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SECOND ORDER REDUCIBLE LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

One of the most effective methods for studying and integrating linear ordinary differential equations (LODE) is the method of factorization of differential operators. Although the method was known as early as in the last century, obstacle to its use in the theory and applications of LODE was the lack of existence theorems and constructive technique of factorization as well. The existence theorems were proved in [1], and a constructive factorization together with changes of the dependent and independent variable (nontrivial combination of both the approaches) was developed by one of the authors [2-4]. In the mentioned works the factorization is used in the generalized Liouvillian extension, allowing to find solution of a given equation in quadrature if possibly. On the base of the approach of [2-4] an algorithm was presented in [5] for obtaining exact solutions of some classes of ODE in quadrature and elementary functions.

The present-day computer algebra systems [6,7] are powerful means to implement exact methods of analysis and integration of differential equations in computers. It reveals the opportunity for the broad user public of practical application of methods until being accessible for the specialists only. Under the conditions the problem of development of constructive mathematical techniques and algorithms and creation of effective programs to implement the algorithms as well acquires a particular significance, thus, for instance, in [8-10] algorithms of constructing Liouvillian solutions for LODE with the rational and Liouvillian coefficients. One of those [8] has been realized in computer algebra systems [11,12].

Recently [13], a project of creation of an integrator of differential equations in **REDUCE** system was proposed relying on algorithmic developments of many authors (see the references in [13]).

In the present work the authors advance their own algorithm implemented in **REDUCE** system for both searching explicit transformation reducing an original second order LODE to that with the constant coefficients, and (if successfully) for finding its factorization and fundamental system of solutions.

2. Factorization

We consider second order linear ordinary differential equations (LODE) of the form

$$Ly \equiv y'' + a_1(x)y' + a_0(x)y=0, \ (')=d/dx, \ x\in I=(a,b), \tag{1}$$

the coefficients $a_1(x), a_0(x)$ of which make up a differential field K[I)] called the basic in the following.

The operator $L(D)$, $D=d/dx$ is called *decomposable* in **K** if it admits a representation as a factorization in terms of first order operators having the coefficients from **K** (generally, over the complex number field):

$$Ly \equiv (D - \alpha_2(x))(D - \alpha_1(x))y = 0. \tag{2}$$

Example 1. The equation

$$y'' - (a + 2x^{-2})y=0, \ a\in R,$$

admits the factorization in the rational function of x field:

$$\left(D - \frac{1}{x} + \frac{\sqrt{a}}{\sqrt{a}\,x - 1} + \sqrt{a}\right)\left(D + \frac{1}{x} - \frac{\sqrt{a}}{\sqrt{a}\,x - 1} - \sqrt{a}\right)y = 0$$

If an operator L is decomposable, its factorization is not the only possible. Thus, in the above example one may replace $\sqrt{a}$ by $-\sqrt{a}$.

For LODE (1) represented by (2) the differential analog of Viets formulas is valid:

$$\alpha_1 + \alpha_2 = - a_1, \quad \alpha_1\alpha_2 - \alpha_1' = a_0, \tag{3}$$

therefore

$$\alpha_1' + \alpha^2 + a_1(x)\alpha + a_0(x) = 0, \tag{4}$$

I) Actually, a smoothness of the coefficients is in order only, e.g., $a_k(x)\in C^k_I$, $k=0,1$.

i.e. α_1 satisfies the Riccati equation. It proves that a transcendental extension of the basic differential field is necessary to proceed factorization in the general case.

Example 2. The Airy equation

$$y'' + xy = 0$$

is indecomposable in its field **K** - the field of the rational functions of x.

Indeed, by (4) a factorization coefficient has to satisfy the Riccati equation here $\alpha_1' + \alpha_1^2 + x = 0$, which it is easy to show, does not admit a rational function as a solution[2)].

One of the most important and frequently used extensions of the basic differential field is the so-called Liouvillian generalized extension [14], which resulting from finite counter of extensions consisted either in joining integral or exponent of integral, or in finite algebraic extension of the field **K**.

By the Picard-Vessiot extension we shall mean the extension of the basic differential field generated by the linearly independent solutions of LODE. Equation (1) is said to be integrable in quadrature if its Picard-Vessiot extension belongs to the Liouvillian generalized extension. Then the factorization coefficients belong to that too.

However, generally neither the equation is solvable in quadrature, nor the factorization coefficients are expressed explicitly. But factorization of equation (1) always exists in itself according to the known Mammana theorem [1] and, moreover, can be expressed in the infinite number of ways.

Representation of LODE (1) in the form (2) is equivalent to the system

$$(D - \alpha_1(x))y = \tilde{y}, \quad (D - \alpha_2(x))\tilde{y} = 0, \qquad (5)$$

the solving of which leads to the fundamental system of solutions (f.s.s.) of LODE (1):

[2)]The fact corresponds to the absence of the Liouvillian solutions of the Airy equation (see [10, 14]).

$$y_1(x)=\exp(\int\alpha_1(x)dx),\ y_2(x)=y_1(x)\int\exp(\exp(\int(\alpha_2-\alpha_1)dx)dx, \qquad (6)$$

i.e. knowledge of a factorization (2) is equivalent to knowledge of f.s.s. of the corresponding LODE. The other way round, if y_1 is a solution of equation (1), then one can take $\alpha_1=y_1'/y_1$. Therefore, by (3), factorization (2) becomes

$$(\ D + a_1 .\ y_1'/y_1\ )(\ D - y_1'/y_1\ )y = 0, \qquad (7)$$

and in the general case, when f.s.s. y , y , of LODE (1) is known:

$$(\ D + a_1 + \frac{py_1' + qy_2'}{py_1' + qy_2'}\ )(\ D - \frac{py_1' + qy_2'}{py_1 + qy_2'}\ )y = 0,\ p,q = \text{const}, \qquad (8)$$

where p=1 or q=1.

3.The Kummer problem

We call so the problem of reducing LODE (1) to an equation of the following form

$$Mz \equiv \ddot{z} + b_1(t)\dot{z} + b_0(t)z = 0,\ (\dot{\ })=d/dt,\ t\in J=(c,d), \qquad (9)$$

$b_k\in C_J^k$, k=0,1, by means of the Kummer - Liouville transformation

$$y = v(x)z,\ dt = u(x)dx;\ u,v\in C_I^2,\ uv\neq 0,\ \forall x\in I. \qquad (10)$$

According to the Stäckel-Lie theorem (10) is the most general point in the local variable transformation which preserves the order and the structure of equation [15].

Purely theoretical interest apart, the Kummer problem - the problem of equivalence of second order LODE - is of great applied significance, for its constructive solution allows in many cases to reduce LODE studied to the equations whose solutions are known in either form.

The main results used later on are given by the following theorem.

Theorem [3,4]. For LODE (1) to be reduced to equation (9) by transformation (10), it is necessary and sufficient feasibility of one of equivalent conditions:

1) equation (1) admits factorization of the form

$$Ly \equiv \left(D - \frac{v'}{v} - \frac{u'}{u} - r_2(t(x))u \right)\left(D - \frac{v'}{v} - r_1(t(x))u\right)y = 0, \tag{11}$$

where $r_1(t)$, $r_2(t)$ are the factorization coefficients of LODE (9):

$$Mz \equiv (D_t - r_2(t))(D_t - r_1(t))z = 0,\ D_t = d/dt; \tag{12}$$

2) the transformation functions $u(x)$, $v(x)$ satisfy the equations

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + B_0(t(x))u^2 = A_0(x), \tag{13}$$

$$A_0 = a_0 - \frac{1}{2}a_1' - \frac{1}{4}a_1^2, \quad B_0 = b_0 - \frac{1}{2}\dot{b}_1 - \frac{1}{4}b_1^2,$$

$$v(x) = |u|^{-1/2}\exp\left(-\frac{1}{2}\int a_1(x)dx + \frac{1}{2}\int b_1(t)dt\right), \tag{14}$$

where A_0, B_0 are semiinvariants of LODE (1) and (9) (i.e. invariants with respect to transformation of the dependent variable only).

To solve (13), it is necessary to know the dependence $t(x)$ which, by (10), in turn is determined by the desired function $u(x)$, so instead of (13), one should actually consider the equation in $t(x)$:

$$\frac{1}{2}\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2 + B_0(t)t'^2 = A_0(x), \tag{15}$$

which will be called the Kummer-Schwarz equation together with (13). Note that the first pair of terms in (15) form the so-called Schwarz derivative.

The stated theorem is constructive: on the one hand, it defines the structure of LODE under consideration in terms of the transformation functions, and on the other - it gives explicit equations for them.

4. Reducibility

A special case of the Kummer problem is very important when (1) is reduced to an equation with constant coefficients, i.e., to (9) with

b_1, b_0 = const. Then the original LODE is called reducible, and its f.s.s. can be written as

$$y_k(x) = v(x)\exp(r_k\int u(x)dx,\ k=1,2,\ r_1\neq r_2,$$
$$y_1(x) = v(x)\exp(r\int u(x)dx),\ y_2(x) = y_1(x)\int u(x)dx,\ r_1=r_2=r, \tag{16}$$

where r_1, r_2 are the characteristic roots of (9) (see also (12)).

The second order LODE are always reducible [2,3]; the question is to find corresponding pair of the transformation functions u(x), v(x) (if only one of vast number of possible ones). As it is clear from the main theorem, the Kummer-Schwarz equation (13) plays a fundamental part here, which we rewrite in the form

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\delta u^2 = A_0(x),\ \delta=b_1^2 - 4b_0 = \text{const}. \tag{17}$$

Since we are free in choosing the coefficients b_1, b_0 of the reduced equation (9), one can consider the discriminant δ as an arbitrary constant and equation (17) - as the first integral of a third order ODE. Then, differentiating (17) with respect to x and substituting $R=u^{-1}$, we arrive at the resolvent equation [4]:

$$R''' + 4A_0(x)R' + 2A_0'(x)R = 0. \tag{18}$$

It is known (see, e.g.,[16]) that the general solution of third order LODE (the equation belongs to those too) can be represented as a quadratic form of f.s.s. of the corresponding second order LODE. In the case we have

$$R(x) = c_1Y_1^2 + c_3Y_1Y_2 + c_2Y_2^2\ ,\ c_1, c_2, c_3=\text{const}, \tag{19}$$

where $Y_1(x)$, $Y_2(x)$ is f.s.s. of LODE

$$Y'' + A_0(x)Y = 0. \tag{20}$$

The latter is obtained from (1) via the transformation

$$y=\exp(-\tfrac{1}{2}\int a_1(x)dx)Y,$$

so via (19) we have finally for $u(x)$:

$$u(x) = \exp(-\int a_1(x)dx)(c_1y_1^2 + c_3y_1y_2 + c_2y_2^2)^{-1}, \; c_3^2 - c_1c_2=\delta, \quad (21)$$

where $y_1(x)$, $y_2(x)$ is f.s.s. of LODE (1).Thus, the Kummer - Schwarz equation (17) has a nonlinear superposition law (21) with respect to solutions of reducible LODE.

According to Viet's formula (3), it follows from (11):

$$-2\frac{v'}{v} + b_1u - \frac{u'}{u} = a_1, \quad (22)$$

so that (11) can be represented as

$$Ly \equiv [\, D + \tfrac{1}{2}(a_1 - \tfrac{u'}{u} + \delta u)][\, D + \tfrac{1}{2}(a_1 + \tfrac{u'}{u} - \delta u)]y = 0. \quad (23)$$

By virtue of (21), the Liouvillian and Picard-Vessiot extensions of the basic differential field **K** are necessary for factorization (23). Formula (22) implies the interrelationship between the functions $u(x)$ and $v(x)$ (see also (14)):

$$v(x) = |u|^{-1/2}\exp(-\tfrac{1}{2}\int a_1(x)dx + \tfrac{1}{2}b_1\int u(x)dx). \quad (24)$$

Taking into account (16) and (24), f.s.s. of reducible LODE becomes

$$y_{1,2}(x) = |u|^{-1/2}\exp(-\tfrac{1}{2}\int a_1dx \pm \tfrac{1}{2}\delta\int udx), \; \delta \neq 0,$$

$$y_1(x) = |u|^{-1/2}\exp(-\tfrac{1}{2}\int a_1dx), \; y_2(x) = y_1(x)\int udx, \; \delta = 0. \quad (25)$$

Note that the function $v(x)$ satisfies LODE

$$v'' + a_1(x)v' + [a_0(x) - b_0u^2(x)]v = 0 \quad (26)$$

as well if $u(x)$ is considered to be a known function, or integro-differential equation (by (24))

$$v'' + a_1(x)v + a_0(x)v - b_0v^{-3}\exp(-2\int a_1dx) \times$$

$$\times [k + b_1\int v^{-2}\exp(-\int a_1dx)dx]^{-2} = 0, \quad (27)$$

where $k=1$ if $b_1=0$, or $k=0$ if $b_1\neq 0$.

For $a_1\equiv 0$, $b_1=0$ (27) converts to the Ermakov equation

$$v'' + a_0(x)v - b_0 v^{-3} = 0 \qquad (28)$$

studied in detail in [17]. The Kummer - Schwarz equation (17) is reduced to a similar equation, but with $A_0(x)$ in place of $a_0(x)$ and $\delta/4$ in place of $-b_0$, by means of the substitution $V=u^{-2}$. It is clear from above that equation (17) has a nonlinear superposition law with respect to f.s.s. of the corresponding reducible LODE of type of (1) as well as equations (26)-(28).

5. Explicit forms of transformations

As has been shown, the Kummer problem (and the reducibility problem in particular) amounts to the solvability problem of the Kummer-Schwarz (15) (or (17)), that in turn, is equivalent to solving LODE under consideration. Sure, one fails to do this for any, arbitrary given equation. However, it is possible to specify transformations sufficiently "powerful" to cover vast classes of LODE. In addition, the Kummer - Schwarz equation serves as a criterion of membership of a given LODE to one of these classes.

A typical example of the mentioned transformation is

$$y = |P|^{-1/4}\exp(-\tfrac{1}{2}\int a_1(x)dx + \tfrac{1}{2}b_1\int\sqrt{|P|}dx)z,\ dt = \sqrt{|P|}dx, \qquad (29)$$

$(P=P(x))$, with a function $P(x)$ being chosen. (Note that in a number of cases it is more convenient to use the modulus under the root in (29) to avoid complex valued $u(x)$, though it is not essential, as the form of factorization (11) implies that the transformation functions $u(x)$, $v(x)$ (10) are determined with a precision of constant multiplier (possibly, complex-valued). Its value affects the values of the coefficients b_1, b_0 only).

Transformation (29) includes a number of substitutions of the most common in the theory and applications of differential equations (see, e.g., [3]) as special cases. Thus, for $b_1=0$, $P(x)=a_0(x)$ we get the Liouvillian transformation

$$y = |a_0|^{-1/4}\exp(-\tfrac{1}{2}\int a_1 dx)z, \quad dt = \sqrt{a_0}\, dx \tag{30}$$

which is corresponded by the factorization of reducible LODE (see (23) for $r_1=r_2=r$)

$$Ly \equiv \left(D + \tfrac{1}{2}a_1 - \tfrac{1}{4}\frac{a_0'}{a_0} + r\sqrt{a_0} \right)\left(D + \tfrac{1}{2}a_1 + \tfrac{1}{4}\frac{a_0'}{a_0} - r\sqrt{a_0} \right)y = 0$$

Taking into consideration that (1) is easy reduced to the semi-canonical form (20), the transformation related to (30) is useful:

$$y = |A_0|^{-1/4}\exp(-\tfrac{1}{2}\int a_1 dx + \tfrac{1}{2}b_1\int\sqrt{|A_0|}dx)z, \quad dt=\sqrt{|A_0|}dx. \tag{31}$$

N.P.Erugin [18] applied the transformation

$$y = z, \quad dt = \sqrt{a_0}\, dx \tag{32}$$

which is confirmed by the factorization

$$Ly \equiv \left(D - \tfrac{1}{2}\frac{a_0'}{a_0} - r_2\sqrt{a_0} \right)\left(D - r_1\sqrt{a_0} \right)y = 0.$$

By way of the example, let us show how one can describe the class of LODE reducible by means of a given transformation of the form (32).

Any pair of the functions $u(x)$, $v(x)$ has to satisfy the system of two equations arised from factorization (11) of LODE (1) according to Viet's formulas (3): equation (22) and

$$\left(\frac{v'}{v} + r_2u + \frac{u'}{u} \right)\left(\frac{v'}{v} + r_1u \right) - \left(\frac{v'}{v} + r_1u \right)' = a_0. \tag{33}$$

The latter suggests immediately the form of the function $u(x)$ for transformation (32) with $v(x)\equiv1$: $r_1r_2u^2=b_0u^2=a_0$ whence, setting $b_0=1$, we come to $u(x)=\sqrt{a_0(x)}$. Equation (22) remains to be fulfilled. In the given case it is the reducibility condition:

$$b_1\sqrt{a_0} - \frac{1}{2}\frac{a_0'}{a_0} = a_1.$$

Thus, LODE reducible by means of (30) have to be of the form

$$y'' + \left(b_1\sqrt{a_0} - \frac{1}{2}\frac{a_0'}{a_0}\right)y' + a_0 y = 0$$

or in the equivalent one:

$$y'' + a_1 y' + [E(x)(c - b_1\int E(x)dx]^{-2}y = 0,\ E(x)=\exp(-\int a_1 dx),$$

where c and b_1 are arbitrary constants and a(x) or a(x) is an arbitrary function.

The setting of classes of LODE reducible by means of various transformations and compilation of an appropriate handbook would be very useful - in fact, it can easy exceed the volume of the Kamke's classical handbook [16] in range of scope. The way, however, has an essential and apparently ineradicable shortage: the worth of such surveys is directly proportional to their completenesses, i.e., volumes, but when they increase, working hours of using the reference books for studying particular equations increase more fast. An alternative consists in peculiar "contraction of information": using one or several transformations of type (29) which are high in their conveniences and generalize a set of the others, in the presence of a uniform test for success relative to any presented equation. The functions u(x) and v(x) can be formally found from the system (22) and (33) whose consequence is (17) if v(x) is expressed in terms of u(x) by means of (22) (see (24)), and (27) otherwise. But since the system (22) and (33) is not solvable in the general case, one is constrained to assign the form of one of the functions u(x) and v(x) determining then another to satisfy the system. In addition, the universal criterion of reaching the aim by means of the chosen transformation can be relation (17) or (27), and another function is obtained from (22). The Kummer-Schwarz equation should be favoured because it does not contain integrals. If u(x) is chosen correctly, there exists such a value of the constant δ (discriminant) that (17) is fulfilled identically with respect to x, i.e.,

$$\frac{1}{u^2}\left[2\frac{u''}{u} - 3\left(\frac{u'}{u}\right)^2 - 4A_0\right] \equiv \text{const} = \delta. \qquad (34)$$

Than the function v(x) is determined by (24), but it is not actually required because the factorization and f.s.s. of reducible

LODE can be obtained from (23) and (25) depending on $u(x)$ and δ only.

For transformation (29) formulas (34), (23) and (25) become respectively:

$$\frac{1}{P}\left[\frac{P''}{P}-\frac{5}{4}\left(\frac{P'}{P}\right)^2-4A_0\right]=\text{const}=\delta, \tag{35}$$

$$Ly\equiv\left[D+\frac{1}{2}\left(a_1-\frac{1}{2}\frac{P'}{P}+\sqrt{\delta P}\right)\right]\left[D+\frac{1}{2}\left(a_1+\frac{1}{2}\frac{P'}{P}-\sqrt{\delta P}\right)\right]y=0 \tag{36}$$

$$\begin{aligned} &y_{1,2}(x)=|P|^{-1/4}\exp\left(-\tfrac{1}{2}\int a_1dx\pm\tfrac{1}{2}\sqrt{\delta}\int\sqrt{P}dx\right),\ \delta\neq0,\\ &y_1(x)=|P|^{-1/4}\exp\left(-\tfrac{1}{2}\int a_1dx\right),\ y_2(x)=y_1(x)\int\sqrt{P}dx,\ \delta=0. \end{aligned} \tag{37}$$

Unfortunately, one fails to algorithmise completely the process of choosing the function $P(x)$ for the given coefficients $a_1(x)$ and $a_0(x)$ of the LODE under investigation. However, the authors have worked out a number of recommendations along these lines; having no opportunity to go through into details here, we quote only some instructive reasons.

Consider (assuming $u^2(x)=P(x)$ is known) associated LODE (26) for $v(x)$:

$$v''+a_1(x)v'+[a_0(x)-b_0u^2(x)]v=0. \tag{38}$$

A way for choosing $P(x)$ is based on the fact that, on the one hand, equation (38) could be solved, and on the other – one would recover a solution $v_0(x)$ coordinated with the chosen $P(x)$ in that relation (24) with $u(x)=\sqrt{P(x)}$ is valid. Then, the pair of the functions $v_0(x)$, $\sqrt{P(x)}$ determine the transformation (29) to be founded.

Let, for instance, the coefficient $a_0(x)$ be representable in the form $a_0(x)=p_0(x)+q_0(x)$ with such $q_0(x)$ that a solution of LODE

$$v''+a_1(x)v'+q_0(x)v=0 \tag{39}$$

is known. Equation (39) is obtained from (38) for $b_0=1$, $P(x)=p_0(x)$ i.e., the corresponding additive part of the coefficient $a_0(x)$ can be advised as $P(x)$. In particular, for $P(x)=a_0(x)$ (29) becomes the generalized Liouvillian transformation (see (30)):

$$y=|a_0|^{-1/4}\exp(-\tfrac{1}{2}\int a_1 dx + \tfrac{1}{2}b_1\int\sqrt{a_0}dx)z,\ dt=\sqrt{a_0}dx. \qquad (40)$$

Semi-canonical form (20) of LODE (1) is corresponded by the equation in $v(x)$

$$v'' + [A_0(x) - b_0P(x)]v = 0,$$

so an additive part of the semi-invariant $A_0(x)$ can be also chosen as $P(x)$. At $P(x)=A_0(x)$ we come to transformation (31).

Example 3 (see [16], 2.115b).

$$y'' + (k + \tfrac{2}{x})y' - (n^2e^{2x} - \tfrac{k}{x} + \tfrac{1-k^2}{4})y=0,\ k,n=\text{const}.$$

Taking $p_0(x)=-n^2e^{2x}$, (39) becomes

$$v'' + (k + \tfrac{2}{x})v' + (\tfrac{k}{x} + \tfrac{k^2-1}{4})v = 0 \qquad (41)$$

By means of the standard transformation of the dependent variable

$$v=\exp(-\tfrac{1}{2}\int a_1(x)dx)V, \qquad (42)$$

where in the case $a_1(x)=k + \frac{2}{x}$, LODE (41) is reduced to the semi-canonical form $V'' - 1/4\ V = 0$ with f.s.s. $V_{1,2}(x)= e^{\pm x/2}$. Hence, by (42), f.s.s. of LODE (41) is $v_{1,2}(x)=\exp(-\frac{1}{2}\int(a + \frac{2}{x})dx \pm \frac{x}{2})$, but only the function $v(x)=1/\sqrt{n}\ v_2(x)$ satisfies (24) with $u(x)=\sqrt{p_0(x)} = \iota ne^x$, $\iota=\sqrt{-1}$, $b_1=0$.

Thus, the desired transformation (29) takes the form

$$y = 1/(x\sqrt{n})e^{-(k+x)/2}z,\ dt=\iota ne^x dx$$

and reduces the initial LODE to $\ddot{z} + z=0$. Formulas (36) and (37) with $P(x)=p_0(x)=-n^2e^{2x}$, $\delta=\iota$ give the factorization and f.s.s.:

$$(D + \tfrac{k}{2} + \tfrac{1}{x} - \tfrac{1}{2}\ ne^x)(D + \tfrac{k}{2} + \tfrac{1}{x} + \tfrac{1}{2} + ne^x)y = 0,$$

$$y_{1,2}(x) = \tfrac{1}{x}\exp(-\tfrac{k+x}{2} \pm ne^x).$$

The above consideration concerning with LODE (38) admits a generalization. Let the coefficient $a_1(x)$ be representable as $a_1(x)=p_1(x) + q_1(x)$. Substituting

$$v = \exp(-\tfrac{1}{2}\int p_1(x)dx)V, \tag{43}$$

transform (38) to the form

$$V'' + q_1(x)V' + [a_0(x) - \tfrac{1}{2}a_1p_1 + \tfrac{1}{4}p_1^2 - \tfrac{1}{2}p_1' - b_0P(x)]V = 0. \tag{44}$$

The further reasonings are analogous to those stated above, i.e., it is advantageous to choose $P(x)$ as an additive part of the expression

$$Q(x) = a_0 - \tfrac{1}{2}a_1p_1 + \tfrac{1}{4}p_1^2 - \tfrac{1}{2}p_1'.$$

For instance, (44) is easy solved when $P(x)=Q(x)$. Hence, the familiar transformations are obtained as special cases: for $p_1=0$, $Q(x)=a_0(x)$ we come to (40) and for $p_1=a_1$, $Q(x)=A_0(x)$ – to (31).

Obviously, the use of the expression $Q(x)$ allows significantly higher scope for choosing the function $P(x)$ and expands the sphere of application of transformation (29), respectively.

Example 4 (see [16], 2.80).

$$y'' - (\tfrac{f'}{f} + 2k)y' + (k\tfrac{f'}{f} + k^2 - n^2f^2)y = 0, \ k,n=\text{const},$$

$f=f(x)$ is an arbitrary function. Assume $p_1=-2k$, $P(x)=Q(x)=-n^2f^2$. Then (44) becomes $V'' - \frac{f'}{f}V = 0$, one of solutions of which is V=const. Setting $V = 1/\sqrt{n}$, we find by (43) $v(x)=1/\sqrt{n}\, e^{K(x)}$ satisfying (24) at $b_1=0$, $u(x) = \sqrt{P(x)} = \imath nf(x)$. Consequently, by means of transformation $y=1/\sqrt{n}\, e^{kx}z$, $dt=\imath nf(x)dx$ the original equation is reduced to $\ddot{z}+z=0$ and has the factorization and f.s.s.

$$(D - \tfrac{f'}{f} - k + nf)(D - k - nf)y = 0,$$

$$y_{1,2}(x) = \exp(kx \pm \exp(n\int f(x)dx)).$$

The examined examples are illustrative; in practice there is no need at all times to consider equation (38) and to search for its solutions satisfying (24) for the chosen $P(x)$. It is more convenient to use just condition (35). Indeed, substitution of (24) to (38) leads to the Kummer – Schwarz equation (17) a corollary of which is (35) for transformation (29). Generally, employment of the universal criterion (34), that is independent on the way of choosing the function $u(x)$

allows to automate the process of searching an adequate transformation.

Example 5 (see [16], 2.81).

$$y'' + \left[\frac{ff'}{f^2+n^2} - \frac{f''}{f} \right]y' - \frac{k^2f'^2}{f^2+n^2}y = 0, \; f=f(x), \; k,n=\text{const}.$$

Putting $P(x)=a_0(x)=-\ k^2f'^2/(f^2+n^2)$ and substituting it into (35), we assure that the condition is fulfilled when $\delta=-4$. Then due to (36) and (37) the factorization and f.s.s. of the initial equation:

$$\left[D + \frac{ff'}{f^2+n^2} - \frac{f''}{f} - \frac{kf'}{\sqrt{f^2+n^2}} \right]\left[D + \frac{kf'}{\sqrt{f^2+n^2}} \right]y = 0,$$

$$y_{1,2}(x) = \left(f + \sqrt{f^2+n^2} \right)^{\pm k}.$$

6.Algorithm description

On the base of the above technique the authors have worked out an algorithm for searching transformation of type (29) which reduces the given LODE (1) to equation (9) with the constant coefficients b_1, b_0 and, if successfully, finding factorization and f.s.s. according to formulas (36) and (37). Below the algorithm is outlined. The notations accepted above hold true.

Problem: to find a function $P(x)$ satisfying condition (35) and thereby to determine transformation (29).

Algorithm:

Input: the coefficients $a_1(x), a_0(x)$ of LODE under consideration and the functions $p_1(x), p_0(x)$.

Output: the coefficients $\alpha_1(x), \alpha_2(x)$ of factorization (2), (36) and f.s.s $y_1(x), y_2(x)$.

Begin

$A_1 := \exp(-\frac{1}{2}\int a_1 dx)$;

if $a_0=0$ then begin $y_1:=1$; $y_2:=\int A_1^2 dx$; $\alpha_1:=0$; $\alpha_2:=a_1$; end;

else $A_0:=a_0-\frac{1}{2}a_1'-\frac{1}{4}a_1^2$;

if A_0=const then begin $\alpha_1:=-\frac{1}{2}a_1+\sqrt{A_0}$; $\alpha_2:=-\frac{1}{2}a_1-\sqrt{A_0}$;
    if A_0=0 then begin $y_1:=A_1$; $y_2:=xA_1$; end;
        else begin $y_1:=A_1\exp(\sqrt{A_0}x)$; $y_2:=A_1\exp(-\sqrt{A_0}x)$; end; end;
else begin $q_0:=a_0-p_0$; $q_1=:a_1-p_1$; $Q:=p_0+q_0-\frac{1}{4}p_1^2-\frac{1}{2}p_1q_1-\frac{1}{4}q_1^2-\frac{1}{2}p_1'-\frac{1}{2}q_1'$;
    generate $\{g_i \neq const, i=1 \div n \leqslant 128\}$-the array of all sorts combinations of terms from Q;
    for i:=1 to n do begin $P:=g_i$; $D:=\frac{1}{P}\left(\frac{P''}{P}-\frac{5}{4}\left(\frac{P'}{P}\right)^2-4A_0\right)$;
      if D=const then begin $\alpha_1:=-\frac{1}{2}\left(a_1+\frac{1}{2}\frac{P'}{P}-\sqrt{D*P}\right)$;
        $\alpha_2:=-\frac{1}{2}\left(a_1-\frac{1}{2}\frac{P'}{P}+\sqrt{D*P}\right)$; end;
        if D=0 then begin $y_1:=A_1|P|^{-1/4}$; $y_2:=y_1\int\sqrt{|P|}dx$; end;
        else begin
        $y_1:=A_1|P|^{-1/4}\exp(\frac{1}{2}\int\sqrt{D*P})$; $y_2:A_1|P|^{-1/4}\exp(-\frac{1}{2}\int\sqrt{D*P}dx)$;
        end; end;
      else return 'no transformation has been found' end;
end

Note that a version of the algorithm described above provides a way of verifying hypotheses of the form of $p_0(x)$ and $p_1(x)$ in the interractive regime. The above algorithm is implemented in the computer algebra system **REDUCE-3.3.**

The program has been tested successfully on examples from the chapter II of the handbook [16]. Our program allows to solve more than 70% of equations. In order to illustrate its effectiveness note, that examples 3 and 4 above take 21 and 22 seconds of ES-1061 running time and about 1000K memory.

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