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SOME EXACT SOLUTIONS OF THE THREE-IDENTICAL-PARTICLE PROBLEM WITH S-WAVE INVERSE SQUARE POTENTIALS

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There are two reasons to investigate the three-identical-particle problem with S-wave potentials

$$V(x) = \alpha x^{-2} , \qquad (1)$$

where α is an arbitrary parameter and x is interparticle distance. The first reason is that the existence theorem of regular solutions to Faddeev differential equations in the case of potentials with singularity $\sim x^{-2}$ is not proved/1/. The other reason is that the modern nucleon-nucleon potentials constructed within field-theoretical models, for example, the Bonn one/2/, contain the short-range singular term $\sim x^{-2}$.

Some exact solutions of the three-identical-particle problem with potential (1) were first obtained by Avishai/3/. Using the separation of variables into the radial and angular one he reduced the problem to two one-dimensional equations. The radial equation was Bessel equation, while the angular equation was an integro-differential one. Avishai has found only some numerical solutions to the latter equation.

In the present work we show that the angular equation with a certain parameter α of potential (1) has analytic solutions.

To describe the positions of three identical particles, we use the hyperradius r and two different sets of hyperspherical angles Ω and Ω'. Our hyperspherical coordinates are associated with two different sets of usual reduced Jacobi vectors /1/ ($\vec{x}$, $\vec{y}$) and ($\vec{x}'$, $\vec{y}'$) by

$$r = (x^2 + y^2)^{1/2} , \quad \Omega = (\varphi, \hat{x}, \hat{y}) , \quad \Omega' = (\varphi', \hat{x}', \hat{y}') ,$$

where

$$\tan\varphi = y/x , \qquad \tan\varphi' = y'/x'$$

and $\hat{a}$ stands for two spherical angles of the vector $\vec{a}$.

The wave function of the three-particle state with total energy E and quantum numbers $\varepsilon = (\ell, \bar{m})$ where ℓ is the total angular momen-

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tum and $\bar{m}$ is its third component, reads in hyperspherical coordinates[4]

$$\Psi^{\mathcal{E}} = 2\,\langle \varphi | \hat{S}^{\ell} | U^{\ell}(\tau,\varphi') \rangle \; \mathcal{Y}^{\mathcal{E}}(\hat{x},\hat{y}) / \tau^2 \sin 2\varphi . \tag{2}$$

Here, the operator $\hat{S}^{\ell}$ is the symmetrization operator in brackets of bispherical harmonics[5]

$$\mathcal{Y}^{\mathcal{E}}(\hat{x},\hat{y}) \equiv Y_{00}(\hat{x})\, Y_{\ell\bar{m}}(\hat{y}) . \tag{3}$$

The operator $\hat{S}^{\ell}$ acts on the variables φ and φ' as the sum of the identity operator and double geometrical operator $\hat{h}^{\ell}$. The mapping of the Faddeev component U^{ℓ} by this operator may be written as the integral[4]

$$\langle \varphi | \hat{h}^{\ell} | U^{\ell}(\tau,\varphi') \rangle = (2/\sqrt{3}) \int_{c_-(\varphi)}^{c_+(\varphi)} d\varphi' \, P_{\ell}^{(0,0)}(u)\, U^{\ell}(\tau,\varphi'), \tag{4}$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial[6] of variable

$$u = \cos(\hat{\vec{y}\,\vec{y}'}) = (\cos 2\varphi + \cos 2\varphi' - 1/2) / 2 \sin\varphi \sin\varphi'$$

and the integral limits are the break-lines

$$c_{\pm}(\varphi) = \min\,(|\varphi \pm \pi/3| \, , \, 2\pi/3 - \varphi).$$

The Faddeev equation[1] has the form[4]

$$(\partial_\tau^2 + \tau^{-1}\partial_\tau - \tau^{-2}\hat{\Lambda}_\varphi^{\ell} + E)\, U^{\ell}(\tau,\varphi) = V(x) \langle \varphi | \hat{S}^{\ell} | U^{\ell}(\tau,\varphi') \rangle , \tag{5}$$

where the operator

$$\hat{\Lambda}_\varphi^{\ell} \equiv -\partial_\varphi^2 + \ell(\ell+1)/\sin^2\varphi \tag{6}$$

is the grand angular momentum operator in brackets of the functions (3) and the variables τ and φ belong to the first quadrant $\mathbb{R}_+^2$ of the two-dimensional plane.

We denote by $\mathfrak{D}$ the class of functions defined in the region $\mathbb{R}_+^2$ having continuous second-order derivatives with respect to the angular variable and vanishing on the rays $\varphi = 0, \pi/2$. The orthogonal angular basis in this class is formed[5] by the regular eigenfunctions of the operator (6). These functions read [6]

$$w_K^{\ell} = N_K^{\ell} (\sin\varphi)^{\ell+1} \cos\varphi \; P_n^{(\ell+1/2,\,1/2)}(\cos 2\varphi) \tag{7}$$

with N_K^{ℓ} being the normalization constant, $K = 2n+\ell$ and $n = 0, 1, \ldots$

At first we try to simplify the problem (5) assuming for some its solutions belonging to the $\mathfrak{D}$ -class the factorized form

$$U^{\ell} = Z_p(\sqrt{E}\,\tau)\, g_m^{\ell}(\varphi) \tag{8a}$$

with the angular function represented by a finite linear combination

$$g_m^{\ell} = \sum_{K=\ell}^{m} b_K^{\ell}\, w_K^{\ell}(\varphi) \tag{8b}$$

of the basis function (7) and numerical coefficients b_K^{ℓ}. We substitute ansatz (8a) into eq.(5) and perform the Avishai[3] separation of variables. As a result we obtain the Bessel equation with index p for the radial function Z_p and the integrodifferential equation

$$(\cos\varphi)^2 (p^2 - \hat{\Lambda}_\varphi^{\ell})\, g_m^{\ell}(\varphi) = \alpha \langle \varphi | \hat{S}^{\ell} | g_m^{\ell}(\varphi') \rangle \tag{9}$$

for the angular function g_m^{ℓ}.

Consider how all the operators of eq.(9) act on the functions of our angular basis (7). Any function w_K^{ℓ} satisfies the equality[5]

$$\hat{\Lambda}_\varphi^{\ell}\, w_K^{\ell} = (K+2)^2\, w_K^{\ell} \tag{10}$$

as well as the equality[4]

$$\hat{S}^{\ell}\, w_K^{\ell} = S_K^{\ell}\, w_K^{\ell} , \tag{11}$$

where $S_K^{\ell} = 1 + 2h_K^{\ell}$ and h_K^{ℓ} is the three-particle Raynal Revai coefficient [7]. The equality

$$(\cos\varphi)^2\, w_K^{\ell}(\varphi) = \sum_{s=-1,0,1} d_{Ks}^{\ell}\; w_{K+2s}^{\ell}(\varphi) \tag{12a}$$

with coefficients

$$d_{K-2\delta_{s1},s}^{\ell} = \delta_{s0}\,[\,1 - \ell(\ell+1)/(K+1)(K+3)\,]/2 + \tag{12b}$$
$$[\,n(2n+1)(n+\ell+1)(K+\ell+1)/K(K+2)\,]^{1/2}/2(K+1)$$

follows from recurrence relations for Jacobi polynomials[6].

Now we substitute ansatz (8b) into eq.(9) and with the help of eqs.(10-12) we obtain the equation $p^2 = (m+2)^2$ for the separation constant and a matrix equation for the column

$$B^{\ell} = (b_{\ell}^{\ell}, b_{\ell+2}^{\ell}, \ldots, b_m^{\ell})^T$$

of unknown coefficients. The latter equation may be written in the two equivalent forms

$$A^{\ell}(\alpha)\, B^{\ell} = 0 , \tag{13a}$$

$$A^{\ell}(0)\,B^{\ell} = \alpha\, S^{\ell}\, B^{\ell} \qquad (13b)$$

with the diagonal matrix

$$\mathrm{diag}\, S^{\ell} = (S^{\ell}_{\ell}, S^{\ell}_{\ell+2}, \dots, S^{\ell}_{m})$$

and the three-diagonal matrix A^{ℓ}. The elements of its main ($s=0$), upper ($s=1$) and lower ($s=-1$) diagonals are equal to

$$a^{\ell}_{Ks}(\alpha) = [\,p^2 - (K+2s+2)^2\,]\, d^{\ell}_{K+2s,-s} - \alpha\, S^{\ell}_{K}\, \delta_{s0}\,, \qquad (14)$$

where index K runs with step two from $\ell+2\delta_{-1,s}$ to $m-2\delta_{1s}$, and the coefficients d^{ℓ}_{Ks} are defined by eq.(12b). Also we reduce the problem (5) with assumed solutions (8) to the matrix problem (13). Next questions are: when the latter problem has solutions and how one can find these solutions?

There are only two cases, namely, $K'=2$, $\ell=0$ and $K'=\ell=1$, when the matrix elements $S^{\ell}_{K'}$, vanish /7/ and the operator (11) acting on the function $w^{\ell}_{K'}$ yields identically zero. In these cases eq.(13) is the identity relation if $m=K'$ and $b^{\ell}_{K}=\delta_{KK'}$ and the corresponding solution (8), i.e. the function

$$U^{\ell} = Z_{K'+2}(\sqrt{E}\,r)\, w^{\ell}_{K'}(\varphi) \qquad (15)$$

satisfies eq.(5) for any parameter α. The solution (15) with $K'=2$, $\ell=0$ was first obtained in ref./8/ and was further analysed in ref./9/.

The homogeneous eq.(13a) has the solutions B^{ℓ} with two or more nonzero elements if and only if the parameter α is a zero of the matrix $A^{\ell}(\alpha)$ determinant /10/. According to eqs.(14) the matrix element a^{ℓ}_{m0} is proportional to the parameter α and the matrix element a^{ℓ}_{K0} is independent of this parameter if $S^{\ell}_{K}=0$. Therefore $\alpha=0$ is a zero of $\det A^{\ell}(\alpha)$ and the number of remaining zeros does not exceed the number

$$n = (m-\ell)/2 - \delta_{\ell 0} - \delta_{\ell 1}\,. \qquad (16)$$

If the matrix $A^{\ell}(\alpha)$ has the diagonal being dominant, then its determinant is not vanishing /10/. Therefore all zeros of this determinant satisfy the set of inequalities

$$|\,a^{\ell}_{K0}(0) - \alpha\, S^{\ell}_{K}\,| < \sum_{s=-1,1} a^{\ell}_{Ks} \qquad (17)$$

with $K=\ell, \ell+2, \dots, m$.

If the matrix A^{ℓ} has a small dimension, then using eqs.(12b), (14) and eq.(36) of ref./7/ for coefficients h^{ℓ}_{K} one can obtain the solutions to eq.(9) in a closed form.

<u>The case $\ell=0$</u>. If $m=0$, then only the trivial solution $B^0=(0)$ exists. If $m=2$, then $B^0=(0,1)^T$ is the solution for any parameter α. For $m=4$ the single solution

$$\alpha = 4\,, \quad B^0 = (-5, 4, 5)^T$$

exists, and for $m=6$ already two solutions appear

$$\alpha = 9 \pm \sqrt{11}\,,\quad B^0 = \left(4,\ \alpha_{\pm}-10,\ \frac{60}{21-\alpha_{\pm}}\,,\ \frac{280}{\alpha_{\pm}(21-\alpha_{\pm})}\right)^T.$$

<u>The case $\ell=1$</u>. The solution $B^1=(1)$ is unique for $m=1$ and any parameter α. If $m=3$, then eq.(13) has only a trivial solution, and if $m=5$, the unique solution reads

$$\alpha = 79/6\,,\quad B^1 = (79\sqrt{3},\ -395,\ 120\sqrt{2}/S^1_5)^T\,.$$

<u>The case $\ell>1$</u>. If $m=\ell$, then only the trivial solution exists. If $m=\ell+2$, then the unique solution reads

$$\alpha = 6/[1+2(-2)^{-\ell}]\,,\quad B^{\ell} = (\sqrt{3(\ell+4)}\, S^{\ell}_{\ell+2},\ \sqrt{2\ell+3}\, S^{\ell}_{\ell})^T\,.$$

In the general case Raynal-Revai coefficients are complicated sums /7/, therefore a more detailed analysis of eqs.(13) with arbitrary indices ℓ and m is difficult. But a numerical solution of these equation is in principle simple. Actually, the methods of numerical solution of the general eigenproblem like eq.(13b) is well-known /11/. Moreover, the elements of the matrix $A^{\ell}(0)$ are simple functions (14) of integer numbers and Raynal-Revai coefficients can easily be calculated by using eqs.(4), (11). Really, the coefficient h^{ℓ}_{K} owing to (7) is the ratio of integral (4) with $U^{\ell}=w^{\ell}_{K}$ to the function w^{ℓ}_{K} taken at the same point φ.

As an illustrative example, we study the problem (13b) in the case $\ell=0$. Some calculated eigenvalues to this problem are listed in the table. From inequalities (17) with $\ell=0$ and any m we have $\alpha>0$.

Table. Some eigenvalues of the problem (13b) with $\ell=0$.

m	α				
8	5.194	12.000	30.806			
10	5.838	16.990	39.401	59.104		
12	6.063	18.953	28.239	73.938	106.807	
14	5.816	17.051	31.851	53.811	103.378	170.427

Hence if $\ell = 0$ and $\alpha < 0$, i.e. the potential (1) is attractive, then eq.(9) has any solution like (8b). Numerical investigation of this equation with a certain negative parameter α was performed by Avishai [3].

Now we substitute each solution (8) of the Faddeev equation (5) into eq.(2) and using eq.(11) write the corresponding solution to the Schrödinger equation in the form

$$\Psi^{\varepsilon} = r^{-2} Z_p(\sqrt{E}\, r) \sum_{K=\ell}^{m} b_K^{\ell} S_K^{\ell} \left[2 w_K^{\ell}(\varphi)\, \mathcal{Y}^{\varepsilon}(\hat{x}, \hat{y}) / \sin 2\varphi \right] \quad (18)$$

with the polyspherical hyperharmonics [5] in the square brackets and $p = \pm(m+2)$. The solution (18) is zero identically if and only if its Faddeev component has the form (15) with $K' = 2$, $\ell = 0$ or $K' = \ell = 1$. The potential (1) in nature is a centrifugal potential, therefore the functions (18) are more close to the well-known solutions of the free Schrödinger equation. These solutions are similar to (18) with $p = \pm(K+2)$ and one nonzero coefficient b_K^{ℓ} and are used as fundamental functinns for studying the three-identical-particle problem with any S -wave potentials. If the potentials are sums of potential (1) and a more smooth one, then the exact solutions (18), of course if they exist, may be used as fundamental functions.

In conclusion, the main result of the present work may be formulated as follows:

<u>Theorem</u>. The three-identical-particle Schrödinger equation with potentials (1) has solutions (18) with the Faddeev component (8) if and only if the parameter α is an eigenvalue of the corresponding problem (13b). For any fixed indices ℓ and m the eigenvalues satisfy ineqs. (17) and their number does not exceed the number n given by eq.(16).

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Некоторые точные решения задачи трех тождественных частиц в случае S-волновых потенциалов обратно пропорциональных квадрату расстояния

В рамках интегродифференциальных уравнений Фаддеева доказан критерий существования решений уравнения Шредингера, представляемых в виде зависящей от гиперрадиуса функции Бесселя, умноженной на конечную сумму гипергармоник.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Some Exact Solutions of the Three-Identical-Particle Problem with S-Wave Inverse Square Potentials

The existence criterion of exact solutions to the Schrödinger equation as a product of the Bessel function of the hyperradius and a finite sum of hyperharmonics is proved within the Faddeev integrodifferential approach.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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