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E5-88-261

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QUASI-PERIODIC SOLUTIONS OF THE INTEGRABLE SYSTEMS RELATED TO THE HILL'S EQUATION

Submitted to "Letters in Mathematical Physics"

1988

1. INTRODUCTION

In the recent years remarkable progress has been achieved in the description of those (quasi)periodic potentials which belong to a given spectrum [1-4] . Many integrable systems of differential equations are shown to be closely connected with the Hill's equation in the case of a finite-gap potential. The coupled Neumann system, the Neumann system, the Rosochatius system are examples of this type [5-8]. In this relation H.Flaschka [9] obtained important results about the algebro-geometrical interpretation of these systems. Recently, [10] R.J.Schilling applied Flaschka's techniques to the generalizations of the Neumann system. In this Note we discusse the relationship between the Hill's equation and the systems of Garnier type. Our intention is to obtain effective finite gap solutions of these systems along the lines of the approach of B.A.Dubrovin, of I.M.Krichever and I.V.Cherednik [11-13] .

In this Note we are concerned with the following completely integrable systems:

(i) The Garnier system [14].

$$\ddot{x}_i = (2\sum_{j=1}^{g} x_j y_i + a_i)x_i \quad , \quad (\tfrac{d}{dt}) \equiv \;\dot{}\; , \tag{1}$$

$$\ddot{y}_i = (2\sum_{j=1}^{g} x_j y_j + a_i)y_i \quad , \tag{2}$$

where $a_1 < a_2 \; \dots < a_g$ are fixed real numbers. The g-dimensional anisotropic harmonic oscillator in a radial quartic potential is obtained when $x_i = y_i$

$$\ddot{x}_i = (2\sum_{j=1}^{g} x_j^2 + a_i)x_i \quad , \quad i=1,\dots,g \,. \tag{3}$$

The complete integrability of the last system was established in [15, 16]. Another interesting integrable system was proposed recently [16], we call it Rosochatius I system. In our context this system is obtained by Deift elimination procedure [5]:

Let $x_i = r_i \exp(\theta_i)$, $y_i = r_i \exp(-\theta_i)$, $c_i = r_i^2 \dot{\theta}_i$ then the equations (1), (2) transform to Rosochatius I system

$$\ddot{r}_i = (2\sum_{j=1}^{g} r_j^2)r_i + a_i r_i - c_i^2/r_i^3 \quad , \; i=1,\dots,g \,. \tag{4}$$

(ii) The coupled Neumann system [9]

$$\ddot{\tilde{x}}_i + \Big(\sum_{j=1}^{g} b_j\tilde{x}_j\tilde{y}_j + \dot{\tilde{x}}_j\dot{\tilde{y}}_j\Big)\tilde{x}_i = b_i\tilde{x}_i , \quad i=0,\dots,g \qquad (5)$$

$$\ddot{\tilde{y}}_i + \Big(\sum_{j=0}^{g} b_j\tilde{x}_j\tilde{y}_j + \dot{\tilde{x}}_j\dot{\tilde{y}}_j\Big)\tilde{y}_i = b_i\tilde{y}_i , \qquad (6)$$

with constraint $\sum_{i=0}^{g} x_i y_i = 1$, where $b_0 < b_1 \dots < b_g$ are fixed real numbers. The Neumann system [5] is obtained when $x_i = y_i$

$$\ddot{\tilde{x}}_i + \Big(\sum_{j=0}^{g} b_j\tilde{x}_j^2 + \dot{\tilde{x}}_j^2\Big)\tilde{x}_i = b_i\tilde{x}_i , \quad \sum_{i=0}^{g}\tilde{x}_i^2 = 1, \; i=0,\dots,g . \qquad (7)$$

This system describes the motion of uncoupled harmonic oscillators $\ddot{\tilde{x}}_i = b_i\tilde{x}_i$, constrained by the force $\sum_{i=0}(b_i\tilde{x}_i^2 + \dot{\tilde{x}}_i^2)$ to move on the unit sphere $\sum_{i=0}^{g}\tilde{x}_i^2 = 1$. Let $\tilde{x}_i = \tilde{r}_i\exp(\tilde{\theta}_i)$, $\tilde{y}_i = \tilde{r}_i\exp(-\tilde{\theta}_i)$, $\tilde{c}_i = \tilde{r}_i^2\dot{\tilde{\theta}}_i$ $i=0,\dots,g$ then by Deift procedure the equations (5),(6) transform to Rosochatius II system [5]

$$\ddot{\tilde{r}}_i = b_i\tilde{r}_i - \Big(\sum_{j=0}^{g} b_j\tilde{r}_j^2 + \dot{\tilde{r}}_j^2 - \tilde{c}_j^2/\tilde{r}_j^2\Big)\tilde{r}_i - c_i^2/\tilde{r}_i^3 , \quad i=0,\dots,g . \qquad (8)$$

In Sect.2 we give the basic facts from the theory of the Baker-Akhiezer (BA) function. More systematic information may be found in the review articles [1, 11]. The spectral interpretation of the systems (1-8) is given in Sect.3 . In sect.4 we obtain the explicit solutions of these systems in terms of the auxiliary spectrum of Hill's periodic theory [1,2] and in sect.5 in terms of Riemann's theta function. In the particular case of the Neumann's system we recover the results of papers [6-9, 17,18] . The rest of the solutions are new.

2. BASIC FACTS FROM THE THEORY OF THE BAKER-AKHIEZER FUNCTION

Let K be the hyperelliptic Riemann surface $y^2 = \prod_{i=0}^{2g}(\lambda - \lambda_i) = R(\lambda)$. The points of K are pairs $P = (\lambda, \sqrt{R})$ and $\lambda(P)$ is the value of the natural projection $P \to \lambda(P)$ of K to the complex projective line CP^1.

Definition 1. For given nonspecial divisor D there is a unique BA function $\Psi(t,P)$, such that

1. the divisor of the poles of Ψ is D ,
2. Ψ is meromorphic on $K\setminus\infty$,
3. when $P \to \infty$

$$\Psi(t,P)\exp(-kt) = 1 + \sum_{s=1}^{\infty} m_s(t)k^{-s} \qquad (9)$$

is holomorphic and $k = \sqrt{\lambda(P)}$ is a local parameter near $P = \infty$.

Proposition 1. There is a unique function $u(t)$ such that

$$\ddot{\Psi} - u(t)\Psi = \lambda(P)\Psi , \qquad (10)$$

where Ψ is BA function.

Proof. Inserting the expansion (9) in (10) we obtain

$$\Psi - 2\dot{m}_1(t)\Psi - \lambda(P)\Psi = \exp(kt)O(k^{-1}) ,$$

and due to the uniqueness of Ψ we prove (10), with $u(t) = 2\dot{m}_1(t)$.

By the Riemann-Roch theorem there exists a unique differential $\tilde{\Omega}$ and a nonspecial divisor D^τ of degree g such that the zeros of $\tilde{\Omega}$ are $D + D^\tau$ and the expansion at $P = \infty$, $\tilde{\Omega}(P) = (1 + O(k^{-2}))dk$.

Proposition 1. For a given D^τ there exists unique dual BA function [13] such that

1. the divisor of the poles of Ψ^τ is D^τ ,
2. Ψ^τ is meromorphic on $K\setminus\infty$,
3. near $P = \infty$, $\Psi^\tau(t,P)\exp(kt) = 1 + O(k^{-1})$ is holomorphic.

Fix τ to be the hyperelliptic involution $P = (\lambda, R) \to P^\tau = (\lambda, -R)$. Then we have $D^\tau = \tau D$, $\Psi^\tau(t,P) = \Psi(t, \tau P)$. Let $\sum_{i=1}^{g}\mu_i(0)$ be the λ-projection of D , and $\sum_{i=1}^{g}\mu_i(t)$ be the λ-projection of the zero divisor of $\Psi(t,P)$. The function $\Psi(t,P)\ \Psi^\tau(t,P)$ is meromorphic on CP^1 and the following identity takes place [1]

$$\Psi(t,P)\ \Psi^\tau(t,P) = \prod_{j=1}^{g}\frac{\lambda - \mu_j(t)}{\lambda - \mu_j(0)} = \frac{F(t,\lambda)}{F(0,\lambda)} . \qquad (11)$$

Introduce the Wronskian $\{\Psi(t,P), \Psi^\tau(t,P)\} = \dot{\Psi}\Psi^\tau - \Psi\dot{\Psi}^\tau = \frac{2\sqrt{R(\lambda)}}{\prod_{j=1}^{g}\lambda - \mu_j(0)}$,

and the differential $\tilde{\Omega}$ is given explicitly by

$$\tilde{\Omega}(P) = \frac{1}{2}\prod_{j=1}^{g}(\lambda - \mu_j(0))/\sqrt{R(\lambda)} . \qquad (12)$$

We assume that $E(p)$ is meromorphic function on K with g+1 simple poles ∞ , $p_1,\dots,p_g$ and at $P = \infty$, $E(P) = k + \dots$, and $\tilde{E}(P)$ is meromorphic on K with g+1 simple poles $q_0, q_1,\dots,q_g$ and at $P = \infty$ $\tilde{E}(P) = k^{-1} + \dots$. We also suppose that the divisors of poles of $E(P)$ and $\tilde{E}(P)$ are different from D, D^τ .

Proposition 2. Let

$$\tilde{x}_j = \tilde{x}_j^0\Psi(t,q_j), \; \tilde{y}_j = \tilde{y}_j^0\Psi^\tau(t,q_j) , \quad \tilde{x}_j^0\tilde{y}_j^0 = \operatorname*{Res}_{P=q_j}\tilde{E}\tilde{\Omega} , \qquad (13)$$

$$b_j = \lambda(q_j) , \quad j=0,1,\dots,g$$

$$x_j = x_j^0\Psi(t,p_j) , \; y_j = y_j^0\Psi^\tau(t,p_j) , \quad x_j^0 y_j^0 = \operatorname*{Res}_{P=p_j} E\tilde{\Omega} , \qquad (14)$$

$$a_j = \lambda(p_j) , \quad j=1,\dots,g$$

then

$$u(t)=-(\sum_{j=0}^{g} b_j x_j y_j + \dot{x}_j\dot{y}_j)\ , \quad \sum_{j=1}^{g} x_j y_j=1 \qquad [8] \qquad (15)$$

$$u(t) = 2\sum_{j=1}^{g} x_j y_j + \text{const} \qquad (16)$$

Proof. Let us construct the meromorphic differential $E\Psi\Psi^T\tilde{\Omega}$. By direct computations we have

$$\sum_{j=0}^{g} \operatorname{Res}_{P=q_j} E\Psi\Psi^T\tilde{\Omega} + \operatorname{Res}_{P=\infty} E\Psi\Psi^T\tilde{\Omega} = \sum_{j=0}^{g} \tilde{x}_j\tilde{y}_j + 1 = 0 \ ,$$

where $\tilde{x}_j^o\tilde{y}_j^o = \operatorname{Res}_{P=q_j} \tilde{E}\tilde{\Omega}$. Differentiating $\sum_{j=0}^{g}\tilde{x}_j\tilde{y}_j = 1$ twice and using Proposition 1 we obtain (15). The eigenvalue equations

$\ddot{\tilde{x}}_j=(\lambda(q_j) + u(t))\,\tilde{x}_j$, $\ddot{\tilde{y}}_j=(\lambda(q_j) + u(t))\,\tilde{y}_j$ by replacing $u(t)$ from (15) are the coupled Neumann system (5),(6). By computations of the same kind we have

$$\sum_{j=1}^{g} \operatorname{Res}_{P=p_j} E\Psi\Psi^T\tilde{\Omega} + \operatorname{Res}_{P=\infty} E\Psi\Psi^T\tilde{\Omega} = \sum_{j=1}^{g} x_j y_j + u(t)/2 + \text{const} = 0 \ ,$$

where $x_j^o y_j^o = \operatorname{Res}_{P=p_j} E\tilde{\Omega}$. The corresponding eigenvalue equations are the Garnier system (1),(2).

3. SPECTRAL INTERPRETATION

Let p be a positive real divisor of degree g on a real hyperelliptic curve $y^2 = R(\lambda) = \prod_{i=0}^{2g}(\lambda-\lambda_i)$, $\lambda_0 < \lambda_1 < \dots < \lambda_{2g}$. The projections $\lambda(p_i)$ lie in the closed lacunae $[\lambda_{2i-1}, \lambda_{2i}]$. The following lemma, due to Jacobi, can be used to construct such a divisor.

Lemma 1. Each divisor p determines and is determined by a system of polynomials $\tilde{A}(\lambda) = \prod_{i=1}^{g}(\lambda - \lambda(p_i))$, $\tilde{C}(\lambda) = \tilde{A}(\lambda)\sum_{i=1}^{g}\frac{\sqrt{R(p_i)}}{\tilde{A}'(p_i)(\lambda-\lambda(p_i))}$, $\tilde{B}(\lambda) = \lambda^{g+1} + \dots$ of degrees $g, g-1, g+1$ respectively, with $R = \tilde{C}^2 - \tilde{A}\tilde{B}$. The complementary divisor q is also determined by this construction. This is the content of the following

Lemma 2. For a given $\lambda_0 = 0 < \lambda_1 < \dots < \lambda_{2g}$ and $\lambda(p_i)\in[\lambda_{2i-1},\lambda_{2i}]$ $i=1,\dots,g$ there exists $\lambda(g_i)$ $i=0,\dots,g$, $\lambda(q_0)\in(-\infty, \lambda_0]$, $\lambda(q_i)\in[\lambda_{2i-1}, \lambda_{2i}]$ such that $R=\tilde{C}^2-\tilde{A}\tilde{B}$, and the projections $\lambda(q_i)$ are the roots of $\tilde{B}$.

For a proof see [20,19] , [17] .

Note that the functions $E(P), \tilde{E}(P)$ are meromorphic on K and the following formulas are immediate

$$E(P) = (\sqrt{R}+\tilde{C})/\tilde{A} = \sqrt{R}/A + \sum_{i=1}^{g} \frac{\sqrt{R(p_i)}}{\tilde{A}'(p_i)(\lambda-\lambda(p_i))}\ , \sqrt{R(p_i)} = \tilde{C}(p_i)\ , \qquad (17)$$

$$\tilde{E}(P) = (\sqrt{R}-\tilde{C})/\tilde{B}\ , \quad \tilde{C}(q_i) = -\sqrt{R(q_i)}\ , \quad (\tfrac{d}{dt}) \equiv {}'\ . \qquad (18)$$

Now we recall some facts from the periodic theory of Hill's equation [1,2] . We suppose that $u(t)$ is a real finite-gap potential i.e. the operator L has only $2g+1$ simple eigenvalues $\lambda_0<\lambda_1<\dots<\lambda_{2g}$ and the rest of the spectrum consists of a double eigenvalues. The periodic spectra of L is determined by the combined eigenvalues of the periodic $Lf_{2i} = \lambda_{2i}f_{2i}$, $f(t+1)=f(t)$, $i=0,\dots,g$ and the antiperiodic $Lf_{2i-1} = \lambda_{2i-1}f_{2i-1}$, $f(t+1)=-f(t)$ $i=1,\dots,g$ eigenvalue equations. The intervals $(-\infty, \lambda_0]$, $[\lambda_{2i-1}, \lambda_{2i}]$ are termed lacunae. The Floquet solutions (BA function) and the corresponding Floquet multipliers are given by [1]

$$\Psi(t,\lambda) = [F(t,\lambda)/F(0,\lambda)]^{1/2}\exp\left(\int_0^t \sqrt{R(\lambda)}\,/F(t',\lambda)dt'\right)\ , \qquad (19)$$

$$\Psi(t+1,\lambda) = \rho_+(\lambda)\,\Psi(t,\lambda)\ ,$$

$$\Psi^T(t,\lambda) = [F(t,\lambda)/F(0,\lambda)]^{1/2}\exp\left(-\int_0^t \sqrt{R(\lambda)}\,/F(t',\lambda)dt'\right)\ , \qquad (20)$$

$$\Psi^T(t+1,\lambda) = \rho_-(\lambda)\,\Psi(t,\lambda)\ , \quad \rho_\pm = \exp(\pm\tilde{p}(\lambda))\ ,$$

where $\tilde{p}(\lambda) = \int_0^1 \sqrt{R(\lambda)}/F(t,\lambda)\,dt$. Note that if λ is in the periodic spectrum, $\Psi(t, \lambda_{2i}) = f_{2i}$, $i=0,\dots,g$ is an periodic eigenfunction, and $\Psi(t, \lambda_{2i-1}) = f_{2i-1}$, $i=1,\dots g$ is an antiperiodic eigenfunction. It is well known that the projections of the zeros of Floquet solution define the auxiliary spectrum of L.

Proposition 3. The following expressions hold

$$x_i y_i = \prod_{j=1}^{g}(\lambda(p_i) - \mu_j(t))/\tilde{A}'(p_i)\ , \quad x_i^o y_i^o = \prod_{j=1}^{g}(\lambda(p_i) - \mu_j(0))/\tilde{A}'(p_i) \qquad (21)$$

$$\tilde{x}_i\tilde{y}_i = \prod_{j=1}^{g}(\lambda(q_i) - \mu_j(t))/\tilde{B}'(q_i)\ , \quad \tilde{x}_i^o\tilde{y}_i^o = \prod_{j=1}^{g}\lambda(q_i) - \mu_j(0))/\tilde{B}'(q_i) \qquad (22)$$

Proof. Using (11-14),(17),(18) we obtain

$$\operatorname{Res}_{P=p_i} E(P)\Psi\Psi^T\tilde{\Omega} = x_i^o y_i^o\,\Psi(p_i)\,\Psi^T(p_i) = \prod_{j=1}^{g}(\lambda(p_i) - \mu_j(t))/\tilde{A}'(p_i)$$

where $x_i^o y_i^o$ is given by (21).

Corollary 1. Let $e_{2i}^2 = \frac{\prod_{i\neq j}^{g}(\lambda_{2i}-\lambda_{2j})}{\prod_{j=1}^{g}(\lambda_{2i}-\mu_j(0))}$, $f_{2i}^2 = \frac{\prod_{j=1}^{g}(\lambda_{2i}-\mu_j(t))}{\prod_{j=1, i\neq j}^{g}(\lambda_{2i}-\mu_j(0))} = \Psi^2(\lambda_{2i})$,

$i=0,\dots,g$ then the expressions (15),(22) are the famous McKean-Moerbeke expansion of the potential u(t) in terms of squares of eigenfunctions

$$u(t)=-(2\sum_{i=0}^{g}\lambda_{2i}f_{2i}^2/e_{2i}^2+\sum_{i=1}^{g}\lambda_{2i-1}-\sum_{i=1}^{g}\lambda_{2i}+\lambda_o) \qquad (23)$$

where the following identity among the squares of eigenfunctions hold on $\sum_{i=0}^{g}e_{2i}^{-2}f_{2i}^2=1$.

The results of this paragraph we may summarise in

Theorem 1. Let u(t) be a real finite-gap potential. There exists g eigenfunctions $\Psi(p_1),\dots,\Psi(p_g)$ and g+1 eigenfunctions $\Psi(q_o),\dots,\Psi(q_g)$ of Hill's equation, corresponding to the eigenvalues $\lambda(p_1),\dots,\lambda(p_g)$ and $\lambda(q_o),\dots,\lambda(q_g)$ respectively, such that

$$\text{(i)}\quad u(t)=2\sum_{i=1}^{g}\Psi(p_i)\Psi^{\tau}(p_i)e_i^{-2}+2\sum_{i=1}^{g}\lambda(p_i)-\sum_{i=0}^{2g}\lambda_i\ , \qquad (24)$$

$$e_i^{-2}=\prod_{j=1}^{g}(\lambda(p_i)-\mu_j(0))/\tilde{A}'(p_i)\ ,\quad i=1,\dots,g.$$

$$\text{(ii)}\quad u(t)=2\sum_{i=0}^{g}\lambda(q_i)\Psi(q_i)\Psi^{\tau}(q_i)\tilde{e}_i^{-2}-2\sum_{i=0}^{g}\lambda(q_i)+\sum_{i=0}^{2g}\lambda_i\ , \qquad (25)$$

$$\tilde{e}_i^{-2}=\prod_{j=1}^{g}(\lambda(q_i)-\mu_j(0))/\tilde{B}'(q_i)\ ,\quad i=0,\dots,g$$

$$\sum_{i=0}^{g}\tilde{e}_i^{-2}\Psi(q_i)\Psi^{\tau}(q_i)=1\ .$$

The corresponding eigenvalue equations are Garnier system and coupled Neumann system.

Corollary 2. Let $e_{2i-1}^2=\prod_{i\neq j}^{g}(\lambda_{2i-1}-\lambda_{2j-1})/\prod_{j=1}^{g}(\lambda_{2i-1}-\mu_j(0))$, $f_{2i-1}^2=\prod_{j=1}^{g}(\lambda_{2i-1}-\mu_j(t))/\prod_{j=1}^{g}(\lambda_{2i-1}-\mu_j(0))$, $i=1,\dots,g$ then we have the following expansion of the potential u(t) in terms of squares of antiperiodic eigenfunctions

$$u(t)=2\sum_{i=1}^{g}f_{2i-1}^2e_{2i-1}^{-2}+2\sum_{i=1}^{g}\lambda_{2i-1}-\sum_{i=0}^{2g}\lambda_i \qquad (26)$$

Definition 2. We call the dynamical systems as in Theorem 1 complementary systems.

4. SOLUTIONS IN TERMS OF AUXILIARY SPECTRUM OF HILL'S EQUATIONS

The idea of the following proposition is due to Garnier [14]

Proposition 4. The solutions of the Garnier system in terms of auxiliary spectrum $\mu_j(t)$, $j=1,\dots,g$ are

$$x_i=x_i^o\left[F(t,a_i)/F(0,a_i)\right]^{1/2}\exp\left(\int_0^t\sqrt{R(a_i)}\Big/\prod_{j=1}^{g}(a_i-\mu_j(t'))dt'\right) \qquad (27)$$

$$y_i=y_i^o\left[F(t,a_i)/F(0,a_i)\right]^{1/2}\exp\left(-\int_0^t\sqrt{R(a_i)}\Big/\prod_{j=1}^{g}(a_i-\mu_j(t'))\right)dt' \qquad (28)$$

where $\mu_j(t)$ satisfies the following system of differential equations $\dot{\mu}_j(t)=2\sqrt{R(\mu_j(t))}/\prod_{j\neq k}^{g}(\mu_j-\mu_k)$, with initial conditions $\mu_j(0)\in[\lambda_{2i-1},\lambda_{2i}]$, and $x_i^oy_i^o=F(0,a_i)/\prod_{i\neq j}^{g}(a_i-a_j)$.

Proof. Differentiating expressions $x_iy_i=x_i^oy_i^o\,\Psi(t,p_i)\Psi^{\tau}(t,p_i)= =\prod_{j=1}^{g}(\lambda(p_i)-\mu_j(t))/A'(p_i)$ and $\dot{x}_iy_i-x_i\dot{y}_i=\{\Psi(p_i),\Psi^{\tau}(p_i)\}\,x_i^oy_i^o$ we have

$$\mathcal{I}(t,P)\Big|_{\lambda=\lambda(p_i)}=\frac{d}{dt}\log x_i(t)=\left[\frac{1}{2}\frac{d}{dt}\prod_{j=1}^{g}(\lambda-\mu_j(t))+\sqrt{R(\lambda)}\right]\Big/\prod_{j=1}^{g}(\lambda-\mu_j(t))\Big|_{\lambda=\lambda(p_i)} \qquad (29)$$

$$\mathcal{I}^{\tau}(t,P)\Big|_{\lambda=\lambda(p_i)}=\frac{d}{dt}\log y_i(t)=\left[\frac{1}{2}\frac{d}{dt}\prod_{j=1}^{g}(\lambda-\mu_j(t))-\sqrt{R(\lambda)}\right]\Big/\prod_{j=1}^{g}(\lambda-\mu_j(t))\Big|_{\lambda=\lambda(p_i)} \qquad (30)$$

Direct integration of (29),(30) gives the solutions (27),(28). The function $\mathcal{I}(t,P)$ has g poles at $\mu_j(t)$ then the numerator of (29) is zero when $\lambda=\mu_j(t)$ and the following system takes place

$$\frac{d}{dt}\left(\prod_{j=1}^{g}(\lambda-\mu_j(t))\right)\Big|_{\lambda=\mu_j(t)}=2\sqrt{R(\mu_j(t))}\ .$$

This is another form of the system $\dot{\mu}_j(t)=2\sqrt{R(\mu_j)}/\prod_{j\neq k}^{g}(\mu_j-\mu_k)$,

In the same way we can obtain the solutions of the coupled Neumann system by replacing $\lambda(p_i)$, $i=1,\dots,g$ with $\lambda(q_i)$ $i=0,\dots,g$ in (27), (28).

Corollary 1. Let $\lambda(p_i)$ be the antiperiodic eigenvalues λ_{2i-1}, $i=1,\dots,g$. Then the exponential function in (27),(28) cancel, $x_i^o=y_i^o$ and the solutions of the g-dimensional oscillator (3) are

$$x_i^2=\prod_{j=1}^{g}(\lambda_{2i-1}-\mu_j(t))/\prod_{i\neq j}^{g}(\lambda_{2i-1}-\lambda_{2j-1})\ . \qquad (31)$$

Corollary 2. Let $\lambda(p_i)$ be in general position i.e. $\lambda(p_i)\in[\lambda_{2i-1},\lambda_{2i}]$ by Deift elimination procedure we may identify r_i with $x_i^oF(t,a_i)/$

$F(0,a_i)]^{1/2}$, θ_i with $\int_0^t \sqrt{R(a_i)}/\prod_{j=1}^{g}(a_i-\mu_j(t'))\,dt'$, $x_i^o = y_i^o$, and hence the solutions of the Rosochatius I system are

$$r_i^2 = \prod_{j=1}^{g}(a_i-\mu_j(t))/\prod_{i\neq j}(a_i-a_j). \qquad (32)$$

Corollary 3. Let $\lambda(q_i)$ be in general position i.e. $\lambda(q_i)\in(-\infty,\lambda_0]$, $[\lambda_{2i-1},\lambda_{2i}]$, $i=0,\dots,g$, by Deift procedure we may identify $\tilde{r}_i$ with $\tilde{x}_i^o F(t,b_i)/F(0,b_i)]^{1/2}$, $\tilde{\theta}_i$ with $\int_0^t \sqrt{R(b_i)}/\prod_{j=1}^{g}(b_i-\mu_j(t'))\,dt'$ and hence the solutions of the Rosochtius II system are

$$\tilde{r}_i^2 = \left[\prod_{j=1}^{g}(b_i-\mu_j(t))/\prod_{i\neq j}(b_i-b_j)\right]. \qquad (33)$$

Now we illustrate the general approach with the simple examples.

Example 1. Let $\wp$ be the Weierstrass $\wp$-function with real period 1, imaginary period and elliptic invariants $e_2<e_1<e_0$. The potential $\wp(t+\omega/2)$ is a one-gap potential with simple periodic spectrum $\lambda_j=-e_j$ $j=0,1,2$ and using first trace formula [1,2] $\mu(t)=\wp(t+\omega/2)$, then the simple solutions of Neumann system (7) are

$$\tilde{x}_0 = \left(\frac{\wp(t+\omega/2)-e_0}{e_2-e_0}\right)^{1/2}, \quad \tilde{x}_1 = \left(\frac{\wp(t+\omega/2)-e_2}{e_0-e_2}\right)^{1/2} \qquad (34)$$

and $x_0^2+x_1^2=1$.

Example 2. Let $u(t)=6\wp(t+\omega/2)$ be the two-gap Lamé potential with simple periodic spectrum $\lambda_0=-\sqrt{3g_2}$, $\lambda_2=-3e_3$, $\lambda_2=-3e_2$, $\lambda_3=-3e_1$, $\lambda_4=\sqrt{3g_2}$. Using first two trace formulas [1,2] we have

$$\mu_1(t) = \tfrac{3}{2}\left[\wp(t+\tfrac{\omega}{2}) + \sqrt{g_2-3\wp^2(t+\tfrac{\omega}{2})}\right] \quad \mu_1(0)=-3e_0,$$

$$\mu_2(t) = \tfrac{3}{2}\left[\wp(t+\tfrac{\omega}{2}) - \sqrt{g_2-3\wp^2(t+\tfrac{\omega}{2})}\right] \quad \mu_2(0)=-3e_1.$$

This is the only information that we need to write the explicit solutions of (1-8).

5. EXPLICIT INTEGRATION IN TERMS OF RIEMANN'S THETA FUNCTION

Let $A_1,\dots,A_g,B_1,\dots,B_g$ be a fixed standard basis of the first homology group on K and $\omega_j=\sum_{k=1}^{g} C_{jk}\lambda^{k-1}/\sqrt{R(\lambda)}\,d\lambda$, $j=1,\dots,g$ the corresponding normalized basis of differentials of the first kind (DFK), with normalization $\int_{A_j}\omega_i=\delta_{ij}$. The period matrix B is $B=(B_{ij})=\int_{B_j}\omega_i$, the Riemann theta function with characteristics $\alpha,\beta\in(\frac{1}{2}Z)^g$ associated to B is

$$\theta[\alpha,\beta](z|B)=\sum_{k\in Z^g}\exp\{\pi i\langle B(k+\alpha),k+\alpha\rangle+2\pi i\langle k+\alpha,z+\beta\rangle\},$$

where $z\in C^g$, $\langle\ ,\ \rangle$ is the scalar product $\langle k,z\rangle=\sum_{i=1}^{g}k_iz_i$, $\langle Bk,k\rangle=\sum_{i,j=1}^{g}B_{ij}k_ik_j$ and ImB is positive definite. Denote by $\tilde{L}$ the lattice of periods $\tilde{L}=\{M+BN,\ M,N\in Z^g\}$. The quotient space $C/\tilde{L}$ is the Jacobi variety J of the Riemann surface K. Define the Abel map $A(P)=(\int_{P_0}^{P}\omega_1,\dots,\int_{P_0}^{P}\omega_g)$, $A: K\to J(K)$, where P_0 is fixed point on K. Let Ω be the differential of the second kind which is normalized by $\int_{A_i}\Omega=0$ and have the following expansion at $P\to\infty$, $\Omega=(d1/k^2+\dots$ Denote by $U=(U_1,\dots,U_g)$ C^g, $U_k=\int_{B_k}\Omega$.

The BA function of a nonspecial divisor D of degree g is given explicitly by

$$\Psi(t,P)=C(P)\exp(t\int_{P_0}^{P}\Omega)\,\frac{\theta(A(P)+z)}{\theta(z)} \qquad (35)$$

where $z=tU+z_0$, $z_0=-A(D)-K$, $C(P)=\theta(z_0)/\theta(A(P)+z_0)$ is a normalization constant, K is the constant Riemann's vector.

Theorem 2. The BA function of a nonspecial divisor D is the solution of the Hill's equation $\ddot{\Psi}-u(t)\Psi=\lambda(P)\Psi$ with the potential $u(t)=2\frac{d^2}{dt^2}\log\theta(z)+C$, where C is a constant. For the proof see [1].

Obviously the dual BA function of a nonspecial divisor D^τ, which is given explicitly by $\Psi^\tau(t,P)=C^\tau(P)\exp(-t\int_{P_0}^{P}\Omega)\,\theta(A(P)-z)/\theta(z)$, $C^\tau(P)=\theta(z_0)/\theta(A(P)-z_0)$ satisfy the Hill's equation with the same potential. From (11), using the explicit expressions of BA function and of dual BA function we obtain two important expressions of $F(t,\lambda)/F(0,\lambda)$ at the points $\lambda(p_i)$ $i=1,\dots,g$ and $\lambda(q_i)$, $i=0,\dots,g$

$$\prod_{j=1}^{g}\frac{\lambda(P_i)-\mu_j(t)}{\lambda(P_i)-\mu_j(0)}=\frac{\theta(z+\Delta_i)\theta(z-\Delta_i)}{\theta^2(z)}\ \frac{\theta^2(z_0)}{\theta(z_0+\Delta_i)\theta(z_0-\Delta_i)} \qquad (36)$$

$$\prod_{j=0}^{g}\frac{\lambda(q_i)-\mu_j(t)}{\lambda(q_i)-\mu_j(0)}=\frac{\theta(z+\tilde{\Delta}_i)\theta(z-\tilde{\Delta}_i)}{\theta^2(z)}\ \frac{\theta^2(z_0)}{\theta(z_0+\tilde{\Delta}_i)\theta(z_0-\tilde{\Delta}_i)} \qquad (37)$$

and one useful identity:

$$\sum_{i=0}^{g}x_i^oy_i^o\ \frac{\theta(z+\tilde{\Delta}_i)\theta(z-\tilde{\Delta}_i)}{\theta^2(z)}\ \frac{\theta^2(z_0)}{\theta(z_0+\tilde{\Delta}_i)\theta(z_0-\tilde{\Delta}_i)}=1. \qquad (38)$$

As we want our solutions to be real let us suppose in addition that

$\sqrt{R(\lambda)}$ has real zeros i.e. on K exists antiholomorphic involution σ : $K \to K$, which commutes with τ and $\lambda(\sigma P)=\lambda(P)$, $\sqrt{R(\sigma P)}=-\sqrt{R(P)}$. Let the projections of zero divisor of BA function lie in the closed lacunae $\lambda_{2i-1} \leq \mu_i(t) \leq \lambda_{2i}$.

Corollary 1. Let $\lambda(q_i)$ be the simple periodic eigenvalues λ_{2i}, $i=0,\ldots,g$. It is well known that in this case $\Delta_i = \frac{1}{2}\sigma''_{2i} + \frac{1}{2}B\sigma'_{2i}, \sigma'_{2i}, \sigma''_{2i} \in Z^g$, and (36) for $i=0,\ldots,g$ are simple periodic eigenfunctions of Hill's equation

$$f_{2i}=c_{2i}\theta[\sigma_{2i}](z)/\theta(z), \quad [\sigma_{2i}] \equiv [\sigma'_{2i}, \sigma''_{2i}], \tag{39}$$

normalized by $c_{2i}=\theta(z_o)/\theta[\sigma_{2i}](z_o)$. Because e_i^2 in (23) are constants fixed by initial conditions, expressions (39) and (22) give solutions of the Neumann system and by simple calculations one obtaines the standard form of these solutions [17,18].

Corollary 2. Let $\lambda(p_i)$ be the simple antiperiodic eigenvalues λ_{2i-1}, $i=1,\ldots,g$ then using (37) we obtain the following solutions of g-dimensional oscillator(3)

$$x_i=x_i^o f_{2i-1}, \quad f_{2i-1}=c_{2i-1}\theta[\sigma_{2i-1}](z)/\theta(z), [\sigma_{2i-1}]\equiv[\sigma'_{2i-1}, \sigma''_{2i-1}], \tag{40}$$

where $c_{2i-1}=\theta(z_o)/\theta[\sigma_{2i-1}](z_o)$, $\Delta_{2i-1}=\frac{1}{2}\sigma''_{2i-1}+\frac{1}{2}\sigma'_{2i-1}B$, $\sigma'_{2i-1}, \sigma''_{2i-1} \in Z^g$.

Corollary 3. Let $\lambda(q_i)$ lie in the closed lacunae $(-\infty, \lambda_o]$, $[\lambda_{2i-1}, \lambda_{2i}]$, $i=0,\ldots,g$ then

$$\tilde{r}_i^2=[\tilde{x}_i^o]^2 \frac{\theta(z+\tilde{\Delta}_i)\theta(z-\tilde{\Delta}_i)}{\theta^2(z)} \frac{\theta^2(z_o)}{\theta(z_o+\tilde{\Delta}_i)\theta(z_o-\tilde{\Delta}_i)} \tag{41}$$

with $\tilde{x}_i^o=\left[\prod_{j=1}^{g}(b_i-\mu_j(0))/\prod_{j\neq i}^{g}(b_i-b_j)\right]^{1/2}$ and constraint (36) give the solution of Rosochatius II system (8).

Corollary 4. Let $\lambda(p_i)$ lie in the closed lacunae $[\lambda_{2i-1}, \lambda_{2i}]$, $i=1,\ldots,g$ then $r_i^2=[x_i^o]^2F(t,a_i)/F(0,a_i)$ and (37) are solutions of Rosochatius I system (4).

Finally from (27,28),(37) we arrive at the general finite-gap solutions of Garnier system.

Conclusions.

In this Note we give new finite gap solutions of the physically significant systems of differential equations. These solutions can be interpreted as eigenfunctions of a suitable differential operator, in our context-Hill's operator. The concept of complementary system is propounded. Such systems are for example, Garnier system and coupled Neumann systems, the systems Rosochatius I and Rosochatius II. The existence of complementary systems is not an isolated phenomenon.

Obviously there exists many completely integrable systems of this type, for example systems related to the discrete spectrum in the case of finite-gap potential of Zakharov-Shabat operator, N-wave operator and many others. We return to the study of the complementary systems in another place.

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Received by Publishing Department on April 21, 1988.