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E5-88-284

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ON A CLASS OF 6j COEFFICIENTS WITH ONE MULTIPLICITY INDEX FOR UNITARY GROUPS

Submitted to "Journal of Mathematical Physics"

1988

I. Introduction

Recently we have developed a new approach in order to obtain an important class of the 6j symbols.[1] In this paper we extend the same method and we find an analogous class of 6j coefficients for the algebras su(2j+1) (j- integer or half-integer). We are interested in the following symbols

$$\Phi(\Omega)^{+}_{m\alpha} = \Phi(\Omega,\Omega^{+}_{m})_{\alpha} = \begin{Bmatrix} 1^{*} & 1 & \Lambda \\ \Omega & \Omega & \Omega^{+}_{m} \end{Bmatrix}_{..\alpha}, \qquad (1.1a)$$

$$\Phi(\Omega)^{-}_{m\alpha} = \Phi(\Omega,\Omega^{-}_{m})_{\alpha} = \begin{Bmatrix} 1 & 1^{*} & \Lambda \\ \Omega & \Omega & \Omega^{-}_{m} \end{Bmatrix}_{..\alpha}, \qquad (1.1b)$$

where Ω stands for the maximal weight vector $\Omega \equiv (\Omega_{j},\Omega_{j-1},\ldots,\Omega_{-j})$ and labels a unitary irreducible representation (UIR). The complex conjugate representation will be denoted by $\Omega^{*}=(-\Omega_{-j},-\Omega_{-j+1},\ldots,-\Omega_{j})$. The quantities $\Omega_{m}-\Omega_{m-1}$ are positive integers and two representations Ω^{1} and Ω^{2} such that $\Omega^{1}_{m}=\Omega^{2}_{m}+c$ (c- a real number) are equivalent. The Λ is the adjoint representation $\Lambda_{0}=(100\ldots0{-1})$ or the scalar representation $(00\ldots0)\equiv(\underline{0})$. Sometimes we omit the parentheses in the notation if the meaning is obvious, as, for example in (1.1a). The α is the multiplicity index, so in the Kronecker product $\Omega\times\Lambda_{0}$ the representation Ω may be found more than once. Other multiplicity indices do not occur in our symbols, so we use dots in their place. We use the same definition of the 6j symbols as in Ref. 2, but in our notation we change the order of the multiplicity indices (see the eq.(2.13)). In the expressions (1.1 a,b) the index m on the left hand side is fixed from Ω^{+}_{m}, Ω^{-}_{m} in the following way

$$\Omega^{\pm}_{m} = (\Omega_{j},\Omega_{j-1},\ldots,\Omega_{m+1},\Omega_{m}\pm1,\Omega_{m-1},\ldots,\Omega_{-j}).$$

We assume that the index α is running up to j' to -j'+1 when

$\Lambda=\Lambda_0$ and it takes the value $-j'$ for Λ equal to $(\underline{0})$. In what follows we will understand that $\Phi(\Omega)^{\pm}$ are the square $2j'+1$ dimensional matrices whose dimensions depend on Ω

$$j' = \frac{1}{2} \sum_m \Delta_m \,, \qquad (1.2a)$$

$$\Delta_m = \begin{cases} 1, & \text{if } \Omega_m > \Omega_{m-1}\,, \\ 0, & \text{otherwise,} \end{cases} \qquad (1.2b)$$

$$\Delta^{+}_{m-1} = \Delta^{-}_{m} = \Delta_m \,, \qquad \text{for } m = j, j-1, \ldots, -j+1, \qquad (1.2c)$$

$$\Delta^{+}_{j} = \Delta^{-}_{-j} = 1 \;. \qquad (1.2d)$$

The unitarity properties of $\Phi(\Omega)^{\pm}$ take the form

$$\sum_m D^{\pm}_m D_a \, \Phi(\Omega)^{\pm}_{ma} \, \Phi(\Omega)^{\pm\,*}_{mb} = \delta_{ab} \,, \qquad (1.3a)$$

$$\sum_a D^{\pm}_m D_a \, \Phi(\Omega)^{\pm}_{ma} \, \Phi(\Omega)^{\pm\,*}_{na} = \delta_{mn} \Delta^{\pm}_m \,, \qquad (1.3b)$$

where D_a is the dimension of the representation $\bar{\Lambda}_a$ ($D_a = 4j(j+1)$ for $a = j', j'-1, \ldots, -j'+1$, $D_{-j'} = 1$) and $D^{\pm}_m$ are the dimensions of the representations $\Omega^{\pm}_m$

$$D^{\pm}_m = D_\Omega \prod_{a(\neq m)} \frac{\omega_m \pm 1 - \omega_a}{\omega_m - \omega_a} \,, \qquad (1.4)$$

$$\omega_m = \Omega_m - \frac{1}{2j+1} \sum_k \Omega_k + m \;. \qquad (1.5)$$

II. The symmetries and phase convention

To obtain the symmetry relations of the investigated 6j symbols we must check the symmetries for the forthcoming 3j symbols. For the above we may assume the simple rules[3] (see also text below (3.16))

$$(\Omega_1 \Omega_2 \Omega_3)_{M_1 M_2 M_3 a} = \mathscr{P}(\Omega_1 \Omega_2 \Omega_3, P)_a \times$$
$$\times (\Omega_{P(1)} \Omega_{P(2)} \Omega_{P(3)})_{M_{P(1)} M_{P(2)} M_{P(3)} a} \,, \qquad (2.1)$$

$$\mathscr{P}(\Omega_1 \Omega_2 \Omega_3, P)_a = (-1)^{S_P (\sum_i [\Omega_i] + \pi_a)} \,,$$

where $S_P = -1$ for the odd and $S_P = 1$ for the even permutation P and

$$[\Omega] = [\Omega^*] = \sum_m m\, \omega_m \,, \qquad (2.2)$$

The factor $\pi_a = \pi(\Omega_1 \Omega_2 \Omega_3)_a$ does not depend on the order $\Omega_1, \Omega_2, \Omega_3$ and we have

$$\pi_a = \begin{cases} 2j+1, & \text{if } a = j' \\ 2j, & \text{if } a = j'-1, j'-2, \ldots, -j'+1, \\ 0, & \text{if } a = -j', \end{cases} \qquad (2.3a,b,c)$$

$$\pi(1, \Omega, (\Omega^{+}_m)^*) = \pi(\Omega^*, 1, \Omega^{-}_m) = 0 \;. \qquad (2.3d)$$

The function $[\Omega]$ is related to the metric tensor $(\Omega)^{M_1 M_2^*}$ which is used to lower and to rise the tensor indices

$$F(\Omega)_M{}^* = F(\Omega)^M = (\Omega)^{M K^*} F(\Omega^*)_{K^*} \,, \qquad (2.4a)$$

$$F(\Omega)_M = F(\Omega^*)^{K^*} (\Omega^*)_{K^* M} \,, \qquad (2.4b)$$

and the standard phase convention is

$$(\Omega)^{M_{max} M^*_{min}} = (\Omega^*)^{M^*_{max} M_{min}} = 1 \,, \qquad (2.5)$$

where M_{max}, M^*_{max} (M_{min}, M^*_{min}) are the maximal (minimal) states respectively for Ω and Ω^*. The phase convention chosen by Gelfand and Cetlin[4] in their expressions for the matrix elements of the generators $A(\Lambda_0)$ leads to the relations

$$(\Omega)^{M_1 M_2^*} = (\Omega)_{M_1 M_2^*} = (-1)^{[\Omega] - (M_1)} \delta(M_1, M_2) \,, \qquad (2.6a)$$

$$(\Omega^*)^{M_1^* M_2} = (\Omega^*)_{M_1^* M_2} = (-1)^{[\Omega] - (M_1^*)} \delta(M_1, M_2) \,, \qquad (2.6b)$$

where the function (M) depends only on the eigenvalues $h(M)m$ of the elements $A^m{}_m$ ($m = j, j-1, \ldots, -j$) with Cartan subalgebra and we have

$$(M) = \sum_m m\, h(M)m \;. \qquad (2.7)$$

The expressions (2.2), (2.6a,b) and (2.7) are rearranged forms of the result obtained earlier by Baird and Biedenharn.[5] The assumption $(\Omega)_{M_1 M_2^*} = D(\Omega)^{1/2}(\Omega\ 0\ \Omega)_{M_1 0 M_2^*}$ leads to (2.5c). Next we introduce the generalization of the Wigner-Eckart theorem

$$\langle \Omega_1 M_1 | T(\Omega_2)_{M_2} | \Omega_3 M_3 \rangle = (-1)^{[\Omega_1]-[\Omega_2]+[\Omega_3]} \times$$

$$\times \sum_{K^*,a} (\Omega)^{M_1 K^*} (\Omega_3 \Omega_2 \Omega_1^*)_{M_3 M_2 K^* a} (\Omega_1 \| T(\Omega_2) \| \Omega_3)^a . \tag{2.8}$$

The above remains invariant with respect to any unitary transformations acting in the multiplicity space, but applying (2.8) to the matrix elements of the su(2j+1) generators $T(\Omega)= A(\Lambda_0)$ we must make some choice for this very peculiar case of the operators

$$(\Omega \| A(\Lambda_0) \| \Omega)^a = \delta^a_{j'}(4j(j+1)C_\Omega D_\Omega)^{1/2} , \tag{2.9}$$

$$C_\Omega = \sum_m (\omega_m^2 - m^2) . \tag{2.10}$$

Taking into the consideration that $h(M)m=-h(M^*)m$ and using (1.13) for $T(\Omega)=A^m{}_m$ we get

$$(-1)^{[\Lambda_0]+\pi(\Omega\Lambda_0\Omega^*)_{j'}} = -1 ,$$

which leads to (2.3a). The results (2.3a,b,c) are, in general, phase dependent, except the case when Ω is a selfcomplex conjugate representation. Here ($\Omega\equiv\Omega^*$) the (2.3a,b,c) should be consistent with predictions which follow from the symmetric and antisymmetric plethysm[6] calculation. For example, for $\Omega=\Lambda_0$ we have

$$(1\underline{0}-1)\otimes(2) = (1\underline{0}-1) + (11\underline{0}-1-1) + (2\underline{0}-2) + (\underline{0}) , \tag{2.11a}$$

$$(1\underline{0}-1)\otimes(11) = (1\underline{0}-1) + (2\underline{0}-1-1) + (11\underline{0}-2) , \tag{2.11b}$$

and we see that both symmetries lead on the r.h.s. to the representation Λ_0, as is seen from (2.11a,b). The general proof of the above consistency will not be presented here. The relation (2.3d)

is elementary, here only the cases when Ω or Ω_m are equal to $(1\underline{0})$ are phase independent and we immediately obtain it for these cases. From (2.1) we find

$$\begin{Bmatrix} \Omega_1\Omega_2\Omega_3 \\ \Lambda_1\Lambda_2\Lambda_3 \end{Bmatrix}_{a_1a_2a_3a_0} = \mathcal{P} \begin{Bmatrix} \Omega_2\Omega_1\Omega_3 \\ \Lambda_2^*\Lambda_1\Lambda_3 \end{Bmatrix}_{a_2a_1a_3a_0} =$$

$$= \mathcal{P} \begin{Bmatrix} \Omega_1\Omega_3\Omega_2 \\ \Lambda_1^*\Lambda_3^*\Lambda_2^* \end{Bmatrix}_{a_1a_3a_2a_0} , \tag{2.12a,b}$$

$$\begin{Bmatrix} \Omega_1\Omega_2\Omega_3 \\ \Lambda_1\Lambda_2\Lambda_3 \end{Bmatrix}_{a_1a_2a_3a_0} = \begin{Bmatrix} \Omega_1^*\Lambda_2\Lambda_3^* \\ \Lambda_1^*\Omega_2\Omega_3^* \end{Bmatrix}_{a_0a_3a_2a_1} =$$

$$= \begin{Bmatrix} \Lambda_1\Lambda_2^*\Omega_3^* \\ \Omega_1\Omega_2^*\Lambda_3^* \end{Bmatrix}_{a_2a_1a_0a_3} , \tag{2.12c,d}$$

$$\mathcal{P} = (-1)^{\pi_{a_1}+\pi_{a_2}+\pi_{a_3}+\pi_{a_0}} .$$

One also gets the general relations[2]

$$\begin{Bmatrix} \Omega_1\Omega_2\Omega_3 \\ \Lambda_1\Lambda_2\Lambda_3 \end{Bmatrix}^*_{a_1a_2a_3a_0} = \begin{Bmatrix} \Omega_1^*\Omega_2^*\Omega_3^* \\ \Lambda_1^*\Lambda_2^*\Lambda_3^* \end{Bmatrix}^{a_1a_2a_3a_0} =$$

$$= G(\Omega_1\Lambda_2^*\Lambda_3)^{a_1b_1} \times G(\Lambda_1\Omega_2\Lambda_3^*)^{a_2b_2} \times G(\Lambda_1^*\Lambda_2\Omega_3)^{a_3b_3} \times G(\Omega_1^*\Omega_2^*\Omega_3^*)^{a_0b_0} \times$$

$$\begin{Bmatrix} \Omega_1^*\Omega_2^*\Omega_3^* \\ \Lambda_1^*\Lambda_2^*\Lambda_3^* \end{Bmatrix}_{b_1b_2b_3b_0} . \tag{2.13}$$

The multiplicity metric tensor (m.m.t.) $G(\Omega_1\Omega_2\Omega_3)$ is unitary, here also symmetric and independent of the order $\Omega_1,\Omega_2,\Omega_3$ (2.1). By the definition we have[2]

$$(\Omega_1\Omega_2\Omega_3)^{M_1M_2M_3a} = G(\Omega_1\Omega_2\Omega_3)^{a\ b} \cdot (\Omega_1\Omega_2\Omega_3)^{M_1M_2M_3}{}_b ,$$

$$(\Omega_1\Omega_2\Omega_3)_{M_1M_2M_3a} = (\Omega_1\Omega_2\Omega_3)_{M_1M_2M_3}{}^b\ G(\Omega_1^*\Omega_2^*\Omega_3^*)_{b\ a} .$$

The one dimensional m.m.t. referring to the triad $(1,\Omega,(\Omega_m^+)^*)$ is equal to 1. Taking into consideration (2.12a), (2.13) and from the fact that the 6j symbols to be obtained in the next section are real, we get

$$G(\Omega,\Omega^{*},0) = 1 \ ,$$

$$G(\Omega,\Omega,\Lambda_{o}) = \mathrm{diag}(1,-1,-1,\ldots,-1) \ .$$

Considering also the fact that $(\Omega^{\pm}_{m})^{*} = (\Omega^{*})^{\mp}_{-m}$ we find

$$\Phi(\Omega)^{\pm}_{m\alpha} = (-1)^{2j+1+\pi_{\alpha}}\ \Phi(\Omega^{*})^{\mp}_{-m\alpha} \ . \tag{2.14}$$

III. Canonical form for the matrices $\Phi(\Omega)^{\pm}$.

Since the multiplicity dependent 3j symbols and hence also the 6j coefficients are not defined uniquely, any unitary transformation acting in the multiplicity space leads to an equivalent set of 3j and respectively 6j symbols. Applying the above to the matrices $\Phi(\Omega)^{\pm}$ we must take into account equation (2.9) and also (2.5). We ad here the point that the above must be preserved and we propose to introduce some further requirements for obtaining the $\Phi(\Omega)^{\pm}$ in the unique form.

Definition

We say that the 6j symbols $\Phi(\Omega)^{+}$ and $\Phi(\Omega)^{-}$ are in the canonical form if the matrices

$$H(\Omega)^{\pm}_{a\ b} = (C_{\Omega}D_{\Omega}D_{a}D_{b}D_{j'})^{1/2}h(\Omega)^{\pm}_{a\ b} \ , \tag{3.1}$$

$$h(\Omega)^{\pm}_{a\ b} = \sum_{m} \mathcal{P}_{ma}\mathcal{P}_{mj'}\mathcal{P}_{mb}\ \mathcal{D}^{\pm}_{m}\ \Phi(\Omega)^{\pm *}_{ma}\ \Phi(\Omega)^{\pm}_{mj'}\ \Phi(\Omega)^{\pm}_{mb} \ , \tag{3.2}$$

$$\mathcal{P}_{m\alpha} = (-1)^{[\Omega] + [\Omega^{\pm}_{m}] + [\Lambda_{\alpha}] + [1\underline{0}]} \ ,$$

take the form

$$H(\Omega)^{\pm} = \begin{bmatrix} h^{\pm}_{j'} & \pm X_{j'-1} & \pm X_{j'-2} & \cdots & \pm X_{-j'+1} & X_{-j'} \\ \pm X^{*}_{j'-1} & h^{\pm}_{j'-1} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pm X^{*}_{-j'+1} & 0 & \cdots & \cdots & h^{\pm}_{-j'+1} & 0 \\ X^{*}_{-j'} & 0 & \cdots & \cdots & 0 & h_{-j'} \end{bmatrix} . \tag{3.3}$$

Theorem:

(A) For the regular representation $(\Omega_{j} > \Omega_{j-1} > \ldots > \Omega_{-j})$ the diagonal elements for the matrices $H(\Omega)^{\pm}$ are of the form

$$h^{\pm}_{\alpha} = \tfrac{1}{2}\,(2j + 1 \mp q_{\alpha}) \ , \ \text{for } \alpha = j, j-1, \ldots, -j+1 \ , \tag{3.4a}$$

$$q_{j} = 0 \ , \tag{3.4b}$$

$$h_{-j} = 0 \ , \qquad q_{-j} = 2j+1 \ , \tag{3.4c,d}$$

and the remaining components of the vector $q = (q_{j}, q_{j-1}, \ldots, q_{-j})$ are calculated from the equation

$$1 = \frac{q + q_{-j}}{q - q_{-j}} \prod_{m} \frac{2\omega_{m} + 1 - q}{2\omega_{m} - 1 - q} \ . \tag{3.5}$$

The off-diagonal elements are obtained from the relation

$$X_{\alpha} = \tfrac{1}{2}\left\{\frac{-\prod_{m}(q_{\alpha} - 2\omega_{m} - 1)}{\prod_{b(\neq \alpha, j)}(q_{\alpha} - q_{b})}\right\}^{1/2} . \tag{3.6}$$

(B) The elements of the matrices $\Phi(\Omega)^{\pm}$ are calculated from the expressions

$$\Phi(\Omega)^{\pm}_{mj} = \mathcal{P}_{mj}\mathcal{N}\,(j \mp \omega_{m}) \ , \tag{3.7.a}$$

$$\Phi(\Omega)^{\pm}_{m\alpha} = \frac{2\,X_{\alpha}}{\pm q_{\alpha} \mp 2\omega_{m} - 1}\ \Phi(\Omega)^{\pm}_{mj} \ , \qquad \text{for } \alpha = j-1, \ldots, -j+1, \tag{3.7b}$$

$$\Phi(\Omega)^{\pm}_{m-j} = \mathcal{P}_{mj}\ ((2j+1)D_{\Omega})^{-1/2} \ , \tag{3.7c}$$

$$\mathcal{N} = \tfrac{1}{2}\,(\,j(j+1)C_{\Omega}D_{\Omega}\,)^{-1/2} \ . \tag{3.7d}$$

(C) All roots of eq.(3.5) are real and if $2\omega_{k}+1 > j \geq 2\omega_{k-1}+1$,

$2\omega_l - 1 \geq -j > 2\omega_{l-1}' - 1$ then the position of the roots are bounded by the inequalities

$$2\omega_{m+1} - 1 \geq q_m \geq 2\omega_m + 1 \ , \qquad \text{for } m = j-1,\ldots,k \ ,$$
$$2\omega_m + 1 \geq q_m > 2\omega_m - 1 \ , \qquad \text{for } m = k+1,\ldots,l+1 \ ,$$
$$2\omega_l - 1 \geq q_l \geq 2\omega_l + 1 \ ,$$
$$2\omega_m - 1 \geq q_m \geq 2\omega_{m-1} + 1 \ , \qquad \text{for } m = l-1,\ldots,-j+1 \ .$$

The roots q_m ($m=k+1,\ldots,l$) reach their maximal values if $2\omega_{m+1}+1$ is decreasing to $2\omega_m+1$ ($\Omega_{m+1} \to \Omega_m$) and q_l tends to its minimal value if the same happens for $m = l-1$ ($\Omega_l \longrightarrow \Omega_{l-1}$).

(D) For the singular representations $\Omega \equiv ([\Omega_{j'}]^{p_{j'}}, [\Omega_{j'-1}]^{p_{j'-1}}, \ldots, [\Omega_{-j'}]^{p_{-j'}})$ ($[\Omega]^p \equiv \Omega, \Omega, \ldots, \Omega$) one may use the abbreviated vector $\omega' \equiv (\omega'_{j'}, \ldots, \omega'_{-j'})$ and $q' = (0, q'_{j'-1}, \ldots, q'_{-j'+1}, 2j+1)$

$$\omega'_k = \Omega_k + j - \sum_{l=k+1}^{j'} p_l + \tfrac{1}{2}(1-p_k) \ . \qquad (3.8)$$

The components of the vector q' and the elements $X_a = X(q'_a)$ are obtained from (3.5) and (3.6) respectively, replacing in both the equations the ingredients $2\omega_m \pm 1$ by the $2\omega'_m \pm p_m$ (and changing the range of product over m and a). The elements $X_b = X(q_b)$ and hence $\Phi(\Omega)_{mb}$ for the removed roots of equation (3.6) (see point C) vanish.

(E) We get very simple formulas for the cases $\Omega \equiv ([\Omega_1]^{p_1}, [\Omega_0]^{p_0}, [\Omega_{-1}]^{p_{-1}})$. Introducing the abbreviated vector $\omega' = (\omega'_1, \omega'_0, \omega'_{-1})$ (3.8) and applying the results obtained in the above points we arrive at relations

$$q_0 = (2j+1)(8\omega'_1\omega'_0\omega'_{-1} + 2\sum_{cycl} p_1 p_0 \omega'_{-1})/M \ , \qquad (3.9)$$

$$q_0 - 2\omega_1 \mp p_1 = p_1(2\omega'_1 - 2\omega'_0 \pm p_1 \pm p_0)(2\omega'_1 - 2\omega'_{-1} \pm p_1 \pm p_{-1})$$
$$(2\omega'_1 \mp p_0 \mp p_{-1})/M \ , \qquad \text{cycl } 1,0,-1 \ , \qquad (3.10a)$$

$$q_0 - 2j - 1 = \frac{2j+1}{M} \prod_{cycl} (2\omega'_1 - p_0 - p_{-1}) \ , \qquad (3.10b)$$

$$X_0 = \frac{1}{2M} \left\{ \frac{p_1 p_0 p_{-1}}{2j+1} \prod_{cycl} [(2\omega'_1 - 2\omega'_0)^2 - (p_1 + p_0)^2] \right\}^{1/2} \ , \qquad (3.11)$$

$$M = 4\sum_{cycl} (p_1 + p_0)\omega'_1\omega'_0 + \prod_{m'} (2j+1-p_{m'}) \ . \qquad (3.12)$$

The above expressions should be substituted into (3.7b) to obtain the 3-dimensional matrices $\Phi(\Omega)^{\pm}$.

Before we pass to the proof, we extend our earlier result and find the following 6j coefficients

$$X(\Omega_A \Omega_B \Omega_C)^p{}_r = (-1)^{[\Omega_A]+[\Omega_B]+[\Omega_C]+[\Lambda_0]} \begin{Bmatrix} \Omega_C \Omega_C^* \Lambda_0 \\ \Omega_B^* \Omega_B^* \Omega_A \end{Bmatrix}^p_{r\, j'_B\, j'_C} \ , \qquad (3.13)$$

for arbitrary simple Lie groups (see eq.(3.2a,b,c), (3.4) and (2.10a,b) in Ref.1). Here the indices p and r refer to the multiplicity space related to the triad $(\Omega_A \Omega_B \Omega_C)$ and the indices j'_A and j'_B are used in the same sense as previously in eq.(2.9). Let us consider the eigenvalue of the operator

$$\hat{F} = (-1)^{[\Lambda_0]} \sum_M A(\Lambda_0, A)_M \, A(\Lambda_0, B)^M ,$$

where $A(\Lambda_0, a)_M$ (a=A,B) are the simple Lee algebra generators and we assume that the above are referred to the two kinematically independent parts of the physical system ($[A(\Lambda_0,A)_M, A(\Lambda_0,B)_K] \equiv 0$). Introducing, a convenient basis $|(\Omega_A\Omega_B)\rho\Omega\, M\rangle$ and the generators which act on the total system

$$A(\Lambda_0)_M = A(\Lambda_0, A)_M + A(\Lambda_0, B)_M$$

we get

$$\hat{F}\,|(\Omega_A\Omega_B)\rho\Omega M\rangle = -\tfrac{1}{2}(C_{\Omega_A} + C_{\Omega_B} - C_\Omega)\,|(\Omega_A\Omega_B)\rho\Omega M\rangle \ , \qquad (3.14)$$

On the other hand using (2.1) one finds

$$\hat{F}\,|(\Omega_A\Omega_B)\rho\Omega M\rangle = (-1)^{\pi(\Omega_A\Omega_B\Omega^*)_p + 1} X(\Omega_A\Omega_B\Omega^*)^r{}_p\,|(\Omega_A\Omega_B)r\Omega M\rangle \ , \qquad (3.15)$$

Comparing the equations (3.14) and (3.15) we get

$$X(\Omega_A\Omega_B\Omega_C)^p{}_r = \tfrac{1}{2}(-1)^{\pi(\Omega_A\Omega_B\Omega_C)_p} \times (C_{\Omega_A}D_{\Omega_A}C_{\Omega_B}D_{\Omega_B})^{-1/2} \times$$

$$\times (C_{\Omega_B} + C_{\Omega_C} - C_{\Omega_A}) \times \delta^r_p \, \Delta(\Omega_A\Omega_B\Omega_C)_p . \qquad (3.16)$$

We want to point out that for the very peculiar case when: a) $\Omega_A=\Omega_B=\Omega_C=\Omega$ and b) $(\Omega)\otimes(21)= p\times(\underline{0})+ \ldots$ the relation (3.15) becomes more complicated (here the p two-dimensional IUR (21) should be introduced into (2.1) in place of the ordinary phase factor $\varphi(\Omega_A\Omega_B\Omega_C P)_\alpha{}^3$). We omit the discussion of the form of eq.(3.16) for this case, because we have not found in the literature any example which refers to the simple Lie group.

Proof: Substituting $\Omega_1=\Omega_2=\Omega_3=\Omega$, $X_1=X_2=X_3=X=(1\underline{0})$ (or $X=(1\underline{0})^*$), $\Lambda_1=\Lambda_2=\Lambda_3=\Lambda_0$, $a_1=a_2=a_3=j'$ into the Biedenharn identity[2]

$$\sum_{Y p_1 p_2 p_3} \ldots \sum \mathcal{P} D_Y \begin{Bmatrix} X_2 X_1^* \Lambda_3 \\ \Omega_1 \Omega_2 Y \end{Bmatrix}^{p_2}_{p_1 a_3 b_3} \begin{Bmatrix} X_3 X_2^* \Lambda_1 \\ \Omega_2 \Omega_3 Y \end{Bmatrix}^{p_3}_{p_2 a_1 b_1} \begin{Bmatrix} X_1 X_3^* \Lambda_2 \\ \Omega_3 \Omega_1 Y \end{Bmatrix}^{p_1}_{p_3 a_2 b_2} =$$

$$= (-1)^{\sum_i([\Omega_i]+[X_i])} \sum_r \begin{Bmatrix} \Lambda_1 \Lambda_2 \Lambda_3 \\ \Omega_1 \Omega_2 \Omega_3 \end{Bmatrix}_{a_1 a_2 a_3 r} \begin{Bmatrix} \Lambda_1 \Lambda_2 \Lambda_3 \\ X_1 X_2 X_3 \end{Bmatrix}^r_{b_1 b_2 b_3} , \qquad (3.17)$$

$$\mathcal{P}= (-1)^{\sum([\Omega_i]+[X_i]+[\Lambda_i]+[Y])} ,$$

we see that the matrix elements of $H(\Omega)^+$ and $H(\Omega)^-$ from the upper block ($a,b \geq -j+1$) are composed of two terms respectively for $r = 1/2$ and $r = -1/2$. Let us note that the second term ($r = -1/2$) in the expression (3.17) changes the sign if we replace the representation $X = (1\underline{0})$ by its complex conjugate (see (2.12a), (2.3b), and also (2.11a,b)) which leads to the transformation $H(\Omega)^+ \longrightarrow H(\Omega)^-$. Taking into consideration also the opening discussion of this section we see that the anticipated forms for the matrices $H(\Omega)^\pm$ are correct and we receive (3.5a,b,c,d) applying (3.16), (3.17) and also the relation

$$H(\Omega)^\pm_{j'j'} = C^{(3)}_\Omega / C_\Omega = \tfrac{1}{2}(2j+1) , \qquad (3.18)$$

where the $C^{(3)}_\Omega$ is the eigenvalue for the third order Casimir. We then proceed in the same way we did for the algebras sp(2N), so(2N) and so(2N+1). Introducing

$$\Psi(\Omega)^\pm_{ma} = \mathcal{P}_{ma}(C_\Omega D_\Omega D_a)^{1/2}\Phi(\Omega)^\pm_{ma}$$

we find from the unitarity requirements (1.3a) that

(i) the eigenvalues of the matrices $H(\Omega)^+$ and $H(\Omega)^-$ are equal to $(\Psi(\Omega)^\pm_{jj}, \Psi(\Omega)^\pm_{j-1\,j}, \ldots, \Psi(\Omega)^\pm_{-jj}) = \{ j\mp\omega_j, j\mp\omega_{j-1}, \ldots, j\mp\omega_{-j}\}$,

(ii) the following relations are hold:

$$(\Psi(\Omega)^\pm_{mj})^p = \sum_b (H(\Omega)^\pm)^{p-1}_{jb} \, \Psi(\Omega)^\pm_{mb} .$$

Equation (3.7a,c) is obtained by applying (3.16). Equations (3.7b,c) has been found from the point (ii). In fact from the system (ii) one gets

$$\Psi(\Omega)^\pm_{ma} / U^\pm_{ma} = \Psi(\Omega)^\pm_{mj} / U^\pm_{mj} ,$$

where $U^\pm_m = (U^\pm_{mj}, U^\pm_{m\,j-1}, \ldots, U^\pm_{m-j})$ are the eigenvectors of the matrices $H(\Omega)^+$ and $H(\Omega)^-$ respectively. Taking into consideration the point (i) one may write the dispersion equations in the form

$$-\frac{(\pm 1)^{2j+1} \prod_m 2(\omega - \omega_m)}{(1 \pm 2\omega - q_{-j}) \times \prod_{a(a\neq -j)} (1 \pm 2\omega \mp q_a)} =$$

$$= -1 + \frac{4}{1 \pm 2\omega} \left\{ \frac{|X_{-j}|^2}{(1 \pm 2\omega - q_{-j})} + \sum_{a=-j+1}^{j-1} \frac{|X_a|^2}{(1 \pm 2\omega \mp q_a)} \right\} . \qquad (3.19a,b)$$

Comparing the residues at different poles of the functions on both sides of the equation (3.19a) we arrive the expression (3.6). Repeating the same for equation (3.19b) one finds eq.(3.5). The points (C,D,E) follow immediately from (A) and (B).

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Received by Publishing Department
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Церкаски М. E5-88-284
Об одном классе 6j коэффициентов
с одним индексом кратности для унитарных групп

Важный класс 6j символов, исследованных ранее для групп SP(2N), So(2N), SO(2N+1)[1], изучается здесь для унитарных групп SU(2j+1). С помощью соответствующего базиса в пространстве индекса кратности получена каноническая форма для этих символов. Найдены простые аналитические выражения для них, зависящие от корней уравнения 2j-1 степени, а также полностью аналитические формулы для некоторых классов представлений. Получены также более элементарные 6j коэффициенты для простых алгебр Ли.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1988

Cerkaski M. E5-88-284
On a Class of 6j Coefficients with
One Multiplicity Index for Unitary Groups

An important class of 6j coefficients that have previously been investigated for groups SP(2N), SO(2N), SO(2N+1)[1] is studied for unitary groups SU(2j+1). An appropriate choice of the basis in the multiplicity space leads to the so-called canonical form for these coefficients. Their expressions depending on the roots of a 2j-1 order equation and explicit expressions for some simple class of the representations are found. Some other elementary 6j coefficients for arbitrary simple Lie algebras are calculated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1988