y_n(.))_{n\ge1}$ *be sequences of operator-valued functions defined on* $\mathcal{X}$ *with values in* $\mathcal{C}_p(\mathcal{H})$, $1 \le p < +\infty$, *such that (2.2) is satisfied. Let* $X(.): \mathcal{X} \to \mathcal{C}_p(\mathcal{H})$ *be an operator-valued function obeying (2.8) and (2.12). If* $\text{s-}\lim_{n\to\infty} X_n(t) = X(t)$, $\text{s-}\lim_{n\to\infty} X_n(t)^* = X(t)^*$, $\|.\|_p\text{-}\lim_{n\to\infty} Y_n(t) = X(t)$ *uniformly in* $t \in \mathcal{X}$ *and*

$$\|X_n(t)\|_p \le \|Y_n(t)\|_p \quad , n = 1,2,3,\ldots, t \in \mathcal{X}, \qquad (2.17)$$

then $\|.\|_p\text{-}\lim_{n\to\infty} X_n(t) = X(t)$ *uniformly in* $t \in \mathcal{X}$.

Proof. In order to apply Proposition 2.4 we have to establish (2.13). Obviously, we get that

$$\|X_n(t)Q_l\|_p \le \|X_n(t)\|_p \le \|Y_n(t)\|_p \quad ,$$

$l = 1,2,\ldots,$ $n = 1,2,\ldots,$ $t \in \mathcal{X}$. Hence, we derive the estimate

$$\begin{aligned}
&|\|X_n(t)\|_p - \|X(t)\|_p| \le |\|Y_n(t)\|_p - \|X(t)\|_p| + \qquad (2.18)\\
&|\|X_n(t)Q_l\|_p - \|X(t)Q_l\|_p| + |\|X(t)Q_l\|_p - \|X(t)\|_p| .
\end{aligned}$$

On account of (2.12) and the estimate

$$|\|X(t)Q_l\|_p - \|X(t)\|_p| \le \|X(t)(I - Q_l)\|_p$$

we find a suitable l such that the last term of (2.18) is sufficiently small uniformly in $t \in \mathcal{X}$. Fixing such an l and applying Proposition 2.1 we obtain that the second term goes to zero uniformly in $t \in \mathcal{X}$ as $n \to +\infty$. Since $Y_n(t) \xrightarrow[n\to\infty]{\|.\|_p} X(t)$ uniformly in $t \in \mathcal{X}$ the first one also converges to zero uniformly in $t \in \mathcal{X}$ as $n \to +\infty$. Thus, we have verified (2.13). Similarly, assuming the other alternative condition of (2.12) we verify (2.13) too. Now, using Proposition 2.4 we complete the proof.∎

Remark 2.7. In particular, Corollary 2.4 holds if $(X_n(.))_{n\geq 1}$, $(Y_n(.))_{n\geq 1}$ and $X(.)$ are independent of t. Since in this case the condition (2.2), (2.8) and (2.12) are automatically satisfied we can omit them.

Furthermore, we will apply a certain generalization of the Lebesgue dominated convergence theorem for $\mathcal{C}_p(\mathcal{H})$-ideals (cf. [13]).

Proposition 2.8. *Let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be sequences of non-negative self-adjoint operators strongly converging to X and Y, respectively. If $Y_n \in \mathcal{C}_p(\mathcal{H})$, n = 1,2,3,..., $Y \in \mathcal{C}_p(\mathcal{H})$, $1 \leq p < +\infty$, $Y_n \xrightarrow[n\to\infty]{\|.\|_p} Y$ and*

$$X_n \leq Y_n \quad , n = 1,2,3,..., \tag{2.19}$$

then $X_n \in \mathcal{C}_p(\mathcal{H})$, n = 1,2,3,..., $X \in \mathcal{C}_p(\mathcal{H})$, $1 \leq p < +\infty$, and $\|.\|_p\text{-}\lim_{n\to\infty} X_n = X$.

Proof. The first two assertions are obvious consequences of (2.19) and the assumed strong convergence. From (2.19) it follows that there are uniquely defined contractions Γ_n: $(\mathcal{R}(Y_n^{1/2}))^- \to \mathcal{H}$ and Γ: $(\mathcal{R}(Y^{1/2}))^- \to \mathcal{H}$ such that $X_n^{1/2} = \Gamma_n Y_n^{1/2}$ and $X^{1/2} = \Gamma Y^{1/2}$, see [17,Corollary 7-2]. Since $X_n^{1/2} \xrightarrow[n\to\infty]{s} X^{1/2}$ and $Y_n^{1/2} \xrightarrow[n\to\infty]{s} Y^{1/2}$ [5] we get

$$\text{s-}\lim_{n\to\infty} \Gamma_n P_n = \Gamma P, \tag{2.20}$$

where P_n and P denote the orthogonal projections of $\mathcal{H}$ onto the subspaces $(\mathcal{R}(Y_n^{1/2}))^-$ and $(\mathcal{R}(Y^{1/2}))^-$, respectively. Moreover, we have $Y_n^{1/2} \in \mathcal{C}_{2p}(\mathcal{H})$, $Y^{1/2} \in \mathcal{C}_{2p}(\mathcal{H})$ and $\lim_{n\to\infty} \|Y_n^{1/2}\|_{2p} = \|Y^{1/2}\|_{2p}$. Then by the Grümm convergence theorem [14]

$$\|.\|_{2p}\text{-}\lim_{n\to\infty} Y_n^{1/2} = Y^{1/2}. \tag{2.21}$$

Therefore, by Proposition 2.1, (2.20) and (2.21) we get that $X_n^{1/2} \xrightarrow[n\to\infty]{\|.\|_{2p}} X^{1/2}$ which implies $X_n \xrightarrow[n\to\infty]{\|.\|_p} X$. ■

Corollary 2.9. *Let $(X_n(.))_{n\geq 1}$, $X(.)$ and $Y(.)$ be operator-valued functions defined on $\mathcal{K}$ such that for any $t \in \mathcal{K}$ the conditions of Proposition 2.8 are satisfied. If in addition $\text{s-}\lim_{n\to\infty} X_n(t) = X(t)$, $\|.\|_p\text{-}\lim_{n\to\infty} Y_n(t) = Y(t)$ uniformly in $t \in \mathcal{K}$ and conditions (2.3) and (2.6) are satisfied, then $\|.\|_p\text{-}\lim_{n\to\infty} X_n(t) = X(t)$ uniformly in $t \in \mathcal{K}$.*

Proof. Obviously, we have the estimate

$$\|X_n(t) - X(t)\|_p \leq \|(X_n(t) - X(t))Q_1\|_p + \|X_n(t)(I - Q_1)\|_p + \|X(t)(I - Q_1)\|_p.$$

Applying the equality

$$\|Z\|_{2p} = \|Z^*Z\|_p^{1/2}, \; Z \in \mathcal{C}_{2p}(\mathcal{H}),$$

$1 \leq p < +\infty$, and the estimate [13]

$$\|X\|_p \leq \|Y\|_p \quad , \; 0 \leq X \leq Y, \; X,Y \in \mathcal{C}_p(\mathcal{H}),$$

$1 \leq p < +\infty$, we find

$$\|X_n(t) - X(t)\|_p \leq \|(X_n(t) - X(t))Q_1\|_p + \|Y_n(t)\|_p^{1/2} \times$$
$$\times \|Y_n(t)(I - Q_1)\|_p^{1/2} + \|Y(t)\|_p^{1/2}\|Y(t)(I - Q_1)\|_p^{1/2}.$$

Taking into account the inequality

$$\|Y_n(t)(I - Q_1)\|_p^{1/2} \leq \|Y_n(t) - Y(t)\|_p^{1/2} + \|Y(t)(I - Q_1)\|_p^{1/2}$$

one calculates

$$\|X_n(t) - X(t)\|_p \le \|(X_n(t) - X(t))Q_1\|_p + \|Y_n(t)\|_p^{1/2} \times$$

$$\times \|Y_n(t)-Y(t)\|_p^{1/2} + \left\{\|Y_n(t)\|_p^{1/2} + \|Y(t)\|_p^{1/2}\right\} \|Y(t)(I-Q_1)\|_p^{1/2}.$$

By virtue of (2.3) and (2.6) we get $\sup_{t\in\mathcal{K}} \|Y(t)\|_p < +\infty$. Since $Y_n(t) \xrightarrow[n\to\infty]{\|.\|_p} Y(t)$ uniformly in $t \in \mathcal{K}$ and $\sup_{t\in\mathcal{K}} \|Y(t)\|_p < +\infty$ there is an integer n_o such that $\{\|Y_n(t)\|_p\}_{n\ge 1, t\in\mathcal{K}}$ is uniformly bounded in $n \ge n_o$ and $t \in \mathcal{K}$. Using this statement and (2.6) we can choose a suitable integer l such that the third term is uniformly small in $n \ge n_o$ and $t \in \mathcal{K}$. For a such fixed l the first term goes to zero uniformly in $t \in \mathcal{K}$ as $n \to +\infty$ on account of the uniformity of the limit $X_n(t) \xrightarrow[n\to\infty]{s} X(t)$ and Corollary 2.2. The second term tends to zero uniformly in $t \in \mathcal{K}$ as $n \to +\infty$ by the uniformity of convergence $Y_n(t) \xrightarrow[n\to\infty]{\|.\|_p} Y(t)$ in $t \in \mathcal{K}$.∎

3. Product formula

3.1. Special case - Kato I

Let $A \ge 0$ and $B \ge 0$ be self-adjoint operators in a separable Hilbert space. Denoting by $Q = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2})$ we do not assume that Q is dense in $\mathcal{H}$. By $\mathcal{H}'$ we denote the closure of Q, i.e. $\mathcal{H}'=Q^-$. In general $\mathcal{H}'$ is a proper subspace of $\mathcal{H}$, i.e. $\mathcal{H} \ne \mathcal{H}'$. The orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}'$ is indicated by Π. We recall that C is the self-adjoint operator in $\mathcal{H}'$ associated with the non-negative closed quadratic form $f \to \| A^{1/2}f \|^2 + \| B^{1/2}f \|^2$, $f \in Q$, i.e. $C = A \dot{+} B$.

Further, we introduce a class of Borel functions f and g defined on $\mathbb{R}^1_+ = \{ t \in \mathbb{R}^1 : t \ge 0 \}$ characterized by

$$\left.\begin{array}{l} 0 \le f(t) \le 1,\ f(0) = 1,\ f'(0) = -1 \\ 0 \le g(t) \le 1,\ g(0) = 1,\ g'(0) = -1 \end{array}\right\} . \qquad (3.1\ )$$

Notice that $f(tA)^\alpha \xrightarrow{s} I$ and $g(tB)^\alpha \xrightarrow{s} I$ as $t \to +0$ for any $\alpha \ge 0$. In addition, throughout this section we assume that

$$0 < f(t),\ t \in \mathbb{R}^1_+, \qquad (3.2\ )$$

and that

$$\left.\begin{array}{l} \varphi(t) = \frac{1}{t}\left(\frac{1}{f(t)} - 1\right) \text{ and } \psi(t) = \frac{1}{t}(1 - g(t)) \\ \text{are monotonously nonincreasing functions.} \end{array}\right\} \qquad (3.3\ )$$

Condition (3.2) is necessary in order to give a correct statement of condition (3.3). The condition (3.3) itself has been firstly used by Kato in [6]. The conditions are satisfied for $f(t) = (1 + kt)^{-k}$, $0 < k \le 1$, and $g(t) = e^{-t}$, for example.

In accordance with Kato [6] we define the family $\{M(t)\}_{t>0}$,

$$M(t) = \frac{1}{t}\,[f(tA)^{-1} - g(tB)], \qquad (3.4\ )$$

of non-negative self-adjoint and in general unbounded operators. Since (3.1) and (3.2) the operators $M(t)$ are well-defined on $\mathcal{D}(M(t)) = \mathcal{R}(f(tA))$, $t > 0$.

Furthermore, we assume that

$$f(tA) \in \mathcal{C}_p(\mathcal{H}),\ t > 0,\ 1 \le p < +\infty. \qquad (3.5\ )$$

Lemma 3.1. *If the conditions* (3.1) - (3.3) *and* (3.5) *are satisfied, then*

$$(\lambda + M(t))^{-1} \in \mathcal{C}_p(\mathcal{H}) \qquad (3.6\ )$$

for $\lambda > 0$ *and* $t > 0$.

Proof. On account of the identity

$$(\lambda + M(t))^{-1} = \quad (3.7)$$
$$t\, f(tA)^{1/2}(I + f(tA)^{1/2}(\lambda t - g(tB))f(tA)^{1/2})^{-1}\, f(tA)^{1/2}$$

the result follows if one proves that the operator in the curved brackets is boundedly invertible for $\lambda > 0$ and $t > 0$. For $\lambda t \geq 1$ we get

$$I + f(tA)^{1/2}(\lambda t - g(tB))f(tA)^{1/2} \geq I.$$

For $0 < \lambda t < 1$ we get the inequality

$$I + f(tA)^{1/2}(\lambda t - g(tB))f(tA)^{1/2} \geq \lambda t I. \blacksquare$$

Owing to the condition (3.3) one concludes that the family $(M(t))_{t>0}$ is monotonously nondecreasing as $t \to +0$. Therefore, the resolvent family $((\lambda + M(t))^{-1})_{t>0}$ for $\lambda > 0$ is monotonously nonincreasing as $t \to +0$. Then, as it has been demonstrated by Kato [6] one has

$$\text{s-}\lim_{t\to+0} (\lambda + M(t))^{-1} = (\lambda + C)^{-1} \oplus 0 , \quad (3.8)$$

$\lambda > 0$, where 0 denotes the null operator on $\mathscr{H} \ominus \mathscr{H}'$.

Lemma 3.2. *If the conditions* (3.1) - (3.3) *and* (3.5) *are satisfied, then for* $\lambda > 0$ $(\lambda + C)^{-1} \in \mathscr{C}_p(\mathscr{H}')$ *and*

$$\|.\|_p\text{-}\lim_{t\to+0} (\lambda + M(t))^{-1} = (\lambda + C)^{-1} \oplus 0. \quad (3.9)$$

Proof. The monotonicity implies

$$(\lambda + C)^{-1} \oplus 0 \leq (\lambda + M(t))^{-1} \in \mathscr{C}_p(\mathscr{H}),\ t > 0,$$

which proves the first conjecture. Furthermore, using (3.8) we get the monotonously nonincreasing convergence of the eigenvalues $\mu_n((\lambda + M(t))^{-1})$ to $\mu_n((\lambda + C)^{-1} \oplus 0)$ as $t \to +0$ for every $n = 1,2,\ldots$. But this fact immediately yields (3.9).$\blacksquare$

Let us introduce the operator-valued functions $F(t) = g(tB)^{1/2}\, f(tA)\, g(tB)^{1/2}$ and $S(t) = \frac{1}{t}\,[I - F(t)] \geq 0$, $t > 0$. Due to (3.5) we have $F(t) \in \mathscr{C}_p(\mathscr{H})$.

Lemma 3.3. *If the conditions* (3.1) - (3.3) *and* (3.5) *are satisfied, then for* $\lambda > 0$ *we have*

$$\|.\|_p\text{-}\lim_{t\to+0} (\lambda + S(t))^{-1}F(t) = (\lambda + C)^{-1} \oplus 0. \quad (3.10)$$

Proof. Notice that $(\lambda + C)^{-1} \in \mathscr{C}_p(\mathscr{H}')$, $\lambda > 0$, by Lemma 3.2. Starting from the identity (3.7) one gets

$$g(tB)^{1/2}\, (\lambda + M(t))^{-1}\, g(tB)^{1/2} = (\lambda + S(t))^{-1}F(t) + \quad (3.11)$$
$$\lambda(\lambda + S(t))^{-1}\, g(tB)^{1/2}\, [I - f(tA)](\lambda + M(t))^{-1}\, g(tB)^{1/2}.$$

Taking into account (3.9) and Proposition 2.1 we find

$$\|.\|_p\text{-}\lim_{t\to+0} g(tB)^{1/2}(\lambda + M(t))^{-1}g(tB)^{1/2} = (\lambda + C)^{-1} \oplus 0. \quad (3.12)$$

Turning to the right-hand side and using the estimate

$$\|\lambda(\lambda + S(t))^{-1}\, g(tB)^{1/2}\, [I - f(tA)](\lambda + M(t))^{-1}\, g(tB^{1/2})\|_p \leq$$
$$\|(I - f(tA))(\lambda + M(t))^{-1}\|_p$$

we prove (3.10) using again Proposition 2.1.$\blacksquare$

Corollary 3.4. *Under the assumptions of* Lemma 3.3 *we have*

$$\|.\|_p\text{-}\lim_{t\to+0} (z - S(t))^{-1}F(t) = (z - C)^{-1} \oplus 0, \quad (3.13)$$

$z \in \mathbb{C}\backslash\mathbb{R}^1_+$.

Proof. From the resolvent identity for $\lambda > 0$ we get

$$R_z(S(t)) - \{R_z(C) \oplus 0\} = [\, I - (\lambda + z)R_z(S(t))] \times$$

$$\times\ [(\lambda + S(t))^{-1} - \{(\lambda + C)^{-1} \oplus 0\}][-I + (\lambda + z)\{R_z(C) \oplus 0\}],$$

$z \in \mathbb{C}\backslash\mathbb{R}^1_+$, where $R_z(.) = (zI - .)^{-1}$ denotes the resolvent. Multiplying from the left by $F(t)$ we obtain

$$R_z(S(t))F(t) - \{R_z(C) \oplus 0\} = \tag{3.15}$$
$$[\, I - (\lambda + z)R_z(S(t))]\ [(\lambda + S(t))^{-1}F(t) - \{(\lambda+C)^{-1} \oplus 0\}] \times$$
$$\times\ [-I + (\lambda + z)\{R_z(C) \oplus 0\}] + [\, I - (\lambda + z)R_z(S(t))] \times$$
$$\times\ (I - F(t))\{(\lambda + C)^{-1} \oplus 0\}[-I + (\lambda + z)\{R_z(C) \oplus 0\}] -$$
$$(I - F(t))\{R_z(C) \oplus 0\}.$$

The first term goes to zero as $t \to +0$ by (3.10). The second term tends to zero as $t \to +0$ by $\operatorname{s-lim}_{t\to+0} (I - F(t)) = 0$, $(\lambda + C)^{-1} \in \mathscr{C}_p(\mathscr{H}')$ and Proposition 2.1. To handle the third term we note the relation

$$R_z(C) = (\lambda + C)^{-1}\{-I_{\mathscr{H}'} + (\lambda + z)R_z(C)\} \in \mathscr{C}_p(\mathscr{H}'), \tag{3.16}$$

$z \in \mathbb{C}\backslash\mathbb{R}^1_+$. Hence, taking into account $\operatorname{s-lim}_{t\to+0} (I - F(t)) = 0$ and Proposition 2.1 we find that the third term converges to zero as $t \to +0$.∎

Corollary 3.5. *Under the assumptions of* Lemma 3.3 *we have*

$$\|.\|_p\text{-}\lim_{\tau\to+0} e^{-tS(\tau)}F(\tau) = e^{-tC} \oplus 0 \tag{3.17}$$

uniformly in t *on compact sets* $\mathscr{K} \subset (0,\infty)$.

Proof. Using the Dunford-Taylor formula one gets

$$e^{-tS(\tau)}F(\tau) - \{e^{-tC} \oplus 0\} =$$
$$\frac{1}{2\pi i}\int_\Gamma dz\ e^{-tz}\ (R_z(S(\tau))F(\tau) - \{R_z(C) \oplus 0\}) \tag{3.18}$$

where Γ is a positively oriented contour in the resolvent set $\mathbb{C}\backslash\mathbb{R}^1_+$ with $\mathbb{R}^1_+$ within Γ. For $t \in [a,b]$, $0 < a < b < +\infty$, we obtain

$$\|e^{-tS(\tau)}F(\tau) - \{e^{-tC} \oplus 0\}\|_p \le \tag{3.19}$$
$$\frac{1}{2\pi}\int_{\Gamma_-} |dz|\ e^{-bRe(z)}\ \|F(\tau)R_z(S(\tau)) - \{R_z(C) \oplus 0\}\|_p +$$
$$\frac{1}{2\pi}\int_{\Gamma_+} |dz|\ e^{-aRe(z)}\ \|F(\tau)R_z(S(\tau)) - \{R_z(C) \oplus 0\}\|_p\ ,$$

where $\Gamma_- = \{z \in \Gamma: Re(z) \le 0\}$ and $\Gamma_+ = \{z \in \Gamma: Re(z) > 0\}$. On account of (3.15) and (3.16) we get the estimate

$$\|R_z(S(\tau))F(\tau) - \{R_z(C) \oplus 0\}\|_p \le C(z,\tau,\lambda)\Big\{\|(\lambda + S(\tau))^{-1}F(\tau) -$$
$$\{(\lambda + C)^{-1} \oplus 0\}\|_p + 2\|(I - F(\tau))\{(\lambda + C)^{-1} \oplus 0\}\|_p\Big\},$$

where $C(z,\tau,\lambda) = \max\Big\{\|(\lambda + S(\tau))(z - S(\tau))^{-1}\|\ \|(\lambda + C)(z - C)^{-1} \oplus$ $\oplus (-I_{\mathscr{H}\ominus\mathscr{H}'})\|, \|(\lambda + C)(z - C)^{-1} \oplus (-I_{\mathscr{H}\ominus\mathscr{H}'})\|\Big\}$. A straightforward calculation shows that

$$0 < C(z,\tau,\lambda) \le 1 + \left(\frac{Re(z) + \lambda}{Im(z)}\right)^2,$$

for $z \in \mathbb{C}\backslash\mathbb{R}^1_+$, and

$$0 < C(z,\tau,\lambda) \le \left(2 + \frac{\lambda}{|Re(z)|}\right)^2,$$

for $z \in \{z \in \mathbb{C}\backslash\mathbb{R}^1_+: Re(z) < 0\}$. Hence, choosing a suitable contour Γ it is possible to guarantee that $C(z,\tau,\lambda)$ will be uniformly bounded in $\tau > 0$ and $z \in \Gamma$ for a fixed $\lambda > 0$. Therefore, $\|R_z(S(\tau))F(\tau) - \{R_z(C) \oplus 0\}\|_p$ is uniformly bounded in $\tau > 0$ and $z \in \Gamma$. Thus, using the Lebesgue dominated convergence theorem for $\mathscr{C}_p$- ideals (Proposition 2.8) and Corollary 3.4 we obtain (3.17) from (3.18).∎

Now we come to our main theorem:

Theorem 3.6. *Let A and B be two non-negative self-adjoint operators defined in the separable Hilbert space $\mathcal{H}$ and let f and g be two Borel functions obeying (3.1)-(3.3). If (3.5) is satisfied, then*

$$\|.\|_p\text{-}\lim_{n\to\infty} (f(\tfrac{t}{n} A)g(\tfrac{t}{n} B))^n = e^{-tC} \oplus 0, \tag{3.20}$$

where $C = A \dot{+} B$, uniformly in t on any compact set $\mathcal{K} \subset (0,\infty)$.

Proof. Notice that $e^{-tC} \oplus 0 \in \mathcal{C}_p(\mathcal{H})$ by Lemma 3.2. Taking into account the definitions $F(t) = (g(tB))^{1/2} f(tA) (g(tB))^{1/2}$ and $S(t) = \frac{1}{t}(I - F(t)) \geq 0$, $t > 0$, one has

$$0 \leq F(t) \leq (I + tS(t) + t^2 S(t)^2 + \ldots)^{-1} \leq e^{-tS(t)},$$

$t > 0$. Hence we get

$$0 \leq (F(\tfrac{t}{n}))^{n+1} \leq e^{-tS(t/n)} F(\tfrac{t}{n}), \quad t > 0. \tag{3.21}$$

By Corollary 3.5 we have

$$\|.\|_p\text{-}\lim_{n\to\infty} e^{-tS(t/n)} F(\tfrac{t}{n}) = e^{-tC} \oplus 0 \tag{3.22}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. On the other hand, by Kato [6] and by the uniformity of the convergences $f(\frac{t}{n} A) (g(\frac{t}{n} B))^{1/2} \xrightarrow[n\to\infty]{s} I$ and $(g(\frac{t}{n} B))^{1/2} \xrightarrow[n\to\infty]{s} I$ in t on bounded sets of $\mathbb{R}^1_+$ we get

$$s\text{-}\lim_{n\to\infty} [F(\tfrac{t}{n})]^{n+1} = e^{-tC} \oplus 0 \tag{3.23}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Then using (3.21) - (3.23) and taking into account Proposition 2.8 we obtain

$$\|.\|_p\text{-}\lim_{n\to\infty} [F(\tfrac{t}{n})]^{n+1} = e^{-tC} \oplus 0 \tag{3.24}$$

for $t > 0$. To prove the uniformity in t we remark that the continuity of the functions $f(.)$ and $g(.)$ at zero and their

boundedness imply

$$s\text{-}\lim_{n\to\infty} F(\tfrac{t}{n}) = I \tag{3.25}$$

uniformly in t on bounded sets of $\mathbb{R}^1_+$. Now a straightforward calculation shows that the uniformity of (3.23) and (3.25) yields the uniformity of

$$s\text{-}\lim_{n\to\infty} F(\tfrac{t}{n})^{n+1} = e^{-tC} \oplus 0 \tag{3.26}$$

in t on compact sets $\mathcal{K} \subset (0,\infty)$. Taking into account (3.21), (3.22), (3.26) and owing to Corollary 2.9 one gets the uniformity in t on compact sets $\mathcal{K} \subset (0,\infty)$ in (3.24). Moreover, by virtue of

$$(f(\tfrac{t}{n} A)g(\tfrac{t}{n} B))^{n+2} = f(\tfrac{t}{n} A)(g(\tfrac{t}{n} B))^{1/2} F(\tfrac{t}{n})^{n+1} g(\tfrac{t}{n} B)^{1/2}$$

and the uniformity of convergences $f(\frac{t}{n} A)(g(\frac{t}{n} B))^{1/2} \xrightarrow[n\to\infty]{s} I$ and $(g(\frac{t}{n} B))^{1/2} \xrightarrow[n\to\infty]{s} I$ in t on bounded sets of $\mathbb{R}^1_+$ we conclude from (3.24) and Corollary 2.2 that

$$\|.\|_p\text{-}\lim_{n\to\infty} (f(\tfrac{t}{n} A)g(\tfrac{t}{n} B))^{n+2} = e^{-tC} \oplus 0$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Due to the uniformity we obtain

$$\|.\|_p\text{-}\lim_{n\to\infty} (f(\frac{s_n(t)}{n} A)g(\frac{s_n(t)}{n} B))^{n+2} = e^{-tC} \oplus 0$$

$(s_n(t) = \frac{n}{n+2} t)_{n\geq 1}$, uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Hence, after substitution we get

$$\|.\|_p\text{-}\lim_{n\to\infty} (f(\tfrac{t}{n+2} A)g(\tfrac{t}{n+2} B))^{n+2} = e^{-tC} \oplus 0$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$.∎

3.2 General case - Kato II

The aim of the present section is to extend the results of the previous section, in particular, to drop the conditions (3.2) and (3.3). To this end we denote by f and g a pair of functions obeying only (3.1). We relate to the pair {f,g} two functions φ_o and ψ_o defined by

$$\left.\begin{array}{l} 0 \le \varphi_o(t) = \inf\limits_{0<s\le t} \varphi(s) = \inf\limits_{0<s\le t} s^{-1}\left(\frac{1}{f(s)} - 1\right) \le 1, \\ 0 \le \psi_o(t) = \inf\limits_{0<s\le t} \psi(s) = \inf\limits_{0<s\le t} s^{-1}(1 - g(s)) \le 1, \end{array}\right\}, \qquad (3.27)$$

where we agree to set $\frac{1}{f(s)} = +\infty$ if $f(s) = 0$. Defining $f_o(t)$ by

$$f_o(t) = \begin{cases} 1 & , t = 0 \\ (1 + t\varphi_o(t))^{-1}, & t > 0 \end{cases} \qquad (3.28)$$

we get $0 < f_o(t) \le 1$ (condition (3.2)) and

$$0 \le f(t) \le f_o(t) \le 1. \qquad (3.29)$$

Since $\lim\limits_{t\to+0} f(t) = 1$, we find $\lim\limits_{t\to+0} f_o(t) = 1$. Moreover, we have

$$1 = \lim_{t\to+0} \varphi(t) = \lim_{t\to+0} \inf \varphi(t) = \lim_{t\to+0} \varphi_o(t) = 1.$$

Thus, using $\lim\limits_{t\to+0} f_o(t) = 1$ we find

$$\lim_{t\to+0} \frac{f_o(t) - 1}{t} = -\lim_{t\to+0} \frac{\varphi_o(t)}{1 + t\varphi_o(t)} = -\lim_{t\to+0} \varphi_o(t) f_o(t) = -1.$$

Consequently, f_o obeys (3.1). Since $t^{-1}\left(\frac{1}{f_o(t)} - 1\right) = \varphi_o(t)$ and $\varphi_o(t)$ is monotonously nonincreasing, by construction the condition (3.3) is satisfied. Let

$$g_o(t) = \begin{cases} 1 & , t = 0 \\ 1 - t\psi_o(t), & t > 0. \end{cases} \qquad (3.30)$$

Since $\psi_o(t) \le \psi(t)$, $t > 0$, we get $0 \le g(t) = 1 - t\psi(t) \le 1 - t\psi_o(t) = g_o(t)$, $t > 0$. Furthermore, the representation

$$0 \le \psi_o(t) = t^{-1}(1 - g_o(t)) \qquad (3.31)$$

implies $0 \le 1 - g_o(t)$ or $g_o(t) \le 1$, $t > 0$. Thus, summing up we obtain $0 \le g(t) \le g_o(t) \le 1$. Taking into account

$$1 = \lim_{t\to+0} \psi(t) = \lim_{t\to+0} \inf \psi(t) = \lim_{t\to+0} \psi_o(t) = 1$$

we find $\lim\limits_{t\to+0} \frac{g_o(t) - 1}{t} = -\lim\limits_{t\to+0} \psi_o(t) = -1$. Hence, g_o obeys (3.1). Since $t^{-1}(1 - g_o(t)) = \psi_o(t)$ is monotonously nonincreasing by construction the condition (3.3) is valid.

Therefore, starting with the functions f and g obeying (3.1) we construct the associated pair $\{f_o, g_o\}$ to use the results of the previous section. To this end we replace the condition (3.5) by

$$f_o(tA) \in \mathscr{C}_p(\mathscr{H}),\ t > 0,\ 1 \le p < +\infty. \qquad (3.32)$$

For example, let $f(t) = e^{-t}$. It is easy to check that in this case $f_o(t) = (1 + t)^{-1}$. Hence, (3.32) means that $(I + tA)^{-1} \in \mathscr{C}_p(\mathscr{H})$, $t > 0$, $1 \le p < +\infty$.

Lemma 3.7. *Let* $F_o(t) = (g_o(tB))^{1/2} f_o(tA) (g_o(tB))^{1/2}$. *If the condition* (3.32) *is satisfied, then* $f(tA) \in \mathscr{C}_p(\mathscr{H})$, $t > 0$, $1 \le p < +\infty$, *and*

$$\| F(t)^m \|_p \le \| F_o(t)^m \|_p \quad , t > 0, \qquad (3.33)$$

for $m \ge 1$.

Proof. The first conjecture follows from (3.29) and (3.32). Now let $\hat{F}(t) = (g(tB))^{1/2} f_o(tA) (g(tB))^{1/2}$. Since $F(t) \le \hat{F}(t)$ we get [13]

$$\|F(t)^m\|_p^p = \mathrm{Tr}\{F(t)^{mp}\} \le \mathrm{Tr}\{\hat{F}(t)^{mp}\} = \|\hat{F}(t)^m\|_p^p \tag{3.34}$$

Let $\check{F}(t) = (f_o(tA))^{1/2}\, g(tB)\, f_o(tA)^{1/2}$ and $\check{F}_o(t) = (f_o(tA))^{1/2} \times g_o(tB)\, (f_o(tA))^{1/2}$. Obviously, one has

$$\|\hat{F}(t)^m\|_p^p = \mathrm{Tr}\{\hat{F}(t)^{mp}\} = \mathrm{Tr}\{\check{F}(t)^{mp}\} = \|\check{F}(t)^m\|_p^p \tag{3.35}$$

and, analogously,

$$\|\check{F}_o(t)^m\|_p^p = \|F_o(t)^m\|_p^p. \tag{3.36}$$

On account of (3.34), (3.35), (3.36) and $\check{F}(t) \le \check{F}_o(t)$ we find

$$\|F(t)^m\|_p \le \|\check{F}(t)^m\|_p \le \|\check{F}_o(t)^m\|_p = \|F_o(t)^m\|_p$$

which proves (3.33).∎

Theorem 3.8. *Let* A *and* B *be two non-negative self-adjoint operators defined in the separable Hilbert space* $\mathcal{H}$ *and let* f *and* g *be two Borel functions obeying* (3.1). *If* (3.32) *is satisfied, then* (3.20) *holds uniformly in* t *on any compact set* $\mathcal{K} \subset (0,\infty)$.

Proof. Firstly, we note that by Kato [7] and again by the uniformity of the limits $f(\frac{t}{n}A)\,(g(\frac{t}{n}B))^{1/2} \xrightarrow[n\to\infty]{s} I$ and $(g(\frac{t}{n}B))^{1/2} \xrightarrow[n\to\infty]{s} I$ in t we obtain

$$\operatorname*{s-lim}_{n\to\infty} F(\tfrac{t}{n})^{n+1} = e^{-tC} \oplus 0 \tag{3.37}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. In addition by Lemma 3.1 the condition

$$\|F(\tfrac{t}{n})^{n+1}\|_p \le \|F_o(\tfrac{t}{n})^{n+1}\|_p \quad , \ t > 0, \tag{3.38}$$

is satisfied. Moreover, from (3.24) we derive

$$\|.\|_p\text{-}\lim_{n\to\infty} F_o(\tfrac{t}{n})^{n+1} = e^{-tC} \oplus 0 \tag{3.39}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Consequently, by Corollary 2.6 we find

$$\|.\|_p\text{-}\lim_{n\to\infty} F(\tfrac{t}{n})^{n+1} = e^{-tC} \oplus 0 \tag{3.40}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. In order to derive (3.20) from (3.40) one has to follow the line of reasoning after (3.26).∎

Corollary 3.9. *If the conditions of* Theorem 3.8 *are satisfied, then the convergence in* (3.20) *really takes place in* $\|.\|_1$*-norm:*

$$\|.\|_1\text{-}\lim_{n\to\infty} (f(\tfrac{t}{n}A)g(\tfrac{t}{n}B))^n = e^{-tC} \oplus 0 \tag{3.41}$$

uniformly in t *on compact sets* $\mathcal{K} \subset (0,\infty)$.

Proof. Let us note that (3.20) can be rewritten as

$$\|.\|_p\text{-}\lim_{\substack{n\to\infty\\(n\ge p)}} (f(\tfrac{t}{pn}A)g(\tfrac{t}{pn}B))^n = e^{-(t/p)C} \oplus 0 \tag{3.42}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Then

$$\|(f(\tfrac{t}{pn}A)g(\tfrac{t}{pn}B))^{np} - (e^{-(t/p)C} \oplus 0)^p\|_1 \le$$

$$\sum_{l=0}^{p-1} \|(f(\tfrac{t}{pn}A)g(\tfrac{t}{pn}B))^n\|_p^l \|(f(\tfrac{t}{pn}A)g(\tfrac{t}{pn}B))^n - (e^{-(t/p)C} \oplus 0)\|_p \times$$

$$\times \|e^{-(t/p)C} \oplus 0\|_p^{p-l-1},$$

$t > 0$, $n \ge p$, and by (3.42) we get

$$\|.\|_1\text{-}\lim_{n\to\infty} (f(\tfrac{t}{pn}A)g(\tfrac{t}{pn}B))^{np} = e^{-tC} \oplus 0$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$. Since $(f(\frac{t}{pn}A)g(\frac{t}{pn}B))^m \xrightarrow[n\to\infty]{s} I$, $0 \le m \le p-1$, uniformly in t we find

$$\|.\|_1\text{-}\lim_{n\to\infty} (f(\tfrac{t}{p[n/p]}A)g(\tfrac{t}{p[n/p]}B))^n = e^{-tC} \oplus 0. \tag{3.43}$$

uniformly in t on compact sets $\mathcal{K} \subset (0,\infty)$ by Corollary 2.2 where [.]

denotes the entire part of a real number. Now, the uniformity of (3.43) in t admits to derive (3.41) from (3.43).■

4. Application and conclusion

As an application of the above results for the Gibbs semigroups we mention continuous systems of QSM in a finite volume.

Let us consider an N-particle system enclosed in a box $\Lambda \subset \mathbb{R}^\nu$ which is a bounded open connected subset of the ν-dimensional Euclidean space with a smooth boundary $\partial\Lambda$. Hence, the appropriate Hilbert space is $\mathscr{H} = L^2(\Lambda^N)$. In our discussion the statistics of particles is not important, therefore, we ignore the symmetry of the wave function $\psi \in \mathscr{H}$. The kinetic-energy operator T_σ for the particles of the mass m is a self-adjoint extension of the sum

$$T_N = \sum_{j=1}^{N} (-\frac{1}{2m} \Delta_j), \quad \Delta_j = \sum_{\alpha=1}^{\nu} \partial^2_{j,\alpha},$$

with domain $\mathcal{D}(T_N) = C_0^\infty(\Lambda^N)$. The domain $\mathcal{D}(T_\sigma)$ is specified by a boundary condition $\sigma \in C(\partial\Lambda)$. Then one can check that T_σ is a p-generator for the self-adjoint Gibbs semigroup $G_{T_\sigma}(t)$ [12], i.e., for $\frac{N\nu}{2} \le p < +\infty$ we have

$$R_z(T_\sigma) \in \mathscr{C}_p(\mathscr{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}^1_+. \qquad (4.1)$$

The stable particle interaction $U_N \ge -NuI$ is a self-adjoint multiplication operator with a real-valued measurable function defined on the domain $\mathcal{D}(U_N) = \{\psi \in \mathscr{H}: U_N\psi \in \mathscr{H}\}$, see e.g. [18].

Therefore, we have to generalize our results a bit to include the semiboundedness of the operator U_N. Moreover, to describe the short-distance behavior of the two-body interaction a hard-core potential is frequently used [18]. Then, the original Hilbert space of the wave functions $\mathscr{H} = L^2(\Lambda^N)$ should be reduced to $\mathscr{H}' = \Pi\mathscr{H} = L^2(\Lambda^N\backslash S_N)$ where S_N is a region forbidden by hard-cores.

Theorem 4.1. *Let $T \ge 0$ and $U \ge -uI$ be self-adjoint operators defined in a separable Hilbert space $\mathscr{H}$. If $Q = \mathcal{D}(T^{1/2}) \cap \mathcal{D}((U + uI)^{1/2})$, then*

$$\|\cdot\|_1\text{-}\lim_{n\to\infty} \{\exp(-\frac{\beta}{n}T)\exp(-\frac{\beta}{n}U)\}^n = e^{-\beta H}\Pi \qquad (4.2)$$

uniformly in the inverse temperature $\beta > 0$ varying in a compact interval bounded away from zero. Here $H = T \dotplus U$ and Π is the orthogonal projection of $\mathscr{H}$ (e.g. $L^2(\Lambda^N)$) onto $\mathscr{H}'$ (e.g. $L^2(\Lambda^N\backslash S^N)$) spanned by Q.

Proof. Let us introduce $A = T \ge 0$ and $B = U + uT \ge 0$. Then by (4.1) we get that $\exp(-\beta T) \in \mathscr{C}_1(\mathscr{H})$ for all $\beta > 0$ and

$$\|\cdot\|_1\text{-}\lim_{n\to\infty} (e^{-(\beta/n)T} e^{-(\beta/n)(U + uI)})^n = e^{-\beta C}\,\Pi \qquad (4.3)$$

uniformly in $\beta > 0$ on any compact interval bounded away from zero by Corollary 3.2. Here $C = H + uI$ and canceling both parts of (4.3) by $\exp(-\beta u)$ one gets (4.2).■

Remark 4.2. We have to remark that our results are not applicable to interactions U which are not semibounded from below (e.g. for Coulomb systems) in spite of the semiboundedness of the operator H.

The reason leading to Remark 4.2 is obvious. It is not so obvious for the unitary group and $\tau = s$, see [8]. We hope to return to this question elsewhere.

References

[1] Trotter, H.F.: On the products of semigroups of operators. Proc. Amer. Math. Soc. **10**, 545-551 (1959).

[2] Chernoff, P.R.: Product formulas, nonlinear semigroups, and addition of unbounded operators. Mem. Amer. Math. Soc. **140**, 1-21 (1974).

[3] Faris, W.G.: The product formula for semigroups defined by Friedrich's extension. Pacific J. Math. **22**, 47-70 (1967).

[4] Simon, B.: Quantum Mechanics for Hamiltonians defined as quadratic forms. Princeton N.J.: Princeton Univ. Press 1971.

[5] Kato, T.: Perturbation theory of linear operators. (2nd ed.) Berlin: Springer-Verlag 1976.

[6] Kato, T.: On the Trotter-Lie product formula. Proc. Japan Acad. **50**, 694-698 (1974).

[7] Kato, T.: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups. Topics in Funct. Anal., Ad. Math. Suppl. Studies, Vol. 3, 185-195 (I.Gohberg and M.Kac eds.). New York: Acad. Press 1978.

[8] Lapidus, M.L.: The problem of the Trotter-Lie formula for unitary groups of operators. Séminaire Choquet, Publ. Math. Univ. Pierre-et-Marie-Curie **46**, 1701-1745 (1982).

[9] Lapidus, M.L.: Product formula for imaginary resolvents with application to a modified Feynman integral. J. Funct. Anal. **63**, 261-275 (1985).

[10] Zagrebnov, V.A.: The Trotter-Lie product formula for Gibbs semigroups. J. Math. Phys. **29**, 888-891 (1988).

[11] Zagrebnov, V.A.: On the families of Gibbs semigroups. Commun. Math. Phys. **76**, 269- 275 (1980).

[12] Zagrebnov, V.A.: Perturbations of Gibbs semigroups. Commun. Math. Phys. **121**, 1-12 (1989).

[13] Simon, B.: Trace ideals and their applications. London Math. Soc. Lecture Notes Ser., Vol. 35. Cambridge: Cambridge Univ. Press 1979.

[14] Grümm, H.R.: Two theorems about $\mathfrak{C}_p$. Rep. Math. Phys. **4**, 211-215 (1973).

[15] Zagrebnov, V.A.: On singular potential interactions in quantum statistical mechanics. Trans. Moscow Math. Soc. **41**, 101-120 (1980).

[16] Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics, Vol. II. New York: Springer-Verlag 1981.

[17] Fuhrman, P.A.: Linear systems and operators in Hilbert space. New York: McGraw Hill Inter. Book Company 1981.

[18] Ruelle, D.: Statistical mechanics. Rigorous results. New York: Benjamin 1969.

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