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V.P.Gerdt, N.A.Kostov

COMPUTER ALGEBRA IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS OF HALPHEN TYPE

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1. INTRODUCTION.

We consider a linear differential equation in spectral parameter λ

$$L\Psi = \left(\frac{d^m}{dx^m} + \sum_{j=1}^{m-1} p_j(x)\frac{d^{m-j}}{dx^{m-j}}\right)\Psi = \lambda\Psi, \qquad (1)$$

where $p_j(x)$ are expressed in terms of elliptic functions. There are two classical problems [1]:

i) For which linear differential equation (1) there is a non-zero family of eigenfunctions $\Psi(x,\lambda,k,\alpha)$, depending smoothly on the eigenfunction parameter λ, such that Ψ is meromorphic function on the algebraic curve

$$C_g: \quad R(k,\alpha)=k^N+\sum_{j=1}^{N-1} k^{N-j}\, r_j(\alpha), \quad \lambda=\lambda(k,\alpha), \qquad (2)$$

where $r_j(\alpha)$ are meromorphic functions on the elliptic curve C_1: $(\wp'(\alpha),\wp(\alpha))$; $[\wp'(\alpha)]^2=4\wp(\alpha)^3-g_2\wp(\alpha)-g_3$; g_2,g_3-elliptic invariants, and $\wp$ is the Weierstrass $\wp$-function. We may view C_g as an N-fold covering of the elliptic curve C_1. Our conventions and notations concerning elliptic functions are those of Whittaker and Watson [2, Chap.XX]. This problem goes back to Halphen [1]. The solution of this problem was given in [1] only when $m=3,4$. The more general Halphen's problems of equivalence and classification of ordinary differential equations are recently solved by Bercovich [3] using the method of factorization of differential operators. These problems are closely related to the problem i). As an illustration we give the following example [1,4]. Let us consider the third order equation

$$\left(\frac{d^3}{dx^3} + 3q_2(x)\frac{d}{dx} + 3q_2'(x)\right)\Psi = \lambda\Psi. \qquad (3)$$

and introduce the so called first and second Halphen's absolute invariants $h=3q_2, I=3q_2'$. There is the following theorem:

<u>Theorem 1</u>. [1] The necessary and sufficient condition of integration

of equation (3) in terms of elliptic functions is the algebraic relation

$$h^3=(1-n^2)l^2/4 + \text{const.}, \quad n\text{-integer number}, \ n \not\equiv 0 \ (\text{mod } 3).$$

Then eq.(3) has the following canonical form (Halphen equation [5])

$$\left(\frac{d^3}{dx^3} + (1-n^2)\wp(x)\frac{d}{dx} + (1-n^2)\wp'(x)/2\right)\Psi = \lambda\Psi \qquad (4)$$

where $\wp(x)$ is the Weierstrass $\wp$-function.

Similar analysis is also possible when $m > 3$. Some particular results are known when $m=4,5$. Below we shall call this family of equations the Halphen type equations. There is another useful approach to generating the equations of Halphen type. Let us recall some results on the algebra of commuting differential operators (Burchnall-Chaundy theory) [6] and corresponding completely integrable systems (the so called Lax-Novikov equations [7-9]). We start with two linear differential operators

$$L_1 = \frac{d^k}{dx^k} + \sum_{i=1}^{k-1} u_i(x)\frac{d^{i-k}}{dx^{i-k}}, \quad L_2 = \frac{d^l}{dx^l} + \sum_{j=1}^{l-1} v_j(x)\frac{d^{l-j}}{dx^{l-j}}. \qquad (5)$$

Then we consider the following nonlinear system of differential equations in u_i, v_j

$$[L_1, L_2]=0, \qquad (6)$$

which is equivalent to a condition of integrability of the system

$$L_1\Psi = \lambda\Psi, \quad L_2\Psi = \mu\Psi. \qquad (7)$$

<u>Theorem 2</u>. (Burchnall-Chaundy) [6], see also [10].

The equation (6) is equivalent to algebraic relation of the following type

$$Q(L_1, L_2) = 0,$$

where Q is a polinomial, such that

1) The eigenfunction $\Psi(x,\lambda)$ is the meromorphic function on the algebraic curve $Q(\lambda,\mu)=0$.

2) The coefficients of $Q(\lambda,\mu)$ are the first integrals of eqs. (6) and are expressed as a differential polinomials of u_i, v_j.

3) When k,l are relatively prime, the space $\mathcal{L}_\lambda$ of Ψ is one dimensional. The system (6) is completely integrable and solutions u_i, v_j are expressed in terms of Riemann θ-function.

<u>Example 1.</u>

Let us consider eqs.(6) when $k=2, l=2k-1$

$$[L, L_i]=0,$$

where $L = -\frac{d^2}{dx^2} + u(x)$, L_i are operators ,which are computed using a relation found by Lax

$$\frac{\partial}{\partial t}L = [L_i, L] \qquad (8)$$

L and L_i are called the Lax pair. The general expression for the L_i is given in [11]

$$L_i = 1/2 \sum_{k=1}^{i} \left[\frac{\partial}{\partial u}H_{k-1}\frac{d}{dx} - 1/2\, X_{k-1}u\right](L)^{i-k} \qquad (9)$$

for example,

$$L_1 = 1/4\frac{d}{dx}, \quad L_2 = -1/4\frac{d^3}{dx^3} + 3u/8\frac{d}{dx} + 3u/16,$$

$$L_3 = 1/4\frac{d^5}{dx^5} - 5u/8\frac{d^3}{dx^3} - 15u'\frac{d^2}{dx^2} - 25u''/32u\frac{d}{dx} + 15uu'/32 - 15/64u'''$$

L_i is a differential operator of degree $2i-1$. For explicit expressions of H_k, X_k see [11]. Using Lame potential $u = i(i-1)\wp(x)$ and formula (9), we can obtain an useful example of Halphen type operators, for example,

$$L_2 = \frac{d^3}{dx^3} - 3\wp(x)\frac{d}{dx} - 3\wp'(x)/2$$
$$\qquad (10)$$
$$L_3 = \frac{d^5}{dx^5} - 15\wp\frac{d^3}{dx^3} - 5.3^2/2\wp'\frac{d^2}{dx^2} - 5^2.3/2\wp''\frac{d}{dx} - 5.3^2/2\wp^2\frac{d}{dx}$$
$$+ 5.3^2/2\wp\wp' - 5.3/2^3\wp'''.$$

The same technique can be applied to the next two examples.

<u>Example 2</u>. Let us consider the generalized Lame equation with potential $u = 2\sum_{i=2}^{i(i-1)/2}\wp(x-x_i)$, where x_i are some constants, fixed by the contion

$$\sum_{i=2}^{i(i-1)/2}\wp'(x_i - x_j)=0, \quad i \neq j$$

Equation of such a type was introduced by Dubrovin and Novikov [12]. By similar technique as in example 1 it is possible to construct new examples of Halphen type operators.

Example 3. Recently Treibich and Verdier [13] found new elliptic potentials u(x) of the following type

$$u(x) = i(i-1)\wp(x)+ 2\sum_{k=1}^{M} g_k(g_k+1)(\wp(x-\omega_k)-e_k), \quad (11)$$

where $0\le g_k\le i-1$, (for M, ω_k see [13]). Let us introduce some of them (i=3, [13])

$$u(x)=6\wp(x) + 2 [\wp(x-\omega_k)-e_k], \; k=1,2,3$$

$$u(x)=6\wp(x) + 2 [\wp(x-\omega_k)-e_k]+ \qquad (12)$$

$$2 [\wp(x-\omega_l)-e_l], \; l\neq k=1,2,3$$

The potentials (11) allows us to obtain new examples of Halphen type operators.

ii). The second problem is to construct the family of eigenfunctions $\Psi(x,\lambda)$. The general form of this function goes back to Hermite [14] (in the case n=2) and to Halphen [1] (in the case n=3). This function was improved by Krichever [15] in the theory of finite-gap integration method especially in the case of generalized Lame equation (see example 2). He also proved that this function satisfies the Baker-Akhiezer (BA) acsiomatics [16]. In the major part of this paper we describe an algorithm of construction of function $\Psi(x,\lambda)$, which we call Hermite-Halphen (HH) algorithm. The particular implementation of HH-algorithm on the computer algebra REDUCE is given. The mathematical background of this algorithm in more details is presented in [17].

In the paper [18] the following problem was studied:

iii) For which linear ordinary differential operators $L=\sum_{j=0}^{l} L_j(x)\frac{d^j}{dx^j}$ there is a non-zero family of eigenfunctions $\Psi(x,\lambda)$

depending smoothly on the eigenfunction parameter λ, which is also an eigenfunctions of a linear differential operator $A =\sum_{r=0}^{m} A_r(\lambda)\frac{d^r}{d\lambda^r}$

$$A\Psi(x,\lambda) = \Theta(x)\Psi(x,\lambda),$$

for an eigenvalue Θ which is function of x. The complete answer was given in the case of Schrodinger operator. Most of the computations in this paper have been carried out using computer algebra system VAXSYMA. The relation between the problems ii) and iii) is under the progress.

In the papers [19,20] the Lame equation was studied from the number theory point of view.

2. NOTATIONS

Let us introduce the functions

$$\Psi(x,\lambda)=\exp(kx)\{a_0(\lambda,k,\alpha)\Phi(x,\alpha)+ \sum_{j=1}^{N} a_j(\lambda,k,\alpha)\frac{d^j}{dx^j} \Phi(x,\alpha)\} \quad (13a)$$

$$\Psi(x,\lambda)=\exp(kx)\{ \sum_{i=2}^{i(i-1)/2} b_i \Phi(x-x_i,\alpha)\} \text{ (I.M. Krichever, [16])} \quad (13b)$$

$$\Psi(x,\lambda)=\exp(kx)\{ \sum_{k=1}^{M} g_k(g_k+1)[a_{0k}\Phi(x-\omega_k,\alpha)]+\sum_{l=1}^{g_k-1} a_{lk} \frac{d^l}{dx^l}\Phi(x-\omega_k,\alpha)$$

(V.Z. Enol'skii) (13c)

where

$$\Phi(x,\alpha)=\sigma(\alpha-x)/(\sigma(\alpha)\sigma(x)) \exp(\zeta(\alpha)x), \quad (14)$$

and σ,ζ are Weierstrass σ,ζ-functions [2].

Recall that the function (14) is a solution of Lame equation

$$(\frac{d^2}{dx^2} - n(n+1)\wp(x))\Psi = \lambda\Psi, \quad (15)$$

when n=2. It is easy to see that the following Laurent series expansion of $\Phi(x,\alpha)$ hold

$$\Phi(x,\alpha)=1/x + \sum_{j=1}^{\infty} \Phi_j x^j, \quad (16)$$

Inserting (16) into the (15,n=2), we have the following recurent formula

$$[j(j-1)-2]\Phi_j -2\wp_{j-1} -2 \sum_{\substack{n,k \\ (n+k=j-2)}} \wp_n \Phi_k = \wp(\alpha)\Phi_{j-2}, \quad j > 2, \quad (17)$$

where we use the well known expansion of $\wp$-function [2]

$$\wp(x) = -1/x^2 + \sum_{j=1} \wp_j x^j. \qquad (18)$$

Some first Φ_j are

$\Phi_1=-1/2\wp(\alpha)$, $\Phi_2=\wp'(\alpha)/6$, $\Phi_3=-\wp(\alpha)^2/8 + g\ /40$, $\Phi_4=\wp(\alpha)\wp(\alpha)''/60, \ldots$

3. DESCRIPTION OF THE ALGORITHM

HERMITE-HALPHEN

Input:

ordinary differential equation of Halphen type.

Output:

function (13a), $a_i=a_i(\lambda,k,\alpha)$,

N-fold covering on the torus C_1 (see (2)).

[1] Inserting (13a) into the ODE (1) and using the expansions (14),(16) generate the system of linear algebraic equations

$$G_m(a_i(\lambda,k,\alpha))=0, \qquad (19)$$

by equating the coefficients at the $1/x^g$ ($g \in N$).

[2] Solve the system (17) and write a_i in terms of k, λ, $\wp(\alpha)$, $\wp'(\alpha)$.

[3] By eliminating a_i in (19) find the following system of nonlinear algebraic equations

$$F_1(k,\lambda,\wp(\alpha),\wp'(\alpha))=0, \quad F_2(k,\lambda,\wp(\alpha),\wp'(\alpha))=0 \qquad (20)$$

where $[\wp'(\alpha)]^2=4\wp^3(\alpha)-g_2\wp(\alpha)-g_3$.

[4] Solve the nonlinear eqs.(20) with respect to k,λ

$k=k(\wp'(\alpha),\wp(\alpha))$, $\lambda=\lambda(\wp'(\alpha),\wp(\alpha))$,

using some appropriate technique, for instance, usual elimination method [21] or Buchberger's approach, based on construction of the Groebner basis [22].

Find the N-fold covering of the type (2).

We have implemented the HH-algorithm on the basis of the computer algebra system REDUCE 3.2 and the programm characteristics are the following:

- computer EC 1061 (IBM 370), operating system TKS,
- high speed storage required, depends on the problem, minimum 800K,
- number of lines 200.

In the last step 4 we use the method of elimination. We test our program with Lame equation $n=2\div9$, and Halphen equation ($n=4,5$). More general implementation is possible using function (13b) (see example 2) and also function (13c) (see example 3). An open problem is the generation of all equtions of Halphen type.

4. EXAMPLES.

Example 4. This example illustrates the basic steps in the realization of Hermite-Halphen algorithm (HH-algorithm) described above.

Let us consider the Lame equation (15,n=4), M=3.

[1] Inserting the function (3) into the Lame equation, n=4 we obtain

$[1/x^5]$	$3ka_3-a_2=0$,
$[1/x^4]$	$3(\lambda-k^2)a_3-6ka_2+7a_1=0$,
$[1/x^3]$	$(\lambda-k^2)a_2-2ka_1+9a_0=0$,
$[1/x^2]$	$3(5\wp^2+g_2)a_3-20/3\ \wp' a_2+(10\wp-k^2+\lambda)a_1-2ka_0=0$,
$[1/x\ ]$	$-8\wp\wp' a_3+5(3\wp^2-g_2)-20/3\ \wp' a_1+(10\wp+k^2-\lambda)a_0=0$.

[2] The solution of the system (17) is

$a_3=1,\ a_2=3k,\ a_1=3k^2-3/7\lambda,\ a_0=k(k^2-3/7\ \lambda)$

[3] After simple manipulations we obtain

$$F_1=35k^4-k^2(30\lambda+210\wp)+140\wp' k+3\lambda^2-105\wp^2 - 21g_2+30\wp\lambda=0,$$

$$F_2=(5\lambda-140\wp)k^3+210\wp' k^2+(-3z+45\wp\lambda+126g_2-420\wp^2)\lambda+70\wp\wp'-25\lambda\wp'=0.$$

[4] Using the method of elimination for the 10-fold covering of the elliptic curve C_1 we have

$$k^{10}-45\wp k^8+120\wp' k^7+(-630\wp^2+399/4\ g_2)k^6+504\wp\wp' k^5+$$
$$(-1050\wp^3+1725/4\ g_3+735/4\ \wp g_2)k^4+(360\wp^2\wp'-165\wp' g_2)k^3+$$
$$(-189/4g_2^2\ -315\wp^4+2205/4\ \wp^2 g_2-855/2\wp g_3)k^2+$$
$$(-163\wp\wp' g_2+125\wp' g_3+40\wp^3\wp')k+$$
$$-9\wp^5-75/4\wp g_2^2-75/4g_2g_3+9/4\wp^2 g_3+309/4\ \wp^3.$$

<u>Example 5</u>.

Let us consider the Halphen equation (4) when $n = 4$

$$\left(\frac{d^3}{dx} -15\wp(x)\frac{d}{dx} + 15/2\ \wp'(x)\right)\Psi = \lambda\Psi. \qquad (21)$$

Assume that Ψ has the following form

$$\Psi = \exp(kx)\left(a_0\Phi(x,\alpha)+ a_1 \frac{d}{dx} \Phi(x,\alpha) +a_2 \frac{d^2}{dx^2} \Phi(x,\alpha)\right) \qquad (22)$$

Inserting (22) in (21) we have

$[1/x^5]$ $\quad 2ka_2-a_1 = 0,$

$[1/x^4]$ $\quad k^2a_2-a_0 = 0,$

$[1/x^3]$ $\quad (2k^3+5\wp'-\lambda)a_2+(6k^2-15\wp/2)a_1-9ka_0=0,$

$[1/x^2]$ $\quad (-5k\wp+3/5\ g_2)a_2 + (-k^3+15/2\ k\wp +\lambda/2)a_1-3k^2a_0=0,$

$[1/x\ ]$ $\quad (45/4\ k\wp^2-3\wp\wp')a_2-(5k\wp'+45/8\ \wp^2)a_1+$

$\quad (k^3-5/2\ \wp'+15/2\ \wp k-\lambda/2)a_0.$

Step by step elimination of a_i gives

$a_0=k^2,\ a_1=2k,\ a_2=1,$ eq.$[1/x^3]\Rightarrow \lambda = 5(k^3-3k\wp+\wp')$, eq.$[1/x^2]\Rightarrow g_2=0$, eq.$[1/x]$ $\rightarrow$

$$k^5-25/2\ k^2\wp'+45/2\wp^2k-3\wp\wp'+15/2\wp k^3-\lambda/2\ k^2=0.$$

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Гердт В.П., Костов Н.А. E5-88-811
Компьютерная алгебра в теории обыкновенных дифференциальных уравнений типа Альфана

Предложен алгоритм построения решений линейных дифференциальных уравнений типа Альфана со спектральным параметром. Условие совместности уравнений данного типа определяет широкий класс нелинейных уравнений типа Лакса-Новикова. Предложенный алгоритм реализован на языке аналитических вычислений REDUCE.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Gerdt V.P., Kostov N.A. E5-88-811
Computer Algebra in the Theory of Ordinary Differential Equations of Halphen Type

We present an algorithm for solving linear differential equations in spectral parameter of the Halphen type. The integrability condition of the pair of equations of the Halphen type gives the large family of nonlinear differential equations of the Lax-Novikov type. This algorithm is implemented on the basis of the computer algebra system REDUCE.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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