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дубна

E5-88-622

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NEW INTEGRABLE SYSTEMS OF INTERACTING NONLINEAR WAVES

Submitted to "Physics Letters A".

1988

In the physical science the following Hamiltonian system

$$i\frac{d}{dt} u_j + \gamma \, |u_j|^2 u_j + \sum_{k=1}^{n} M_{jk} u_k = 0 \ , \ j=1,\dots,n \qquad (1)$$

is now widely recognized as a generic model of self-trapping in condensed matter physics. In the case n=2 this system is completely integrable in the Liouville sense, due to the existence of the following conserved quantities $H=-\sum_{j=1} |u_j^4| - \sum_{j,k} M_{j,k}\, u_j u_k$ $N=\sum_{j=1} |u_j^2|$ which are in involution. In the case of $n>2$ there is conjecture that this system is not integrable.

The system (1) is a very special case of the following general vector equations

$$i\frac{d}{dt}u_j = \sum_{k} \gamma^{(1)}_{jk} \, u_k + \sum_{k,l} \gamma^{(2)}_{jkl} \, u_k u_l + \sum_{k,l,m} \gamma^{(3)}_{jklm} \, u_k u_l u_m + \dots \qquad (2)$$

which describe nonlinear interaction of n waves. These equations have a large number of applications in condensed matter physics, hydrodynamics, metereology, molecular dynamics, etc. The structure of constant tensors $\gamma^{(\alpha)}$, $\alpha=1,\dots,\theta$ is determined by the particular physics of the processes under investigation.

After the pioneer work [2] many works of physical and mathematical character [3-7] have been devoted to the problems of classification of symmetries, reductions, and applications of the system (2). As special cases of the integrable equations of type (2) we may identify almost all finite dimensional integrable dynamical systems known at the present time, for example, one dimensional reductions of generalized matrix nonlinear Schrodinger equations [5], Toda lattice dynamical systems [6], equations of motion of n-dimensional rigid body in the external gravitational and electromagnetic fields [6] and many others.

The aim of the present article is the construction and analysis of reductions leading to two new integrable dynamical systems of the special type which belong to the class (2).One of these in the two-dimensional case is isomorphic to the nonperiodical Toda lattice; the other in the case n= 2 is the so called dymer system equations [1] In the later case we obtain the general solutions using the finite-gap integration method. These solutions have been obtained in [1] by direct methods.

We start the analysis of the dymer equations (system (2) with n=2) with the Lax representation

$$\frac{d}{dt} L = [L,A], \qquad (3)$$

where the matrices L,A have the following form

$$L = \begin{bmatrix} 0 & \varepsilon & u_1 & 0 \\ \varepsilon\lambda & 0 & 0 & u_2 \\ iu_1^* & 0 & 0 & -\varepsilon/\lambda \\ 0 & iu_2^* & -\varepsilon & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -i|u_1|^2-\varepsilon^2\lambda & 0 & 0 & -\varepsilon u_1/\lambda-\varepsilon u_2 \\ 0 & -i|u_2|^2-\varepsilon^2\lambda & -\varepsilon u_1\lambda-\varepsilon u_2 & 0 \\ 0 & -i\varepsilon u_1^*-i\varepsilon u_2^*/\lambda & i|u_1|^2+\varepsilon^2/\lambda & 0 \\ -i\varepsilon\lambda u_2^*-i\varepsilon u_1^* & 0 & 0 & i|u_2|^2+\varepsilon^2/\lambda \end{bmatrix}$$

The central idea of the method of finite-gap integration [8] is the construction of the Baker-Akhiezer function (BA-function). By definition the BA-function is the solution of the following matrix linear equations:

$$L(\lambda)\Psi = \mu\Psi, \quad \frac{d}{dt}\Psi = A(\lambda)\Psi \ . \qquad (4)$$

Generally, BA-function is explicitly written in terms of Riemann's theta functions which are associated with the affine part of some algebraic curve. In our case this curve is hyperelliptic of the genus 3:

$$K: \quad \mu^2\varepsilon\,(\lambda + \lambda^{-1}) + P_2(\mu^2) = 0. \qquad (5)$$

It is easy to see that K is equivalent to

$$K_1: \ w^2 = (\xi^4-(u^2+u^{-2})\xi^2+1)(\xi^4-(v^2+v^{-2})\xi^2+1), \qquad (6)$$

where u,v are the rational functions of the first integrals of (2) when n=2, these integrals are the coefficients of the polynomial in (5). We point out that K is exactly the spectral curve of the two-particle Toda system associated with the Kac-Moody algebra $D_2^{(1)}$, see [11]. The spectral curve K_1 possesses the dyhedral group of automorphysms and due to this fact the 3-dimensional theta functions, which naturally exist in finite-gap method [10] reduce to one dimensional ones i.e. reduce to elliptic functions. The simplest way to demonstrate this reduction is the following: all Abelian integrals of the first type associated with the curve K_1 are easily related to $\int d\varkappa\, (\varkappa + \sqrt{\eta\nu})\, [\varkappa(\varkappa-1)(\varkappa-\eta)(\varkappa-\nu)(\varkappa-\eta\nu)]^{-1/2}$ which after the rank 2 Jacobi reduction goes into elliptic integrals. These results agree with the results of paper [1].

Owing to the conjecture of nonintegrability of (2) the following problem naturally arises: Are there integrable cases of dynamical systems of the type (1) ?
The structure of (1) gives us a possibility to find the Lax representation in the following form:

$$L = \begin{bmatrix} l_1 & q \\ r & -l_2 \end{bmatrix}, \quad A = \begin{bmatrix} \mu_1 & \chi_1 \\ -\chi_2 & \mu_2 \end{bmatrix} \qquad (7)$$

where $l_{1,2}, \mu_{1,2}, q, r, \chi_{1,2}$ are matrices of the type n x n . The Lax representation is equivalent to the following matrix equations:

$$\frac{d}{dt}l_1 = [l_1,\mu_1]-(q\chi_2+\chi_1 r), \quad \frac{d}{dt}l_2 = [l_2,\mu_2]-(r\chi_1+\chi_2 q), \qquad (8)$$

$$\frac{d}{dt}q = q\,\mu_2 - \mu_1 q + (l_1\chi_1 + \chi_1 l_2),$$
$$\frac{d}{dt}r = r\,\mu_1 - \mu_2 r + (l_2\chi_2 + \chi_2 l_1). \qquad (9)$$

Is it possible to specify the structure of $(l_\alpha, \mu_\alpha, \chi_\alpha)$ in such a way that (9) reduces to an equation of first order with cubic nonlinearities and (8) reduces to identity? The answer to this question is positive. The simplest construction arises after the following choice of the matrices entering in (8),(9):

$$\varkappa_1 = -l_1 q - q l_2, \qquad \varkappa_2 = l_2 r + r l_1,$$

$$\mu_1 = -qr - m_1, \qquad \mu_2 = rq - m_2.$$

where $[m_1, l_1] = [m_2, l_2] = 0$, l_1 and l_2 are constant matrices. The system of equations (9) for q and r in this case has the form

$$\frac{d}{dt}q = 2qrq - [(l_1^2+m_1)q + 2l_1 q l_2 + q(l_2^2+m_2)],$$
$$\frac{d}{dt}r = -2rqr + [(l_2^2+m_2)r + 2l_2 r l_1 + r(l_1^2+m_1)]. \qquad (10)$$

Using phase transformation it is possible to cancel in (10) terms of the following type $r(l_1^2+m_1)$, $q(l_2^2+m_2)$ and so we shall put in (10) $m_1=-l_1^2$, $m_2=-l_2^2$. We get:

$$\frac{d}{dt}q = 2qrq - 2\, l_1 q\, l_2, \qquad (11)$$

$$\frac{d}{dt}r = -\, 2rqr - 2\, l_2 r\, l_1,$$

where the matrices l_1 and l_2 are arbitrary. By similarity transformation it is possible to reduce one of them to the Jordan normal form. It is easy to see that it is impossible to put (10) in the form of the self-trapping equation. Indeed, the sum with cubic terms is absent only when the matrices q and r are diagonal, i.e.

$$q_{ij} = Q_i\, \delta_{ij}, \qquad r_{ij} = R_i\, \delta_{ij}, \qquad i,j=1,\dots,n \qquad (12)$$

From (10) we obtain, that the matrices $l_1 q l_2$, $l_2 r l_1$ must be diagonal too. Up to the phase and gauge ambiguities this condition is satisfied only for the two sets of matrices (l_1, l_2). Each choice corresponds to the irreducible system of equations, i.e. this system does't split to independent subsystems:

(A) $$(l_1)_{jk} = \sqrt{\varepsilon}\ \delta_{j,k+1}, \qquad (l_2)_{jk} = \sqrt{\varepsilon}\ \delta_{j,k+1} \qquad (13)$$

(B) $$(l_1)_{jk} = \sqrt{\varepsilon}\ (\delta_{j+1,k} + \lambda\ \delta_{j,n}\delta_{k,1}),$$

$$(l_2)_{jk} = \sqrt{\varepsilon}\ (\delta_{j,k+1} + \lambda^{-1}\ \delta_{j+1,k}\ \delta_{k,n}). \qquad (14)$$

The corresponding systems (10) may be written in the following form

$$\frac{d}{dt}Q_j = 2(Q_j^2 R_j - \varepsilon Q_{j+1}), \qquad \frac{d}{dt}R_j = -2(Q_j R_j^2 - \varepsilon R_{j-1}). \qquad (15)$$

In the case (A) we have $Q_{n+1}=0, R_0=0$ ("nonperiodic conditions"), and in the case (B) we impose the cyclic conditions $Q_{n+1}=Q_1$, $R_0 = R_n$. The Lax matrices in the case (B) contain an explicit dependence on a spectral parameter.

The simplest reduction of the system (15) is obtained by imposing the following cyclic conditions $R_j = i\, Q^*_{n+1-j}$, $\varepsilon = i\,\delta$, where δ is a real number.

$$\frac{d}{dt}Q_j = i\, (Q_j^2 Q^*_{n+1-j} - \delta\ Q_{j+1}) \qquad (16)$$

The system (16) has a different structure of nonlinear term in comparison with nonintegrable self-trapping equations. Its physical applications (if any) are unknown.
In the case n=2 another reduction of the system (15) is possible $R_1 = i\, Q_1^*$, $R_2 = i\, Q_2^*$. This reduction immediately gives the completely integrable system (16) in the so called "periodic" case. When $-\infty < j < \infty$ we may consider the system (15) as a differential-difference equations of the first order. We shall present the cnoidal-type solutions of this system elsewhere.

Now we want to investigate only the finite dimensional case. It is easy to see that the systems (15),(16) are Hamiltonian and the Hamiltonians have the form $H = 1/4\ \mathrm{tr}\ (L^4)$. The later integrals are the part of more general integrals of motion $I_k = (1/2k)\mathrm{tr}(L^{2k})$.

The leading terms of these integrals have the form $k^{-1}\sum_{j=1}^{n} Q_j^k R_j^k$ i.e. they are polynomials of k-th order of the variables $\{Q_j, R_j\}$ which proves that the integrals in the set $\{I_k\}$ are functionally independent. Using the standard method (see for example [9]) it is easy to prove that these integrals are in involution with each other. This completes the proof of the integrability of the system (15) in the Liouville sense.

For the dynamical systems of type (B) we may write the spectral curve $\det(L(\lambda) - \mu E) = 0$ in the more suitable form $|W_{ij}| = 0$, where we introduce the notation:

$$W_{ij} = (Q_i R_i + \varepsilon Q_i/Q_{i-1} - \mu^2)\delta_{i,j} - \delta_{i,j+1} Q_i/Q_{i-1} \mu\sqrt{\varepsilon} + \\ + \mu\delta_{i+1,j}\sqrt{\varepsilon} - \sqrt{\varepsilon}\,\mu/\lambda\ \delta_{i,1}\delta_{j,n} Q_1/Q_n + \sqrt{\varepsilon}\,\mu\lambda\ \delta_{i,n}\delta_{j,1}$$

It is easy to express the explicit dependence of the spectral curve on the spectral parameter λ

$$K : \mu^n \varepsilon^{n/2}(\lambda + \lambda^{-1}) + P_n(\mu^2) = 0$$

where P_n is the polynomial of degree n in μ^2. Its coefficients constitute the full set of involutive integrals of motion. The genus of the curve equals $2n-1$, the number of involutions on K are different for odd and even n

$$n = 2k, \quad T_1: \lambda \to -\lambda, \quad T_2: \mu \to -\mu$$
$$n = 2k-1, \quad T: \lambda \to -\lambda, \ \mu \to -\mu$$

When n=2 besides T_1, T_2, an additional involution exists (see the previous text).The system (15) is similar (in the sence of linearization of these systems on the hyperelliptic algebraic curves) to the Adler-van Moerbeke generalized Toda system [11] associated with the Kac-Moody algebra $D_n^{(1)}$. The linearization of these dynamical systems occurs on the Prym varieties of the Jacobeans ($\mathrm{Prym}(K_0) \subset \mathrm{Jac}(K)$). The structure of the $\mathrm{Prym}(K_0)$ will presumably allow us to express the explicit solutions of (15) in terms of the Prym theta functions.

An open problem is the relation of the systems (15), (16) with the periodic and nonperiodic Toda systems [11,12]. In the simplest case n=2 we are able to find the following isomorphism between the system of the type (A) and the usual nonperiodic Toda lattice with the canonical variables (p_1,x_1,p_2,x_2):

$$Q_1 = p_1 e^{x_1}/2, \quad Q_2 = e^{x_2}, \quad R_1 = e^{-x_1}, \quad R_2 = p_2 e^{-x_2}/2.$$

In our opinion, it is of great interest to find the Lie algebra interpretation of the Lax matrix representation in the case (A) and the Kac-Moody algebra interpretation in the case (B). The investigation of the quasiperiodic dynamics of the system (B), including the construction of the BA-function on the Prym varieties of the spectral curve will be given in a separate paper. We also expect that these dynamical systems are a subclass of a more complicated family of integrable equations of the nonlinear wave interaction type. Their classification is far from being complete.

Acknowledgements.

We are very much indebted to A.V. Mikhailov for many discussions and suggestions.

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Received by Publishing Department
on August 11, 1988.