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ASYMPTOTIC EXPANSIONS OF THREE-IDENTICAL-PARTICLE WAVE FUNCTIONS AT SMALL HYPERRADIUS AND S-WAVE POTENTIALS

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I. Introduction

Asymptotic expansion of a three-body wave function at small hyperradius was first oonsidered by Fock [1] for helium in the ^1S symmetry. A complete list of references to the investigations and generalizations of the Fock expansion of the Schrödinger equation solutions for the atomic systems is given in papers [2]. In those papers the Fock expansions of three-body atomic wave functions are investigated in detail.

In ref. [3] the Faddeev equations were derived in the three-dimensional configuration space and the generalization of the Fock expansion was begun to the case of arbitrary spherical-symmetric potentials and any particle-masses. In the framework of the Faddeev integrodifferential equations in the two-dimensional configuration space [4] this generalization has recently been performed in ref. [5], where the asymptotic expansions for the fundamental system of regular solutions to these equations were constructed at small hyperradius.

The present work is devoted to the analysis of asymptotic expansions for regular solutions to the integrodifferential equations and corresponding wave functions but in the case of identical particles and s-wave potentials represented at sufficiently small interparticle distances x as series

$$V(x) = \sum_{n=-1}^{\infty} v_n x^n \tag{1}$$

This analysis seems to be urgent now. Really, in the low-energy nuclear physics the two-body interactions are usually replaced by the S-wave ones, most of which having the asymptotics (I) as $x \to 0$. On the other hand, the asymptotic expansions of the wave functions is the first step on the way to the analytic solutions of the three-body problem, and the calculation of these functions with a given accuracy seems to be impossible without a detailed knowledge of all their asymptotics.

The contents of the present work is as follows: Section 2 contains the relevant basic facts and formulae of the special function theory and of the integrodifferential three-body equations, in Sec.3 the structure of asymptotic expansions of a fundamental system of regular solutions to the above equations is explored depending on the coefficients of series (I), i.e., the following three cases are considered:

A) $v_{-1} \neq 0$, B) $v_{-1} = 0,\ v_1 \neq 0$, C) $v_{2n-1} = 0,\ n = 0, 1, \ldots$

in Sec.4 the asymptotic expansions of the wave functions are investigated; the main results are summarised in Sec.5 and the relevant technical details are listed in the Appendix.

2. Preliminaries

To describe the positions of three identical particles in their c.m.s., we use the hyperradius r , its logarithm $s = \ln r$ and two different sets of hyperspherical angles Ω and Ω' along with two different sets $(\vec{x}, \vec{y})$ and $(\vec{x}', \vec{y}')$ of the usual reduced Jacobi vectors [4]. Our hyperspherical coordinates are defined by [6]

$$r = (x^2 + y^2)^{1/2}, \quad \Omega = (\varphi, \hat{x}, \hat{y}), \quad \Omega' = (\varphi', \hat{x}', \hat{y}'),$$

where

$$\tan\varphi = y/x, \quad \tan\varphi' = y'/x'$$

and $\hat{a}$ stands for two spherical angles of any vector $\vec{a}$.

The wave function of the three-particle state with total energy E and quantum number $\varepsilon = (\ell, \ell_3)$, where ℓ is the total angular momentum and ℓ_3 is its third component, reads in hyperspherical coordinates as

$$\Psi^{\varepsilon}(r,\Omega) = 2(r^2 \sin 2\varphi)^{-1} \langle \varphi | S^{\ell} | U^{\ell}(r,\varphi') \rangle \, \mathcal{Y}^{\varepsilon}(\hat{x},\hat{y}) . \qquad (2)$$

Here, the operator S^{ℓ} is the symmetrization operator [4] in the brackets of bispherical harmonics [7]

$$\mathcal{Y}^{\varepsilon}(\hat{x},\hat{y}) = Y_{00}(\hat{x})\, Y_{\ell\ell_3}(\hat{y}) . \qquad (3)$$

The operator S^{ℓ} acts on the variables φ and φ' as the sum of the identity operator and double geometric operator h^{ℓ} . The mapping of the Faddeev component U^{ℓ} by this operator may be written as the integral [8]

$$\langle \varphi | h^{\ell} | U^{\ell}(r,\varphi') \rangle = (2/\sqrt{3}) \int_{c(\varphi)}^{c_1(\varphi)} P_{\ell}(u)\, U^{\ell}(r,\varphi')\, d\varphi' , \qquad (4)$$

where P_{ℓ} is the Legendre polynomial [9] in the variable

$$u = \cos(\hat{y}\hat{y}') = (\cos 2\varphi + \cos 2\varphi' - 1/2)/2 \sin\varphi \sin\varphi'$$

and the integral limits are the break-lines

$$c_1(\varphi) = min(|\varphi - \pi/3|, 2\pi/3 - \varphi) .$$

The variables (s,φ) belonging to the band

$$B = \{(s,\varphi): -\infty < s < \infty,\ 0 < \varphi < \pi/2\}$$

are more convenient for our purposes; therefore, we rewrite the Faddeev equation [4] as follows:

$$
(\partial_S^2 - \Lambda_\varphi^\ell) U^\ell(s,\varphi) = \exp(2S)\{-E\, U^\ell(s,\varphi) + V(x) \langle \varphi | S^\ell | U^\ell(s,\varphi') \rangle \} , \tag{5}
$$

where

$$
x = \exp(S) \cos\varphi
$$

and the grand angular momentum operator in the brackets of the functions (3) is denoted by

$$
\Lambda_\varphi^\ell = -\partial_\varphi^2 + \ell(\ell+1)/(\sin\varphi)^2 .
$$

As is known, eq. (5) has a unique solution in the well-defined $\mathcal{R}$ -class of functions. This class is formed by functions belonging to the C_B^2 -class vanishing, owing to the regularity of the wave functions (2), at the triple collision point $S = -\infty$, on the rays $\varphi = 0, \pi/2$ and satisfying physical boundary conditions for $S \to \infty$.

In the next section we shall construct a fundamental system of regular solutions to eq. (5) as asymptotic series, each term belonging to a wide class $\mathcal{H} \supset \mathcal{R}$. The $\mathcal{H}$ -class is formed by the functions vanishing as $S \to -\infty$, on the rays $\varphi = 0, \pi/2$ and having second-order continuous derivatives with respect to both the variables S and φ . Our constructions are based on the choice of a convenient orthogonal angular basis and formulae (7–9) to be written below.

As basis functions we use the regular eigenfunctions of the operator Λ_φ^ℓ . These functions read [6]

$$
w_\kappa^\ell(\varphi) = N_\kappa^\ell (\sin\varphi)^{\ell+1} \cos\varphi\, P_n^{(\ell+1/2,\, 1/2)}(\cos 2\varphi) , \tag{6}
$$

where N_κ^ℓ is the normalization constant, $P_n^{(a,b)}$ is the Jacobi polynomial [9] , $\kappa = \ell + 2n$ and $n = 0, 1 \ldots$ There are two important facts following from the recurrence relations for the Jacobi polynomials [9], namely all the coefficients $a_{mp}^{\kappa\ell}$ of the equalities

$$
(\cos\varphi)^m\, w_\kappa^\ell(\varphi) = \sum_{p=\ell}^{\infty} a_{mp}^{\kappa\ell}\, w_p^\ell(\varphi) , \tag{7}
$$

where $m = -1, 0, \ldots,$ are nonzero if m is an odd number and vanish if m is an even number and $p > \kappa + m$ or $p < max(\ell, \kappa - m)$.

Any basis function (6) satisfies the equality [6]

$$
\Lambda_\varphi^\ell\, w_\kappa^\ell = (\kappa+2)^2\, w_\kappa^\ell \tag{8}
$$

as well as the equality [8]

$$
\langle \varphi | S^\ell | w_\kappa^\ell(\varphi') \rangle = S_\kappa^\ell\, w_\kappa^\ell(\varphi) , \tag{9}
$$

where $S_\kappa^\ell = 1 + 2 h_\kappa^\ell$ and h_κ^ℓ is the three-particle Raynal-Revai coefficient [10] , i.e. the matrix element of unitary transformation between polyspherical hyperharmonics [6]

$$
Y_\kappa^\varepsilon(\Omega) = 2\, cosec\, 2\varphi\;\, w_\kappa^\ell(\varphi)\; \mathcal{Y}^\varepsilon(\hat{x}, \hat{y}) \tag{10}
$$

written in the coordinates Ω and Ω'.

The formulae for calculating the normalization constants of the functions (6), coefficients of the series (7) and the coefficients h_κ^ℓ are given in the Appendix.

3. Fundamental system of solutions of the integrodifferential equation

In the $\mathcal{H}$ class all the solutions $U^{\kappa\ell}$ of eq.(5) with the potential (1) are identified by the asymptotically ($S \to -\infty$) conserved quantum number $\kappa = \ell, \ell+2, \ldots$ or by the leading terms of their asymptotics

$$U^{\kappa\ell} \to \exp\{(\kappa+2)s\}\, w^{\ell}_{\kappa}(\varphi) \;, \quad s \to -\infty \;. \qquad (11)$$

These terms are the solutions of the characteristic equation [11] for the operator of the left-hand side of eq. (5). Using the main idea of ref. [12] we look for every κ-solution, i.e. the solution of eq. (5) with asymptotics (11) as the generalized power series expansion

$$U^{\kappa\ell} = \sum_{n=0}^{\infty} r^{\kappa+n+2}\, U^{\kappa\ell}_{n}(s,\varphi) \;, \qquad (12)$$

where, according to eq.(11), $U^{\kappa\ell}_{0} = w^{\ell}_{\kappa}$ and the remaining unknown functions are supposed to be linearly independent of hyperradius powers. The substitution of the expansions (1) and (12) into (5) gives the following recurrent equations

$$D^{\kappa\ell}_{n}\, U^{\kappa\ell}_{n}(s,\varphi) = -E\, U^{\kappa\ell}_{n-2}(s,\varphi) + \qquad (13)$$

$$\sum_{p=0}^{n-1} v_{n-p-2}\, (\cos\varphi)^{n-p-2} \,\langle \varphi | S^{\ell} | U^{\kappa\ell}_{p}(s,\varphi')\rangle \;,$$

where n=0,1,...; according to the ansatz (12), $U^{\kappa\ell}_{n} = 0$ if $n < 0$ and

$$D^{\kappa\ell}_{n} = (\partial_s + \kappa + n + 2)^2 - \Lambda^{\ell}_{\varphi} \;. \qquad (14)$$

Equation (13) for the function $U^{\kappa\ell}_{n}$ will be called the n-equation. Its right-hand side $R^{\kappa\ell}_{n}$ contains only solutions of n'-equations with $n' < n$. Any solution of the n-equation may be represented as a sum [11] of its particular solution in the $\mathcal{H}$-class and a general solution of the corresponding

homogeneous equation in the same class. The latter solution is a linear combination of the functions

$$\exp(ps)\, w^{\ell}_{\kappa'}(\varphi) \;, \quad p = \kappa' - \kappa - n \geqslant 0 \;,$$

each being inserted into eq.(12) generates a leading term of the asymptotics (11) of the other κ'-solution. Therefore, without loss of generality, we investigate only a particular solution of eq.(13) belonging to the $\mathcal{H}$-class. Let us prove that these solutions are polynomials in the variable

$$U^{\kappa\ell}_{n} = \sum_{m=0}^{\bar{m}(n)} s^{m}\, U^{\kappa\ell}_{nm}(\varphi) \qquad (15)$$

of finite degree depending on the coefficients of series (1)

$$\bar{m}(n) \leqslant \begin{cases} [n/2] & \text{A} \qquad (16a) \\ [n/6] - \theta(\lfloor (n-1)/6 \rfloor - (n-1)/6) & \text{B} \qquad (16b) \\ 0 & \text{C} \;, \qquad (16c) \end{cases}$$

where [t] is an integer part of number t and $\theta(t)$ is the step function equal to unity at $t \geqslant 0$ and vanishing in the opposite case. We shall also find some of the most slowly vanishing terms of the series (12) as $s \to -\infty$ and obtain the simple set of ordinary second-order differential equations

$$[(\kappa+n+2)^2 - \Lambda^{\ell}_{\varphi}]\, U^{\kappa\ell}_{nm}(\varphi) - 2(\kappa+n+2)(m+1)\, U^{\kappa\ell}_{n,m+1}(\varphi)$$

$$-(m+1)(m+2)\, U^{\kappa\ell}_{n,m+2}(\varphi) = -E\, U^{\kappa\ell}_{n-2,m}(\varphi) + \qquad (17)$$

$$\sum_{p=0}^{n-1} v_{n-p-2}\, (\cos\varphi)^{n-p-2} \,\langle \varphi | S^{\ell} | U^{\kappa\ell}_{pm}(\varphi')\rangle \;,$$

where n=0,1... ; m = 0, ..., m(n), and according to eqs.(15), (16), $U^{\kappa\ell}_{nm} = 0$ providing $n,m < 0$ or $m > \bar{m}(n)$.

Further, the superscript κ will be fixed and ℓ – omitted where it is possible. Suppose that the right-hand side of the n-equation is a finite degree polynomial in the variable s, i.e.

$$R^{\kappa}_{n} = \sum_{m=0}^{m'(n)} s^{m} R^{\kappa}_{nm}(\varphi) \quad .$$

Let us g^{κ}_{np} are the coefficients of expansion of a solution to n-equation over the basis (6). Using (8), for these coefficients we obtain the uncoupled set of the Gauss equations [11]

$$\left\{ \partial^2_s + 2(\kappa+n+2)\partial_s + d^{\kappa}_{np} \right\} g^{\kappa}_{np}(s) = \langle w_p(\varphi) | R^{\kappa}_{n}(s,\varphi) \rangle . \qquad (18)$$

Here and further the Dirac brackets denote the integration with respect to angle φ on the interval $0 \le \varphi \le \pi/2$ and the part of the operator (14) independent of the variable s in the brackets of the functions w_p is equal to

$$d^{\kappa}_{np} = (\kappa+n+2)^2 - (p+2)^2 \qquad (19)$$

and vanishes if and only if n is an even number and $p = \kappa + n$. This important fact follows from the definition (6) of index κ which everywhere is an even (odd) number if index ℓ is an even (odd) number. As is known [11], if $d^{\kappa}_{np} = 0$, then the solution of eq.(18) is a polynomial of degree $m'+1$ and with the highest term $\beta^{\kappa}_{n,m'+1} s^{m'+1}$ providing of course that the coefficient

$$\beta^{\kappa}_{n,m'+1} = \langle w_{\kappa+n} | R^{\kappa}_{nm'} \rangle / 2(\kappa+n+2)(m'+1) \qquad (20)$$

is nonzero. In all other cases the solution of eq.(18) is a polynomial of degree m' . Therefore, if the right-hand side of n-equation (13) is a polynomial in the variable s of degree m', then its solution is a polynomial in the variable s of degree m' or $m'+1$ under the condition that n is an even or odd number. In the last case the highest term of the solution is

$$\beta^{\kappa}_{n,m'+1} s^{m'+1} w_{\kappa+n} \quad .$$

Supposing that $S^{\ell}_{\kappa} \neq 0$, i.e., the right-hand side of eq.(9) with chosen fixed indices κ and ℓ is nonzero, we start the investigation of the set (13) with a more general form A of the potential (I). Using (9) and putting $t=1$ we write first eq. (13) in the form

$$D^{\kappa}_{t} U^{\kappa}_{t} = v_{t-2} S_{\kappa} (\cos\varphi)^{t-2} w_{\kappa}(\varphi) \quad . \qquad (21)$$

The solution of this equation is independent of s and reads

$$U^{\kappa}_{t} = U^{\kappa}_{t0} = v_{t-2} S_{\kappa} \begin{cases} \sum_{p=\ell}^{\infty} a^{\kappa}_{t-2,p} \, w_p(\varphi) / d^{\kappa}_{tp} & (22a) \\ (\cos\varphi)^{t} \sum_{p=\ell}^{\kappa} c^{\kappa}_{tp} \, w_p(\varphi) . & (22b) \end{cases}$$

To derive eq. (22a) we have used eqs. (7-9), (14), (18) and (19). The coefficients c^{κ}_{tp} of the series (22b) are the solutions of the linear set, given in the Appendix. Using (8) we write the second eq. (13) as follows:

$$D^{\kappa}_{2} U^{\kappa}_{2} = (v_0 S_{\kappa} - F) w_{\kappa}(\varphi) + v_1 \sec\varphi \langle \varphi | S | U^{\kappa}_{1}(\varphi') \rangle = R^{\kappa}_{2} \qquad (23)$$

and look for the solution in the form

$$U^{\kappa}_{2} = s\, U^{\kappa}_{21}(\varphi) + U^{\kappa}_{20}(\varphi) \quad .$$

where

$$U^{\kappa}_{21}(\varphi) = \beta^{\kappa}_{21}\, w_{\kappa+2}(\varphi) .$$

Then, we obtain the equation

$$D^{\kappa}_{2}\, U^{\kappa}_{20} = R^{\kappa}_{2} - 2(\kappa+4)\, \beta^{\kappa}_{21}\, w_{\kappa+2}$$

which is soluble [11] if and only if its right-hand side is orthogonal to the function $w_{\kappa+2}$. This condition uniquely determines the coefficient

$$\beta^{\kappa}_{2t,1} = v_{t-2} \langle w_{\kappa+2t}(\varphi)(\cos\varphi)^{t-2} | h | U^{\kappa}_{t}(\varphi') \rangle / (\kappa+2t+2) . \qquad (24)$$

To derive eq. (24) with t=1 we have used eqs. (9), (20), (22b) and (23). Both the functions in the matrix element (24) are, according to eqs. (7) and (22a), infinite series of the eigenfunctions (6) of the operator h. Hence, in general the coefficient (24) is nonzero. Continuing the analysis one can easily be convinced that the coefficient (20) with any even n is proportional to $(V_{-1})^n$, contains in the matrix element the function $\sec\varphi\, w_{\kappa+n}(\varphi)$ and in general is nonzero due to the same reasons as for the coefficient (24). Therefore, the right-hand side of n-equation (13) is a polynomial in the variable s whose degree is not greater than $[(n-1)/2]$ and its solution is a polynomial in the variable s of degree less or equal to $[n/2]$. Also, eqs. (15) and (16a) are proven.

Now we put the expansion (15) into (13) and thus obtain eqs. (17) which are written in order of increasing index n and in order of decreasing index m for any fixed n. Then, the set (17) turns out to be recurrent. Its solutions with an

even n and any $m=0, \ldots \bar{m}(n)$ have the form of the sums

$$U^{\kappa}_{nm} = \beta^{\kappa}_{nm}\, w_{\kappa+n}(1-\delta_{m0}) + u^{\kappa}_{nm}(1-\delta_{m\bar{m}(n)})$$

of two orthogonal terms. The coefficient β^{κ}_{nm} is defined by the condition of orthogonality of the right-hand side of the next equation to the function $w_{\kappa+n}$. Numerical solution of the cut-up set (17) presents no special difficulties since the corresponding homogeneous equations are reduced [6] to the hypergeometrical one the Green functions of which are well-known [9].

Let us study the set (13) in the case B. As now $V_{-1}=0$, then from eqs. (21)–(23) it follows that $U^{\kappa}_{1}=0$ and

$$U^{\kappa}_{2} = U^{\kappa}_{20} = (v_0 S_{\kappa} - E)\, w_{\kappa} / 4(\kappa+3) . \qquad (25)$$

The third eq. (13) and its solution have respectively the forms of equations (21) and (22) in which t=3. The right-hand side of the fourth eq. (13) is independent of S and with the help of (7) and (9) is reduced to a linear combination of the functions w_p with the subscripts $p \neq \kappa+2$. Hence, the coefficient β^{κ}_{41} of (20) vanishes and the solution U^{κ}_{4} does not depend on s. Analogously, the right-hand side of the sixth eq.(13) is reduced to a sum of two terms. One of them contains the function U^{κ}_{3} of (22b) and the other is a linear combination of the functions w_p with subscripts $p \neq \kappa+4$. Therefore, eq. (20) defining the coefficient β^{κ}_{61} is reduced to eq.(24) with t=3. In general $\beta^{\kappa}_{61} \neq 0$ owing to the same reason as β^{κ}_{21} in the considered case A. Thus, the first term depending on s of the κ-solution (12) is $s\, U^{\kappa}_{61} = s\, \beta^{\kappa}_{61}\, w_{\kappa+6}$. The solution to the seventh eq.(13) and its right-hand side are independent of s. The term depending on s in the right-hand side of the eight eq.(13) is $s\,(V_0 S - E)\,U_{61}^{\kappa}$ and owing to (9) is proportional to $s\,w_{\kappa+6}$. Hence, the coefficient β_{82}^{κ} of (20) vanishes and the solution U_8^{κ} is a polynomial in the variable s with the highest term $\sim s\,w_{\kappa+6}$. The solutions U_n^{κ} with subscripts $n=9,10,11$ are also linear functions of s. The right-hand side of the twentieth eq. (13) contains the term

$$s V_1 \cos\varphi \,\langle \varphi | S | U_{91}^{\kappa}(\varphi')\rangle$$

which in general is not orthogonal to the function $w_{\kappa+12}$. Consequently, U_{12}^{κ} is a polynomial in the variable s of degree $\bar m(12) \leqslant 2$. Continuing the analysis, one can be convinced that all the coefficients (20) with subscripts n divisible by six are in general nonzero since they contain in the matrix elements the functions $\cos\varphi\; w_{\kappa+n}(\varphi)$ and $S\,U_{n-3,m'}^{\kappa}$ representable, according to (7), as infinite series of the functions (6). Thus, the solution U_n^{κ} is a polynomial (15) of degree (16b) and the right-hand side of n-equation (13) is a polynomial in the variable s of degree

$$m'(n) = \bar m(n) \quad \theta([n/6] \quad n/6) \; .$$

Similarly, it may be shown that the solutions of the set (13) depend on s beginning from a definite number n if and only if series (1) contains one or more terms with odd indices. In the case C, when the potential (1) is the function of the square of distance x, all such terms are absent. Therefore, the right-hand sides of eqs.(13) with odd (even) numbers n contain the solutions of the preceding equations with odd(even)

numbers. Solutions (22) of the first (t=1) and third (t=3) eqs. (13) vanish. Further, by induction we obtain $U_{2n+1}^{\kappa}=0$ for any n . Solution to the second eq. (13) has the form (25). Using (7)-(9), (19) and (20), also by induction we deduce

$$U_{2n}^{\kappa} = U_{2n,0}^{\kappa} = \sum_{p=q_-(\kappa,n)}^{q_+(\kappa,n)} f_{np}^{\kappa}\, w_p \quad , \qquad (26)$$

where n=0,1,..., the limits are

$$q_{\pm}(\kappa,n) = \max\{\,\ell,\ \kappa \pm 2\,(n-1+\delta_{n0})\,\}$$

and f_{np}^{κ} are the numerical coefficients satisfying the recurrence relations given in the Appendix.

To finish the analysis of set (13) we consider the remaining special cases. Assume that indices κ and ℓ are such that $h_{\kappa}^{\ell} = -1/2$, i.e., $S_{\kappa}^{\ell} = 0$. Then, due to eq. (9) the operator S^{ℓ} mapps the function w_{κ}^{ℓ} into identity zero. Using this fact one can easily show that for any coefficient V_n of the series (1) the set (13) becomes

$$D_n^{\kappa} U_n^{\kappa} = E\, U_{n-2}^{\kappa} \quad \cdot \quad U_{n-1}^{\kappa} \quad 0 ,$$

where n=2,4,..., $U_0^{\kappa} = w_{\kappa}^{\ell}$ and κ-solution (12) read

$$U^{\kappa\ell} = J_{\kappa+2}(\sqrt{E}\, r)\; w_{\kappa}^{\ell}(\varphi) \qquad (27)$$

with $J_{\kappa+2}$ being the Bessel function [9]. As follows from eq. (36) of ref.(10) the Raynal-Revai coefficient h_{κ}^{ℓ} satisfies the equality $h_{\kappa}^{\ell} = -1/2$ if $\kappa = \ell - 1$ or $\kappa = 2$, $\ell = 0$. Hence, in these two cases κ-solution (12) of eq. (5) has the form (27) for any potential.

4. Asymptotic expansions of the wave functions

Every κ-solution (12),(15), (16) of eq.(5) is put into correspondence by formula (2) to κ-solution of the Schrödinger equation

$$\Psi^{\kappa\varepsilon} = \sum_{n=0}^{\infty} r^{\kappa+n} \sum_{m=0}^{\tilde{m}(n)} s^m \; \Psi^{\kappa\varepsilon}_{nm}(\Omega) \quad , \tag{28}$$

where the angular functions are

$$\Psi^{\kappa\varepsilon}_{nm} = 2\, cosec\, 2\varphi \langle \varphi | S^{\ell} | U^{\kappa\ell}_{nm}(\varphi') \rangle \, \mathcal{Y}^{\varepsilon}(\hat{x}, \hat{y}) \; . \tag{29}$$

If κ-solution of eq.(5) has the form (27), then all the functions (29) and κ-solution (28) are identity zero. It is possible if $\kappa = \ell = 1$ or $\kappa = 2$, $\ell = 0$. A nontrivial formal solution (27) with $\kappa = 2$, $\ell = 0$ corresponding to the trivial solution of the Schrödinger equation, has first been obtained in ref.[13] and further studied in ref.[14].

In the $\mathcal{K}$-class the general solution of eq.(5) and the relevant solution of the Schrödinger equation are in general the infinite series in the corresponding κ-solutions and any numerical coefficients C_κ , i.e.

$$U^{\ell} = \sum_{\kappa=\ell}^{(\infty)} C_\kappa U^{\kappa\ell} \quad , \quad \Psi^{\varepsilon} = \sum_{\kappa=\ell}^{(\infty)} C_\kappa \Psi^{\kappa\varepsilon} \; . \tag{30}$$

This asymptotic series are only formal solutions at small hyperradius. To construct the physical solutions it is necessary to prove the convergence of the first series (30) in the $\mathcal{K}$-class and to find the coefficients C_κ from the boundary condition for $r \to \infty$. This construction based on the solution of the infinite set (17) is undoubtedly interesting but seems very complex. More practical is the calculation of the component U^{ℓ} and wave function Ψ^{ε} with right asymptotics at small hyperradius. Inside a sphere centred at r=0 with a sufficiently small radius we apply the asymptotic representations as a sum of terms of the series (30) most slowly vanishing as $r \to 0$. This permits us to put the boundary condition on the sphere and further, sewing together the numerical solution of the problem outside the sphere with the analytical one inside it, to define the unknown coefficients C_κ . This construction will be performed elsewhere. Let us investigate the three terms of the wave function (30) most slowly vanishing as $r \to 0$ assuming all the coefficients C_κ to be nonzero. Substitute the functions $U^{\kappa\ell}_{nm}$ found in sec.3 into the κ-solutions (28) and (29) and them - into the second series (30). As a result, we obtain the wave function asymptotics around the triple collision point ($r \to 0$)

$$\Psi^{\varepsilon} = r^{\kappa} \begin{cases} \sum_{n=0}^{1} r^n \Psi^{\kappa\varepsilon}_{n0} + r^2 s \, \Psi^{\kappa\varepsilon}_{21} + O(r^2) & \text{A} \quad (31a) \\ \sum_{n=0}^{3} r^n \Psi^{\kappa\varepsilon}_{n0} \delta_{n1} + C_{\kappa+2} \, r^2 \, \Psi^{\kappa+2,\varepsilon}_{00} + O(r^4) & \text{B} \quad (31b) \\ \sum_{p=\kappa}^{\kappa+4} C_p \sum_{n=0}^{2+(\kappa-p)/2} r^{2n} \, \kappa \, p \, \Psi^{p\varepsilon}_{2n,0} + O(r^6) \; . & \text{C} \quad (31c) \end{cases}$$

Here $C_\kappa = 1$ and $\kappa = \ell + 2\delta_{1\ell}$ because κ-solution (28) with $\kappa = \ell = 1$ is identity zero; the functions (29) with an even first subscript, owing to eqs. (9),(11) and (23-26), are the finite sums of hyperharmonics (10)

$$\Psi^{\kappa\varepsilon}_{21} = S^{\ell}_{p} \, \beta^{p\ell}_{21} \, Y^{\ell}_{p} \quad , \quad p = \kappa + 2 \; ,$$

$$\Psi^{\kappa'\varepsilon}_{2n,0} = \sum_{p=q(\kappa',n)}^{q_{+}(\kappa',n)} S^{\ell}_{p} \, f^{\kappa'\ell}_{np} \, Y^{\ell}_{p} \quad . \tag{32}$$

where $\kappa' = \kappa, \kappa+2, \kappa+4$ and $n = 0, 1, 2$;

the functions (29) with n=1,3, owing to eqs.(9), (22) can be represented either as an infinite series of the hyperharmonics (10) or as products of the bispherical harmonic (3) and a finite sum of functions of angle φ:

$$\Psi^{\kappa\varepsilon}_{n0} = v_{n-2}\, S^{\ell}_{\kappa} \begin{cases} \sum\limits_{\rho=\ell}^{\infty} \left(a^{\kappa\ell}_{n-2,\rho}\, S^{\ell}_{\rho} / d^{\kappa}_{n\rho} \right) Y^{\varepsilon}_{\rho}(\Omega) & (33a) \\ 2\, \mathcal{Y}^{\varepsilon}_{\ell}(\hat{x},\hat{y}) \sum\limits_{\rho=\ell}^{\kappa} c^{\kappa\ell}_{n\rho} \langle \varphi | S^{\ell} | (\cos\varphi')^{n} w^{\ell}_{\rho}(\varphi') \rangle / \sin 2\varphi . & (33b) \end{cases}$$

According to eq.(31), the wave function of S-state ($\ell=0$) in general does not vanish at r=0. Therefore, let us explore its asymptotics in more detail. At $\ell=0$, $\kappa=2$ the functions $S^{\ell} w^{\ell}_{\kappa}$ and (29) as well as the κ-solution (28) are identity zero. Hence the sum (31a) does not contain a term $\sim r^2 S$ and the first four terms of the asymptotics of the wave function (30) are the terms of the κ-solution (28) with $\kappa = \ell = 0$. Three of them are found by the formulae (32) and (33b), the fourth by the solution of eq.(23). Also, we use the known closed expression

$$h^{0}_{\kappa} = (2/n\sqrt{3})\, \sin(2\pi n/3), \quad n = \kappa/2+1, \quad \kappa = 0, 2, \dots$$

for the Raynal-Revai coefficients [10]. As a final result, in the case A we derive

$$\Psi^{\varepsilon} = 1 + v_1 \{ x + \theta_- r^2/x + \theta_+ (x^2+9y^2)/3\sqrt{3}y \}/2 + r^2 \{ 3v_1^2 + 6v_0 - 2E + v_{-1}^2 r^{-2} \langle \varphi | S^{0} | W(\varphi') \rangle / xy \}/24 + O(r^3) . \quad (34a)$$

Here and further

$$\theta_{\pm} = \theta(x \pm \sqrt{3}y) = \theta(\pm\varphi \mp \pi/6)$$

the matrix element $\langle \varphi | S^{0} | W \rangle$ is a very complex but finite sum in various elementary functions. Therefore, we give only the function

$$W(\varphi) = \theta_-\, \sin 4\varphi \{ z_- + 3(\sec 2\varphi - 2\ln\cos\varphi)/2 + 2\varphi\, \mathrm{cotan}\, 4\varphi \} + \theta_+ \{ z_+ \sin 4\varphi + \sqrt{3}(15 - 16\cos 2\varphi) + (9\pi - 31\sqrt{3} - \varphi)\cos 4\varphi \}/36 .$$

The coefficients $z_{\pm}$ are determined from the condition of continuity of the functions W and $\partial_{\varphi} W$ at the point $\varphi = \pi/6$.

In the case B, adding to eq.(31b) the term $\sim r^4$ calculated by eqs.(26),(28), (29) and the leading term (11) of the asymptotics of κ-solution with $\kappa=4$, we derive

$$\Psi^{\varepsilon} = 1 + (3v_0 - E) r^2/12 + v_1 \{ x^3 + \theta_- (5x^4 + 30x^2y^2 + 9y^4)/20x - \theta_+ (15x^4 + 2x^2y^2 - 37y^4)/30\sqrt{3}y \}/12 + \{ (3v_0 - E)^2 + 18 v_2 \} r^4/384 + C_4 r^{-2} (4x^2 - 3r^2)(4y^2 - 3r^2) + O(r^5) . \quad (34b)$$

The asymptotics of the wave function for the case C is given by (34b) with $v_1 = 0$.

Formulae (34) demonstrate one general property of the asymptotic expansion (30), namely, the functions (15) and (29) have in general different functional dependence of the angle φ on the segments $[0, \pi/6]$, $[\pi/6, \pi/3]$ and $[\pi/3, \pi/2]$.

This fact is caused by the sign changing of derivatives of the integral (4) limits at the points $\varphi = \pi/6, \pi/3$.

5. Summary and conclusion

Let us summarize our main results. In the $\mathcal{H}$-class of functions all linearly independent solutions of eq.(5) are represented at small hyperradius as series (12),(15), (16) in powers of the hyperradius, its logarithm powers and also the functions of one hyperspherical angle. These functions satisfy the simple recurrent set of ordinary second-order differential equations (17). Solutions of the first three eqs.(17) may be found by formulae (11) and (22-26). If the potentials (1) depend on the square of distance (the case C), then the solution of any equation of the set (17) is a finite linear combination (26) of the functions (6) and coefficients satisfying recurrent relations. All linearly independent solution to the Schrödinger equation with the Faddeev components belonging to the $\mathcal{H}$-class are represented at small hyperradius as series (28) with the functions (29) of the hyperspherical angles. These three functions, corresponding to the terms of the asymptotics (31) of the wave functions are found by formulae (32) and (33).

The wave function of three identical particles cannot vanish as $r \to 0$ slower than $r^{\ell+2\delta_{\ell 1}}$.
In the case $\ell=0$ its asymptotics (34) does not contain a term ρ^2s.

In conclusion, we should like to point out two facts. The derived asymptotic expansions are useful both for the analytical and numerical investigations of three-identical-particle problem with S-wave potentials (1). On the other hand, all our results

may be easily generalized to the case of nonidentical particles and S-wave potentials with spin-isospin dependence. For this purpose one needs to write the initial eq.(5) and its solutions in the matrix form and then to repeat all the carried out derivations.

Appendix

Squares of the norm factors of the functions (6) read

$$(N_\kappa^\ell)^2 = 2(\kappa+2)\Gamma(n+1)\Gamma(\ell+n+2)/\Gamma(\ell+n+3/2)\Gamma(n+3/2)$$

with Γ being the Gamma-function [9].

The coefficient h_κ^ℓ, owing to eq.(9), is the ratio of the integral (4) with $U^\ell = w_\kappa^\ell$ to the function w_κ^ℓ taken at the same point φ.

The coefficients of the sum (7) are equal to the matrix elements

$$a_{m\kappa'}^{\kappa\ell} = \langle w_{\kappa'}^\ell(\varphi) | (\cos\varphi)^m \, w_\kappa^\ell(\varphi) \rangle$$

which are reduced to a sum of table integrals [15] by representating the Jacobi polynomial in one of the functions $w_{\kappa'}^\ell$ or w_κ^ℓ as a hypergeometric series [9]. The final result reads

$$a_{m\kappa'}^{\kappa\ell} = N_{\kappa'}^\ell N_\kappa^\ell \{(-1)^{n'+n}/2\Gamma(n'+1)\Gamma(n+1)\}(3/2)_n$$

$$\sum_{\rho=0}^{n} \{ (n)_\rho (\ell+n+2)_\rho (m/2 \ \ \rho)_{n'}$$

$$B(m/2+\rho+3/2, \ell+n'+3/2)/(3/2)_\rho \Gamma(\rho+1)\},$$

where $(a)_n$ is the Pokhgammer symbol B is the Beta-function [9] and $\kappa' \ \ \ell+2n'$, $\kappa \ \ \ell+2n$; $n', n \ \ 0, 1, \ldots$; $m = 1, 0, \ldots$

At fixed indices κ, ℓ, t the column of the coefficients $c^{\kappa\ell}_{tp}$ of sum (22b) ordered in the growth of the index $p = \ell, \ell+2, \dots$ is a solution of a set of linear equations with the three-diagonal matrix. Its main diagonal elements are

$$N^{\ell}_{p}\{t(p+t+1-\ell(\ell+1)/(p+1)) + Q^{+}_{pt}(1-\ell(\ell+1)/(p+1)(p+3))\}$$

and the elements of the upper (−) and lower (+) diagonals read

$$N^{\ell}_{p}\, Q^{\pm}_{pt}(2p-2\ell+3\pm1)(2p+2\ell+5\pm3)/8(p+2)(p+2\pm1)\ ,$$

where

$$Q^{\pm}_{pt} = \{(\kappa+t+2)^2 - (p\pm t+2)^2\}/2 \quad .$$

The elements of column of the right-hand side of this set are equal to $\delta_{\kappa p}$. The set is derived after substitution of ansatz (22b) into (21) by representing the functions

$$(1-u^2)\,\partial_u\, P^{(\ell+1/2,\ 1/2)}_{n}(u) \quad , \qquad u\, P^{(\ell+1/2,\ 1/2)}_{n}(u) \quad ,$$

where $u = \cos 2\varphi$, in the form of sums of the Jacobi polynomials with subscripts $n, n\pm1$ [9]. Solutions of this set with $\kappa = \ell$ are $c^{\kappa\ell}_{t\ell} = (5t-3)^{-1}$ and in the case $\kappa = \ell+2$ have the form

$$c^{\kappa\ell}_{t\ell} = \sqrt{3(\ell+4)}\{(14\ell+1)t - 44\ell - 6\}/60(\ell+3)\sqrt{2\ell+3}\ ,$$

$$c^{\kappa\ell}_{t\kappa} = \delta_{t1}(4\ell+15)/12(\ell+3) + \delta_{t3}(\ell+4)/20\ .$$

In the sums (26) the coefficients with subscript $n=0$ and $n=1$, owing to eqs.(11) and (25), are equal to

$$f^{\kappa\ell}_{np} = \delta_{\kappa p}\{\delta_{n0} + \delta_{n1}(v_0 S^{\ell}_{\kappa} - 1)/4(\kappa+1)\} \quad .$$

The remaining coefficients satisfy the recurrent relations

$$f^{\kappa\ell}_{n+1,p} = (1/d^{\kappa}_{2(n+1),p}) \sum_{m=0}^{n} \sum_{q=q_-(\kappa,m)}^{q_+(\kappa,m)} \{(v_{2(n-m)} S^{\ell}_{q} - E\,\delta_{nm})\, a^{\kappa\ell}_{2(n-m),p}\, f^{\kappa\ell}_{mq}\}\ ,$$

where the coefficients d^{κ}_{np} are given by eq.(19) and the functions $q_{\pm}$ are the same as in the sums (26).

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Асимптотические разложения волновых функций трех тождественных частиц при малых значениях гиперрадиуса и S-волновых потенциалах

В рамках интегродифференциальных уравнений построены асимптотические разложения волновых функций в виде рядов, содержащих степени гиперрадиуса, его логарифма и неизвестные функции гиперсферических углов. Для вычисления этих функций получена рекуррентная система обыкновенных дифференциальных уравнений второго порядка. Исследована зависимость асимптотических разложений от величины полного углового момента и от поведения потенциалов на малых расстояниях.

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Asymptotic Expansions of Three-Identical-Particle Wave Functions at Small Hyperradius and S-Wave Potentials

In the framework of the integrodifferential approach the asymptotic expansions of the wave functions are constructed as series in powers of the hyperradius, its logarithm powers and unknown functions of hyperspherical angles. For calculation of these functions a recurrence chain of ordinary second-order differential equations is obtained. The dependence of the asymptotic expansions on the total angular momentum and behaviour of potentials at small distances is investigated.

The investigation has been performed at the Laboratory of Theoretical Physics JINR.

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