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TRAPPING MODES IN A CURVED ELECTROMAGNETIC WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

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1. Introduction

The solution of the wave equation in an infinitely long curved planar strip was often discussed in the electromagnetic and/or acoustic waveguide literature [1-6]. In all these works, however, people restricted their attention only to the "scattering" solutions of the corresponding equations, calculating the transmission and reflection coefficients for particular waveguide configurations. It was silently supposed that in a strip with a constant non-zero width solutions of another type do no exist. In particular, it was supposed that there are no square integrable solutions. Actually, this assumption can be mathematically confirmed in the case of sound propagation (Neumann boundary conditions on the strip boundary). It seems to be Atkinson [7] who first showed that the acoustic wave equation in a tube of a constant width has a continuous spectrum only.

On the other hand, from the theory of surface waves (a wave equation with mixed boundary conditions $f + \sigma\frac{\partial f}{\partial n} = 0;\ \sigma \neq 0$) we know that trapping modes (square integrable solutions) appear even on the surface of a straight infinite canal - see [8] and also [9] for the experimental verification ; the name "trapping mode" was invented by F.Ursell in [8].

In this context it seems to be interesting to investigate the existence of the trapping modes inside a bent electromagnetic waveguide with perfectly conducting walls (Dirichlet boundary conditions). It is the aim of the present paper to show that such solutions really exist in this case, provided the waveguide is bent enough.

The wave equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0 \qquad (1)$$

with Dirichlet boundary conditions on boundaries of an infinite extent has been investigated in the mathematical literature for a long time. For instance, for a semiinfinite tube it was shown that the spectrum of k^2 is purely discrete when the tube becomes infinitely narrow at infinity [10] and purely continuous when the tube is conical [11], i.e., if the outward normal at every point of the semiinfinite tube makes an angle not less than $\frac{\pi}{2}$ with a fixed direction. No results are known, however, for the case of a bent tube of a constant width.

Recently the possible existence of square integrable solutions of the wave equation was discussed by Popov [12], who investigated TM waves propagating in a straight waveguide which has a protrusion over a finite length but being otherwise of uniform width. In distinction to this paper we focus our attention to waveguides which are everywhere of a constant width but possibly bent.

Our paper is organized as follows: The main result is formulated in Section 2. Section 3 contains some concluding remarks, while the proofs of the theorems are sketched in the Appendix.

2. The main results

Let us investigate the electromagnetic wave propagating inside an infinite rectangular waveguide with a constant cross-section, the lower and upper walls of which lie in two parallel planes (Fig.1). We suppose the walls of the waveguide being perfectly conducting and we restrict ourselves to the TM-type waves only.

Because of the constant height of the waveguide the problem can be reduced to investigation of the two-dimensional wave equation (1) in a curved planar strip Ω (being the projection of the waveguide to the lower plane) with Dirichlet boundary conditions [13]

$$f(x,y) = 0 \quad ; \; (x,y) \in \partial\Omega \tag{2}$$

on the strip boundary. Here f denotes the z component of the electric field, $f = E_z$.

We are interested primarily in the spectral properties of the equation (1) with the boundary conditions (2). It is therefore reasonable to rewrite it in the operator form

$$\Delta_D f = k^2 f \tag{3}$$

and to investigate its spectrum applying the methods of functional analysis (borrowed from the Schroedinger operator theory). Here Δ_D

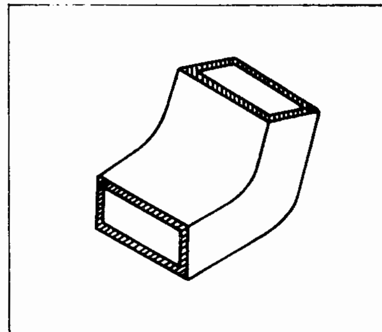

Fig.1

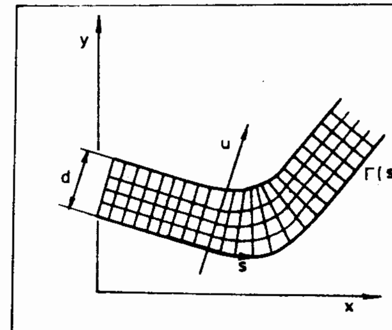

Fig.2

is the Dirichlet Laplacian in the space $L^2(\Omega)$, i.e. it is the Friedrichs extension [14] of the map

$$f \longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f \tag{4}$$

defined on $C_0^\infty(\Omega)$. The strip Ω is supposed to be smooth and of a constant width. Using one of its boundaries as a reference curve Γ, we can introduce natural curvilinear coordinates (s,u) as

$$\begin{aligned} x &= a(s) - ub'(s) \\ y &= b(s) + ua'(s), \end{aligned} \tag{5}$$

where a,b are smooth functions characterizing the curve Γ

$$\Gamma = \{ (a(s),b(s)); \; s \in \mathbb{R} \}. \tag{6}$$

We assume, moreover,

$$a'(s)^2 + b'(s)^2 = 1 \; , \tag{7}$$

so s is the arc length of Γ and u means the distance of the point (x,y) from Γ (Fig.2).

The coordinates (s,u) are locally orthogonal. Therefore the metrics in Ω expresses with respect to them through a diagonal metric tensor,

$$dx^2 + dy^2 = g_{ss} ds^2 + g_{uu} du^2 \tag{8}$$

with

$$\begin{aligned} g_{ss} &\equiv g = (1+u\gamma(s))^2 \\ g_{uu} &= 1, \end{aligned} \tag{9}$$

where $\gamma(s)$ is the signed curvature of the reference curve Γ

$$\gamma(s) = b''(s)a'(s) - a''(s)b'(s) \; . \tag{10}$$

We suppose that the width d of the strip is restricted by the inequality $d\gamma(s) > -1$ for all $s \in \mathbb{R}$ and that the waveguide is bent in a finite region only, i.e. that $\gamma \in C_0^\infty(\mathbb{R})$.

The first step in proving the existence of the trapping modes is to transform the operator Δ_D to the coordinates (s,u). Using the unitary map $U : L^2(\Omega) \longrightarrow L^2(\mathbb{R}\times(0,d))$ given by

$$(Uf)(s,u) = g^{1/4}(s,u)f(s,u) \qquad (11)$$

we get after straightforward differentiation that Δ_D is unitarily equivalent to an operator A defined on $L^2(\mathbb{R}\times(0,d))$ by the differential expression

$$Af = \left[\frac{\partial}{\partial s} g^{-1} \frac{\partial}{\partial s} + \frac{\partial^2}{\partial u^2} \right] f + V(s,u)f \qquad (12)$$

and by the Dirichlet boundary conditions $f(s,0) = f(s,d) = 0$ for all $s \in \mathbb{R}$. Here

$$V(s,u) = \frac{1}{2} g^{-3/2} \frac{\partial^2 \sqrt{g}}{\partial s^2} - \frac{5}{4} g^{-2} \left[\frac{\partial \sqrt{g}}{\partial s} \right]^2 - \frac{1}{4} g^{-1} \left[\frac{\partial \sqrt{g}}{\partial u} \right]^2 . \qquad (13)$$

Hence to prove that the wave equation (1) has square integrable solutions , one has to check that the operator A has a non-empty discrete spectrum.

<u>Theorem 1</u>

Let us suppose that the strip Ω is bent nontrivially , i.e.,that $\gamma(s)$ is non-zero for some s. Then there is a positive number d_o such that for each $d < d_o$, A has at least one bound state.

The magnitude of d_o can be estimated by

<u>Theorem 2</u>

The operator A has at least one bound state if

$$\int_{\mathbb{R}} \int_0^d \left[- \frac{\gamma(s)^2}{(1+u\gamma(s))^2} + \frac{u^2 \gamma'(s)^2}{(1+u\gamma(s))^4} \right] \sin\left(\frac{\pi u}{d} \right) du\, ds < 0 . \qquad (14)$$

Proofs of these two Theorems are sketched in the Appendix.

3. Discussions

Let us now discuss some physical consequences of Theorem 1. First of all we have to mention that as far as we know the existence of trapping modes inside a curved waveguide has been overlooked in both the theoretical and mathematical literature. On the other hand, the trapping modes can manifest themselves in an experimentally verifiable way. In order to illustrate this we would briefly discuss the energy transmission through a bent waveguide of a finite length .

Mathematically speaking, once we deal with a waveguide of a <u>finite length</u> L the trapping modes disappear from the spectrum turning into resonances the imaginary parts of which approach zero as $L \longrightarrow \infty$. (This can be rigorously proven using the asymptotic perturbation theory and spectral concentration results [14].) Physically it means that the trapping modes would contribute to the energy transfer through the truncated waveguide. This contribution would be substantial especially for waves with frequency below the first transversal mode of the waveguide, leading in such a way to a resonant peak in the energy transfer plot. The peak is supposed to be sharp for waveguides long enough, when the imaginary part of the trapping mode resonance tends to zero.

A similar resonant peak was observed in a sound transmission through bent pipes (see [15] for experimental and [16] for theoretical results). Moreover the peak was localized below the frequency of the second transversal mode. This is in a close analogy with our results, since a direct computation shows that for a rectangularly bent waveguide the trapping mode appears at $k^2 = 0.93\, k_1^2$, $k_1 = \frac{\pi}{d}$ being the first transversal mode.

Let us now comment briefly on Theorem 2. Using it one can obtain simple bounds on the width d_o below which at least one trapping mode **appears.** For a simply bent strip (i.e., $\gamma(s) \ge 0$ for all $s \in \mathbb{R}$) we find that

$$d_o \ge \frac{1}{2\gamma_+} \left[\sqrt{1+4\gamma_+\sqrt{z}} - 1 \right] , \qquad (15)$$

where

$$z = \left[\int_{\mathbb{R}} (\gamma'(s))^2 ds \right]^{-1} \int_{\mathbb{R}} \gamma^2(s)\, ds \qquad (16)$$

and $\gamma_+ = \max \gamma(s)$. This bound shows that the inverse d_o^{-1} of the critical width is of the same order of magnitude as the maximal curvature γ_+ .

Appendix

Before proving Theorems 1 and 2 we formulate a simple Lemma concerning the operator $H_\lambda = - \Delta_D + \lambda V$ on $L^2(\mathbb{R}\times[0,d])$, where Δ_D is the Dirichlet Laplacian.

Lemma

Suppose that $V:\mathbb{R}\times[0,d] \longrightarrow \mathbb{R}$ is a bounded and measurable function such that

$$\int_{\mathbb{R}}\int_0^d V(x,y)\,(1+x^2)\,dx\,dy < \infty . \tag{17}$$

Then H_λ has at least one bound state $E(\lambda)$ below the bottom of the continuous spectrum, $E(\lambda) < (\pi^2/d^2)$, if

$$\int_{\mathbb{R}}\int_0^d V(x,y)\,\sin^2(\tfrac{\pi}{d}\,y)\,dx\,dy < \infty . \tag{18}$$

Proof: We use the Birman-Schwinger principle [14] according to which $E(\lambda)$ is a bound state of H_λ if and only if the operator λK_α has an eigenvalue -1 for $\alpha^2 = E(\lambda)$. Here K_α is an integral operator with the kernel

$$K_\alpha(x,y,x',y') = |V(x,y)|^{1/2}R_0(\alpha,x,y,x',y')V(x',y')^{1/2}, \tag{19}$$

where $R_0(\alpha,\dots)$ is the kernel of $R_0(\alpha) = (-\Delta_D - \alpha^2)^{-1}$ and $V^{1/2} = |V|^{1/2}\operatorname{sgn} V$. Separating the variables we can decompose $R_0(\alpha,x,y,x',y')$ as [17]

$$R_0(\alpha,x,y,x',y') = \sum_{n=1}^{\infty} \chi_n(y)\; r_n(\alpha,x,x')\; \chi_n(y'), \tag{20}$$

where $\chi_n(y) = (2/d)^{1/2}\sin(\frac{\pi n}{d}y)$ is the n-th transversal-mode wavefunction and $r_n(\alpha,x,x')$ is the kernel of

$$\left(-\frac{d^2}{dx^2} + \frac{\pi^2 n^2}{d^2} - \alpha^2\right)^{-1}.$$

Using (20) we can divide the kernel (19) into two parts

$$K_\alpha = M_\alpha + L_\alpha, \tag{21}$$

where

$$M_\alpha(x,y,x',y') = |V(x,y)|^{1/2}\sum_{n=2}^{\infty} \chi_n(y)\,\frac{\exp(-k_n(\alpha)|x-x'|)}{2k_n(\alpha)}\,\chi_n(y')V(x',y')^{1/2}$$

$$+\; |V(x,y)|^{1/2}\chi_1(y)\,\frac{\exp(-k_1(\alpha)|x-x'|) - 1}{2k_1(\alpha)}\,\chi_1(y')V(x',y')^{1/2}\;; \tag{22}$$

$$L_\alpha(x,y,x',y') = \frac{1}{2k_1(\alpha)}\,|V(x,y)|^{1/2}\chi_1(y)\chi_1(y')\;V(x,y)^{1/2}$$

and

$$k_n(\alpha) = \left(\frac{n^2\pi^2}{d^2} - \alpha^2\right)^{1/2}.$$

The operator M_α is bounded for $\alpha \in [0,\pi/d]$. Therefore for $\alpha \in [0,\pi/d]$ and λ sufficiently small we have

$$\|\lambda M_\alpha\| < 1 \tag{23}$$

so

$$(1+\lambda K_\alpha)^{-1} = [\,1+\lambda(1+\lambda M_\alpha)^{-1}L_\alpha\,]^{-1}(1+\lambda M_\alpha)^{-1}. \tag{24}$$

Hence λK_α has the eigenvalue -1 iff the same is true for $\lambda(1+\lambda M_\alpha)^{-1}L_\alpha$. This operator is, however, of rank one and has therefore just one non-zero eigenvalue $\zeta(\lambda)$ given by

$$\zeta(\lambda) = \frac{\lambda}{2k_1(\alpha)}\int_{\mathbb{R}}\int_0^d V(x,y)^{1/2}\chi_1(y)\left[(1+\lambda M_\alpha)^{-1}|V|^{1/2}\chi_1\right](x,y)\,dxdy. \tag{25}$$

Using the formula (25) we find for λ small

$$\zeta(\lambda) = \frac{\lambda}{2k_1(\alpha)}\left[\int_{\mathbb{R}}\int_0^d V(x,y)\chi_1(y)^2\,dxdy + O(\lambda)\right] \tag{26}$$

and solving the equation $\zeta(\lambda) = -1$ we get finally

$$k_1(\lambda) = -\frac{\lambda}{2}\int_{\mathbb{R}}\int_0^d V(x,y)\chi_1(y)^2\,dxdy + O(\lambda^2)\;. \tag{27}$$

Moreover $k_1(\lambda)$ corresponds to a bound state of H_λ iff $k_1(\lambda) > 1$, which proves the Lemma

Proof of Theorem 1:

We estimate first the operator A from above by an operator A_+

$$A_+ = -\frac{1}{(1+d\gamma_-)^2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + V(s,u)\,, \tag{28}$$

where $\gamma_- = \inf \gamma(s)$. The function $V(s,u)$ fulfils the condition (17) of the Lemma. We find in such a way that A_+ has at least one bound state if

$$\int_{\mathbb{R}}\int_0^d V(s,u)\,\sin^2(\frac{\pi u}{d})\,ds\,du < 0 \tag{29}$$

which is obviously true for d small enough. Since $A \le A_+$, and both A and A_+ have the same continuous spectra, the existence of the bound state of the operator A_+ implies the existence of the bound state of the operator A.

Proof of Theorem 2:

As already mentioned in the proof of Theorem 1, we have only to check (29) and to apply the Lemma. The part of $V(s,u)$ containing the second derivative of $\gamma(s)$ may be integrated by parts, which yields the condition of the Theorem.

References:

1. Campbell J.J., Jones W.R.:IEEE Transactions on Microwave Theory and Technique, MTT 16 (1968) 517
2. Kirilenko A.A., Rudj L.A., Tkatchenko V.I.: Radiotechnika i Elektronika 30 (1985) 918 (in Russian)
3. Kirilenko A.A., Rudj L.A.,Shestopalov: Prikladnaja Elektronika, Vyschaja Shkola, Moscow 1978 (in Russian)
4. Kirilenko A.A., Rudj L.A.,Shestopalov: Radiotechnika i Elektronika 4 (1974) 687 (in Russian)
5. Mehran R: IEEE Transactions on Microwave Theory and Technique MTT 26 (1978) 400
6. Marcuritz M.: Waveguide Handbook, Dover Publ., New York 1964
7. Atkinson F.V.: Phil.Mag.40 (1949) 645
8. Ursell F.: Proc.Camb.Phil.Soc. 47 (1951) 347
9. Ursell F.: Proc.Roy.Soc. 214 (1952) 79
10. Rellich F.:in Studies and Essays Presented to R.Courant; New York, 1948
11. Jones D.S.: Proc.Camb.Phil.Soc. 49 (1953) 668
12. Popov A.N.: Journal of Technical Physics 56 (1986) 1916 (in Russian)
13. Lewin L.: Theory of Waveguides; Butterworth Publ. London 1975
14. Reed M.,Simon B.: Methods of Modern Mathematical Physics vol.4, Academic Press, New York 1978
15. Lippert W.K.R.: Acustica 5 (1955) 274
16. Kacenelenbaum B.Z.: The Theory of Nonregular Waveguides with Slowly Varying Parameters; Moscow 1961 (in Russian)
17. Petlenko V.A., Khizhniak N.A.:Izvestija VUZ 21 (1978) 1325 (in Russian)

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Экснер П., Шеба П. E5-88-379

Существование связанного состояния в изогнутом волноводе с идеально проводящими стенками

Пользуясь методами, разработанными в теории операторов Шредингера, мы показываем, что волновое уравнение для изогнутого электромагнитного волновода с идеально проводящими стенками имеет квадратично интегрируемые решения /запертые моды/.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Trapping Modes in a Curved Electromagnetic Waveguide with Perfectly Conducting Walls

Using methods developed in the Schroedinger operator theory we show the existence of square integrable solutions (trapping modes) of the wave equation inside a curved electromagnetic waveguide with perfectly conducting walls.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1988