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ON THE SOLUTION OF THE INVERSE SCATTERING PROBLEM ON THE HALFAXIS

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We shall consider the Schroedinger equation on the halfaxis

$$-y'' + V(x)\,y = k^2 y. \tag{1}$$

A potential $V(x)$ is supposed to be locally intagrable function decreasing quite rapidly such that the condition

$$\int^{\infty} x\,|V(x)|\,dx < \infty$$

is held. The solution $\phi(x,k)$ of Eq.(1) with the boundary conditions $\phi(0,k) = 0$, $\phi'(0,k) = 1$ has an asymtotics for large x and real k,

$$\phi(x,k) = \frac{A(k)}{k}\sin(kx - \eta(k)) + O(1).$$

The inverse scattering problem consists in the $V(x)$ potential determining via the given scattering $\eta(k)$ phase. Lots of investigations have been devoted to the ISP formulated in such a way. For instance, the works containing the main results and the history of the problem have to be noted /1,2/.

Our goal here is to obtain the ISP representation in a closed integral form, i.e., to give the expression for the $V(x)$ potential by way of scattering data.

In general case, the scattering data are not exhausted by the scattering phase and include the information on the bound states:

$$Sc(V) = \{\eta(k),\ 0 \le k < \infty,\ \kappa_m,\ c_m,\ m = 1,2,\dots,n\}. \tag{2}$$

Here κ_m define the position of the bound states on the imaginary axis $k_m = i\kappa_m$ and c_m are correspondingly normalizing constants

$$c_m^{-1} = \int_0^{\infty} \phi^2(x, i\kappa_m)\,dx.$$

Function $S(k) = \exp(-2i\eta(k))$ is called "S-function". In the framework of scattering data $S(k)$ may be used instead of $\eta(k)$.

Let us formulate the necessary and sufficient conditions of the ISP resolving in terms of the S-function[1].

Any $S(k)$ function for which the following conditions hold

1) $|S(k)| = S(\infty) = S(0) = 1$

2) $S(-k) = \overline{S(k)} = S^{-1}(k)$

3) $S(k) = 1 + \int_{-\infty}^{\infty} F(t)e^{ikt}\,dt, \quad \int_{-\infty}^{\infty} |F(t)|\,dt < \infty$

4) $\arg S(k) \int_{-\infty}^{\infty} = -4i\pi n,\ n \geq 0$

is the S-function of some operator of the type (1) that has a continuous spectrum on the halfaxis $(0,\infty)$ and n negative eigenvalues. To fulfil condition $\int^{\infty} x|V(x)|\,dx < \infty$ it is necessary and sufficient that $\int^{\infty} x|F'(x)|\,dx < \infty$. If $n > 0$, the $V(x)$ potential is not uniquely determined and there exists a set of the potentials depending on n-parameters. Conditions 1)-4) are supposed below to be held. From condition 4) it follows that the definition of the $\eta(k)$ function contains the information whether the bound states are available.

The case, when the bound states are lacking and the scattering data (2) only reduce to the $\eta(k)$ scattering phase, is the main one. The ISP generally formulated may be held to this case with the Cramm-Krein procedure. Thereby, it is sufficient to investigate the problem without bound states.

There are some approaches to solving the ISP by Marchenko[3,4], Gel'fand-Levitan[5,6], and Krein[7]. All of them are related to each other. In each of them the solution of the integral equation to be defined on the basis of the initial data of the problem is required. Here we shall proceed from the Krein's line of the approaches.

As the scattering data have been supposed to contain only the $\eta(k)$ phase, the $V(x)$ potential depends only on $\eta(k)$.

The Krein's method lies in the following steps. If for the phase $\eta(k) = (i/2)\ln S(k)$ then $\eta(0) = \eta(\infty) = 0$ and, thereby, there exists a function $\gamma(t)$ such that

$$\eta(k) = -\int_0^{\infty} \gamma(t)\sin kt\,dt, \quad \int_0^{\infty} |\gamma(t)|\,dt < \infty. \tag{3}$$

The kernel $H(t)$ is determined from the equation

$$1 + \int_{-\infty}^{\infty} H(t)e^{ikt}\,dt = \exp(-2\int_0^{\infty} \gamma(t)\cos kt\,dt). \tag{4}$$

The $H(t)$ function is determined via the inverse Fourier transform. The function is even and integrable all over the axis

$$\int_{-\infty}^{\infty} |H(t)|\,dt < \infty.$$

The Krein's equation has the form

$$\Gamma_{2x}(t) + \int_0^{2x} H(t-s)\Gamma_{2x}(s)\,ds = H(t), \quad 0 \leq t \leq 2x. \tag{5}$$

Here x is a parameter of the $(0,\infty)$ interval. The $V(k)$ potential is determined by the solution $\Gamma_{2x}(t)$ of Eq.(1)

$$V(x) = 2\frac{d}{dx}[\Gamma_{2x}(0) - \Gamma_{2x}(2x)]. \tag{6}$$

The solution $\phi(x,k)$ is also represented by way of $\Gamma_{2x}(t)$

$$\phi(x,k) = \frac{1}{k}\,\mathrm{Im}\,[e^{ikx}(1 - \int_0^{2x} \Gamma_{2x}(t)e^{-ikt}\,dt)].$$

As it follows from the expression the potential of Eq.(1) as well as the solution of Eq.(1) are found out irrespective of one another via Eq.(5). It shows that the Krein's equation is of great importance to the ISP theory.

Further investigation is concerned with the study of Eq.(5). A special feature of this type is that the kernel depends on the difference of the arguments. A number of works deals with the equations of this type. The Krein's paper[8] is of particular importance.

In the study we use the Fourier transform as the main one

$$\hat{F}(k) = \int_{-\infty}^{\infty} F(t)e^{ikt}\,dt.$$

Let R_+ be the ring of functions

$$\hat{F}(k) = c + \int_0^{\infty} F(t)e^{ikt}\,dt,$$

where $F \in L_1(0,\infty)$. If $\hat{F} \in R_+$ and $f(z)$ is an analytic function in the domain containing all the values of $\hat{F}(k)$, then $f(\hat{F}(k)) \in R_+$ namely,

$$f(\hat{F}(k)) = d \int_0^\infty G(t)\, e^{ikt}\, dt, \quad G \in L_1(0,\infty), \tag{7}$$

and d is a number. The similar proposition is true for the ring of functions R_-

$$\hat{G}(k) = c + \int_{-\infty}^0 G(t)\, e^{ikt}\, dt.$$

This proposition is of great significance and is called the Wiener-Levi Theorem [9].

Now let us adduce the functions factorization result on the axis [8].

Lemma. Let the $G(k) \neq 0$ function be represented as

$$G(k) = 1 + \hat{F}(k)$$

then

$$G(k) = G_+(k)\, G_-(k),$$

where $G_+(k) \in R_+$, $G_-(k) \in R_-$.

Proof follows from the chain of equalities based on the Wiener-Levi Theorem. Let $\ln z$ be the $f(z)$ function

$$G(k) = 1 + \hat{F}(k) = \exp\Big(\int_{-\infty}^{\infty} F(t) e^{ikt}\, dt\Big) =$$

$$= \exp\Big(\int_{-\infty}^{0} F(t)\, e^{ikt} dt\Big) \exp\Big(\int_{0}^{\infty} F(t)\, e^{ikt}\, dt\Big) =$$

$$= \Big(1 + \int_{-\infty}^{0} F(t)\, e^{ikt} dt\Big)\Big(1 + \int_{0}^{\infty} F(t)\, e^{ikt} dt\Big) = G_+(k)\, G_-(k).$$

Now introduce the integral operators P_+ and P_- in a following manner:

$$P_+\Big(c + \int_{-\infty}^{\infty} F(t)\, e^{ikt}\, dt\Big) = c + \int_0^{\infty} F(t)\, e^{ikt}\, dt,$$

$$P_-\Big(c + \int_{-\infty}^{\infty} F(t)\, e^{ikt}\, dt\Big) = c + \int_{-\infty}^{0} F(t)\, e^{ikt} dt.$$

The $P_+(P_-)$ operator is identical on the functions from $R_+(R_-)$.

Before to result the solution of Eq.(5) it should be noted that in the frames of ISP the inequality

$$1 + \hat{H}(k) > 0,$$

holds for all real k [7] . That means positiveness of the integral operator defined by the left side of Eq.(5) and tends to the unique solvability of the equation. The same inequality leads to solving Eq.(5) by means of factorization of the left side.

On the base of Lemma there exist expansions:

$$1 + \hat{H}(k) = g_{0+}(k)\, g_{0-}(k), \tag{8}$$

$$g_{0+}(k)\, e^{-2ikx} = g_{2x+}(k)\, g_{2x-}(k). \tag{9}$$

Here the functions g_{0+} and $g_{2x+} \in R_+$, and g_{0-}, $g_{2x-} \in R_-$.

Theorem. Solution of the Eq.(5) has the form

$$\hat{\Gamma}_{2x}(k) = g^{-1}_{2x}(k)\, P_-(e^{-2ikx} g^{-1}_{2x+}(k)\, P_+(g^{-1}_{0-}(k)\, \hat{H}(k))). \tag{10}$$

Proof. Let us introduce a function

$$B(t) = \begin{cases} \int_0^{2x} H(t-s)\, \Gamma_{2x}(s)\, ds & \text{for } t \notin [0, 2x], \\ 0 & \text{for } t \in [0, 2x]. \end{cases}$$

and let $\Gamma_{2x}(t) = H(t) = 0$ be for $t \notin [0, 2x]$. Then the Krein's equation can be written in the form

$$\Gamma_{2x}(t) + \int_{-\infty}^{\infty} H(t-s)\, \Gamma_{2x}(s)\, ds = H(t) + B(t), \quad -\infty < t < \infty.$$

Here the integral is a convolution of $H(t)$ and $\Gamma_{2x}(t)$. Now let us apply the Fourier transform to both the sides of the equality. Taking into consideration that the Fourier transform of convolution of two functions equals their Fourier transform product we obtain

$$\hat{\Gamma}_{2x}(k)(1 + \hat{H}(k)) = \hat{H}(k) + \hat{B}(k). \tag{11}$$

Now we eliminate the term depending on B . Substituting (7) in (11) gives

$$g_{0+}(k)\hat{\Gamma}_{2x}(k) = g_{0-}^{-1}(k)\hat{H}(k) + g_{0-}^{-1}(k)\hat{B}(k).$$

Then let us apply the P_+ operator to both the sides of the equality obtained

$$g_{0+}(k)\hat{\Gamma}_{2x}(k) = P_+(g_{0-}^{-1}(k)\hat{H}(k)) + P_+(g_{0-}^{-1}(k)\hat{B}(k)). \qquad (12)$$

Now substitute (9) into (12):

$$g_{2x+}(k)\hat{\Gamma}_{2x}(k) = e^{-2ikx}\ g_{2x+}(k)P_+(g_{0-}^{-1}(k)\hat{H}(k)) +$$

$$+ e^{-2ikx}\ g_{2x+}^{-1}P_+(g_{0-}^{-1}(k)\hat{H}(k)).$$

Now we apply the P_- operator to the equality obtained. The last term belongs to R_+ and disappears, and, thus, we obtain the equality

$$g_{2x-}(k)\hat{\Gamma}_{2x}(k) = P_-(e^{-2ikx}\ g_{2x+}^{-1}(k)P_+(g_{0-}^{-1}(k)\hat{H}(k))),$$

which is equivalint to (10). The available factor e^{-2ikx} here is associated with the necessity to fit the semiinfinite interval $(-\infty, 2x)$ to the P_- operator. Now the proof is complete.

By means of the theorem mentioned above one can give the explicit ISP solution in the case of lack of the bound states. It has to be noted that the sine and cosine Fourier transform,

$$\int_0^\infty \gamma(t)\sin ktdt \quad \text{and} \quad \int_0^\infty \gamma(t)\cos ktdt,$$

are harmonic conjugate functions on the real axis,i.e., the functions equal to the values of the real and imaginary parts of the analytic function on the upper halfplane. They are related to each other by means of the Hilbert integral

$$1 + \int_{-\infty}^{\infty} H(t)e^{ikt}\,dt = \exp\left(-\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\eta(k')}{k-k'}\,dk'\right). \qquad (13)$$

Insert (13) into (10)

$$\hat{\Gamma}_{2x}(k) = (g_{2x}^{-1}(k)P_-(e^{-2ikx}\ g_{2x+}^{-1}(k)P_+(g_{0-}^{-1}(k)(1-\exp(-\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\eta(k')}{k-k'}dk'))))).$$

The $V(x)$ potential is determined by means of differentiation (6).

The representation obtained may be used, for example, to study the problem qualitatively. For instance, when having information on the properties of $\eta(k)$ and the bound states as well one can analyse the behaviour of $V(k)$.

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