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STABILITY OF THE MOVING BUBBLES IN THE SYSTEM OF INTERACTING BOSONS

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1. INTRODUCTION

The gas of bosons interacting via the 2-body attractive and 3-body repulsive δ-function potential is described quasi-classically by the so-called $\psi^3-\psi^5$ nonlinear Schrödinger equation[1]:

$$i\psi_t + \Delta\psi - a_1\psi + a_3\psi|\psi|^2 - a_5\psi|\psi|^4 = 0, \tag{1}$$

where a_3, $a_5 > 0$, and $\Delta \equiv \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_D^2$. An equivalent scaled form which we prefer to work with reads[2]:

$$i\phi_t + \Delta\phi + (|\phi|^2 - 1)(2A + 1 - 3|\phi|^2)\phi = 0. \tag{2}$$

Apart from the physical systems that can be modelled by the Bose gas with this type of interaction (such as superfluid helium[3]) the $\psi^3-\psi^5$ NLS arises in a number of independent applications including quantum crystals[4], one-dimensional ferromagnetic[5] and molecular chains[6], nonlinear optics[7], nuclear hydrodymanics[8] and many others. The physically interesting dimension D varies from 1 to 3 while the appropriate boundary conditions can be both of zero and "condensate" type, i.e., both $\phi(\vec{x},t) \to 0$ and

$$|\phi(\vec{x},t)| \to 1 \quad \text{as} \quad \vec{x}^2 \to \infty. \tag{3}$$

In ref.[2] the equation (2) was found to possess, under the condition (3), a new type of soliton solutions that were called "bubbles". These nontopological solitons exist at $0 < A < 1$ and have the remarkable property to survive passing to arbitrary higher dimensions. (In the case of lumps this property is not surprising but it becomes nontrivial for solitons with non-vanishing boundary conditions. To compare, note that the kinks and vortices of the repulsive ψ^3 NLS *do not* have localized stationary analogues in D = 3). However, regardless of the dimension, the *quiescent* "bubbles" proved unstable[2,9].

In the present study we address ourselves the question of whether the *travelling* "bubble" can be stable. Here our treatment will be restricted to the one-dimensional situation since in this case the corresponding solution of Eq.(2) is known explicitly[2]:

$$\phi_b(x,t) = \phi_b(\tilde{x}) = \frac{\sqrt{2}\cosh(\xi/2 - i\mu)}{[(2-A)(A^2+v^2)^{-1/2}+\cosh\xi]^{1/2}}, \qquad (4)$$

where $\tilde{x} = x - vt$, $\xi = (c^2 - v^2)^{1/2}\tilde{x}$, $c = 2(1 - A)^{1/2}$ stands for the velocity of sound, and v is the velocity of the soliton, $-c < v < c$. The "twisting angle" μ is also defined through v:

$$\sin 2\mu = \frac{1}{2} v[(c^2 - v^2)/(A^2 + v^2)]^{1/2}, \quad -\pi/4 < \mu < \pi/4.$$

In subsequent publications we plan to analyse the 2 and 3 dimensional situations which we expect to have much in common with the D = 1 case.

2. LINEARIZED STABILITY

Linearizing Eq.(2) in the vicinity of the "bubble" (4) and taking the infinitesimal perturbation in the form $\delta\phi(x,t) = [f(\tilde{x}) + ig(\tilde{x})]e^{\lambda t}$ with f, g, λ real, we arrive at the eigenvalue problem

$$H_v y = \lambda J y(x), \qquad (5)$$

$$y(\pm\infty) = 0, \qquad (6)$$

where

$$H_v \equiv -\frac{d^2}{dx^2} I + v\frac{d}{dx} J - U_v(x), \qquad (7)$$

I is the 2x2 identity matrix, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$U_v(x) = \begin{pmatrix} F + 2F_\rho r^2 & 2F_\rho rj \\ 2F_\rho rj & F + 2F_\rho j^2 \end{pmatrix} \qquad (8)$$

Next, in (5)-(8) r and j stand for the real and imaginary part of Eq.(4): $\phi_b(\tilde{x}) = r(\tilde{x}) + ij(\tilde{x})$, $\rho = |\phi_b| = (r^2 + j^2)^{1/2}$. Finally, $F = F(\rho) = (\rho - 1)(2A + 1 - 3\rho)$, $F_\rho = 2(A + 2 - 3\rho)$, $y = (f, g)^T$, and the tilde over x has been omitted in (5)-(8).

The eigenvalue problem (5), (6) was analysed numerically at 20 equally spaced values of A from the range (0,1). Having

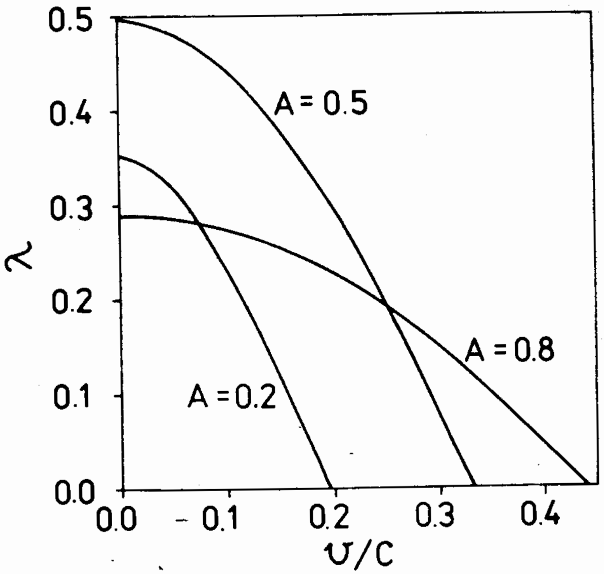

Fig.1. The instability growth rate λ_1 versus v/c, the velocity of the soliton in units of the sound velocity ($c = 2\sqrt{1 - A}$).

taken into account the symmetry $H_v(x) = H_{(-v)}(-x)$, we confined ourselves to positive velocities v. For each A a single positive eigenvalue $\lambda = \lambda_1$ was found, depending continuously on v.

The function $\lambda_1(v)$ which looks similar for each A, is depicted in Fig.1. It is seen that there is a certain critical velocity v_c such that for $v \geq v_c$ only zero eigenvalue $\lambda_0(v) \equiv 0$ exists corresponding to the translational symmetry. The soliton is therefore stable for $v \geq v_c$. To describe the stability domain it remains only to find the precise value of v_c.

The difficulty in determining v_c is related to the fact that for $v \geq v_c$ the eigenvalue $\lambda_1(v)$ immerses into the continuous spectrum of $J^{-1}H_v$ which occupies the imaginary axis of λ. That is, as $v \to v_c - 0$ and $\lambda_1(v) \to 0$ the functions f, g become non-localized.

Indeed, asymptotically as $x \to \pm\infty$ we have $r(\pm\infty) = \cos\mu$, $j(\pm\infty) = \mp\sin\mu$ so that

$$U_v(\pm\infty) = c^2 \begin{pmatrix} \cos^2\mu & \mp\sin\mu\cos\mu \\ \mp\sin\mu\cos\mu & \sin^2\mu \end{pmatrix}$$

and the characteristic equation for Eq.(5) is readily calculated to be

$$k^4 - c^2 k^2 + (\lambda_1 + vk)^2 = 0. \qquad (9)$$

It is not difficult to show that (at least in a finite vicinity of $\lambda_1 = 0$) Eq.(9) has 4 real roots, 2 of them being positive and 2 negative. Let $k_\pm$ be the two roots with the minimal modúli, $k_+ > 0$ and $k_- < 0$. Then $y = (f, g)^T$ obeys

$$y(x) \to y_0^{\pm} \exp(-k_\pm x) \quad \text{as} \quad x \to \pm\infty. \tag{10}$$

Now it is straightforward to obtain from (9) that when $\lambda_1 \to 0$ we have $k_\pm \to 0$ implying that the solution $\{\lambda_1, y\}$ of the problem (5), (6) ceases to exist at $v = v_c$. One cannot therefore determine v_c simply as such v for which $\lambda_1(v)$ vanishes; some sort of extrapolation procedure has to be applied instead. Our extrapolation was based on a Taylor series expansion

$$\lambda_1(v) = \kappa_1(v - v_c)^2 + \kappa_2(v - v_c)^2 + \dots \tag{11}$$

in the vicinity of v_c (for $v \leq v_c$). The validity of (11) was verified numerically.

3. NUMERICAL COMPUTATION

Suppose v approaches $v_c - 0$ and $k_\pm \to 0$. Then if we wanted the boundary conditions (6) to be satisfied to some reasonable accuracy, we would have to extend the integration interval to infinity. This computational difficulty can be circumvented by invoking the asymptotics (10) and passing to the conditions of the form $(dy/dx + k_\pm y)|_{x=\pm\infty} = 0$. Next, for $\lambda_1 \to 0$ Eq.(9) yields $k_\pm = \pm(c \mp v)^{-1}\lambda_1 + O(\lambda_1^2)$ so that in the vicinity of the critical velocity the latter conditions simplify to

$$\{dy/dx \pm (c \mp v)^{-1} \lambda y\}|_{x=\pm\infty} = 0. \tag{12}$$

Table

The critical velocities (velocities of stabilization), v_c, *for different values of the parameter* A. v_c *is given in units of the sound velocity,* $c = 2\sqrt{1-A}$.

A	0.1	0.2	0.3	0.4	0.5
v_c/c	0.1301	0.1943	0.2461	0.2911	0.3317

A	0.6	0.7	0.8	0.9
v_c/c	0.3692	0.4044	0.4377	0.4695

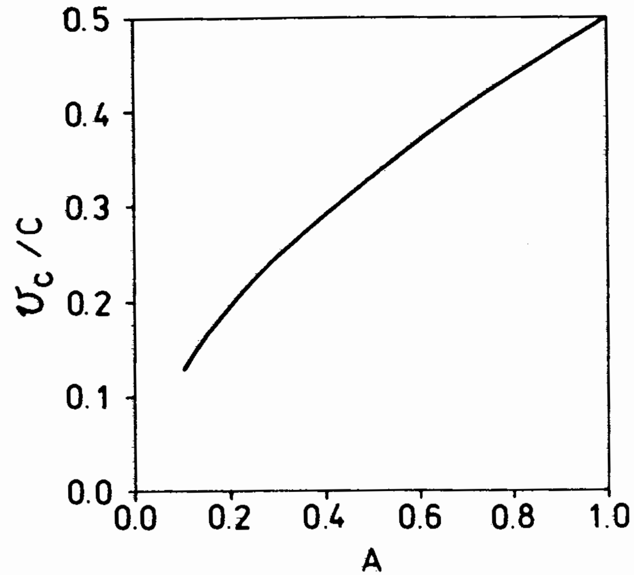

Fig.2. The critical velocity v_c *versus* A.

Our computational policy was to solve the *inverse* spectral problem, i.e., for several sufficiently small values of λ we solved the problem (5), (12) to determine the corresponding v. We have found that the expansion (11) is indeed correct, and that $\kappa_1 \neq 0$. The critical velocity was obtained then by the linear extrapolation of the resulting curve $\lambda_1(v)$ to $\lambda_1 = 0$. The values of v_c pertaining to different A are collected in the Table; the dependence $v_c(A)$ is illustrated by Fig.2.

4. DISCUSSION

So we have shown that the *one-dimensional* bubble-like solitons stabilize when moving sufficiently fast. It would be intesting to find out whether analogous critical velocities exist for 2 and 3-dimensional "bubbles" which are also known to be unstable at rest. In this connection it is worthwhile to recall[2] that propagation of *transonic* "bubbles" is governed approximately by the Korteweg - de Vries equation at D = 1, and by the two and three-dimensional Kadomtsev - Petviashvili equations at D = 2 and 3. On the other hand, the KdV soliton as well as the 2-dimensional KP lump is known to be stable while the 3-dimensional lump unstable[10]. This suggests that the critical velocity v_c such that the $\psi^3 - \psi^5$ NLS "bubble" is unstable for $v < v_c$ and stable at $v \geq v_c$, exists only for D = 1 and 2. The 3-dimensional "bubble", conversely, is expected to be unstable for any velocity.

Finally, let us remark that when we analyse NLS equations, the dependence of soliton's stability on its velocity is inherent only for solitons with non-vanishing boundary conditions, i.e.,

$$\phi(x,t) \to e^{\mp i\mu} \quad \text{as} \quad x \to \pm\infty. \qquad (13)$$

Really, consider the case of the vanishing conditions $\phi(x,t) \xrightarrow[|x|\to\infty]{} 0$ and suppose we have a lump $\phi_\ell(x,t)$ moving with velocity v. The Galilean transformation

$$\phi(x,t) \to \tilde{\phi}(x,t) = \exp\{-\frac{1}{2}iv(x+\frac{1}{2}vt)\}\phi(x+vt,t), \qquad (14)$$

takes then the soliton ϕ_ℓ to the rest frame. Furthermore, this transformation can be applied to any nearby soliton, the stability problem being reduced to the one for $v = 0$. On the contrary, in the case of the boundary conditions (13) the transformation (14) does not take the travelling soliton to the static one; the dependence on the velocity is more complicated here, see e.g. Eq.(4).

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Барашенков И.В. и др. E5-88-547
Устойчивость движущихся пузырьков
в системе взаимодействующих бозонов

Рассматривается $\psi^3-\psi^5$ нелинейное уравнение Шредингера, описывающее бозе-газ с двух- и трехчастичным взаимодействием и обладающее решениями в виде солитоноподобных пузырьков. Известно, что статические "пузырьки" неустойчивы; в настоящей работе исследуется устойчивость движущихся солитонов. Показано, что существует критическая скорость v_c такая, что "пузырек" устойчив при $v \geq v_c$ и неустойчив при $v < v_c$.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1988

Barashenkov I.V. et al. E5-88-547
Stability of the Moving Bubbles
in the System of Interacting Bosons

The static bubble-like soliton of the $\psi^3-\psi^5$ nonlinear Schrödinger equation describing the boson gas with 2- and 3-body interactions, is known to be unstable. Here we study stability of the moving "bubbles". Our conclusion is that certain critical velocity exists, v_c such that the "bubble" is stable for $v \geq v_c$ and unstable otherwise.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1988