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ONE-PARAMETER GROUPS ARISING FROM REAL SUBSPACES OF SELF-DUAL HILBERT W*-MODULI

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§1 Introduction

A (left) pre-Hilbert A-module over a certain C^*-algebra A is an A-module $\mathcal{H}$ equipped with an A-valued non degenerate conjugate-bilinear mapping $\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow A$, $\langle \cdot , \cdot \rangle$ being A-linear at the first argument. $\mathcal{H}$ is Hilbert if it is complete with respect to the norm $\| \cdot \| = \| \langle \cdot , \cdot \rangle \|_A^{1/2}$. We suppose always that the linear structures of A and of $\mathcal{H}$ are compatible. For basic facts concerning Hilbert C^*-moduli we refer to [7]. A Hilbert A-module $\mathcal{H}$ over a C^*-algebra A is called self-dual if every bounded module map $r: \mathcal{H} \longrightarrow A$ is of the form $\langle \cdot , \bar{a} \rangle$ for some $\bar{a} \in \mathcal{H}$. In this paper we restrict our attention mainly to Hilbert W^*-moduli . For them some more facts are known as in the general case. We need the following ones :

Definition 1.1.: [4,Def.7]

Let A be a W^*-algebra, $\mathcal{H}$ be a pre-Hilbert A-module and P be the set of all normal states on A. The topology induced on $\mathcal{H}$ by the semi-norms

$$f(\langle \cdot , \cdot \rangle)^{1/2}, \; f \in P,$$

is denoted by τ_1. The topology induced on $\mathcal{H}$ by the linear functionals

$$f(\langle \cdot , \bar{y} \rangle) , \; \bar{y} \in \mathcal{H}, \; f \in P,$$

is denoted by τ_2.

We remark that, in general, the topology τ_2 is weaker than the topology τ_1, and that they are both weaker than the norm topology. Throughout this paper we use the following notation. If $\mathcal{K}$ is a subset of the Hilbert W^*-module $\mathcal{H}$, $[\mathcal{K}]_\tau^-$ denotes the set $\{\lambda\bar{x} : \lambda \in \mathbb{R}_+, \bar{x} \in \mathcal{K}_0\}$ where $\mathcal{K}_0$ is the τ_1-completion of the set $\{\bar{x} \in \mathcal{K} : \|\bar{x}\| \leq 1\}$.

Theorem 1.2.: [4,Th.9]
Let A be a W^*-algebra and $\mathcal{H}$ be a Hilbert A-module. The following conditions are equivalent :

(i) $\mathcal{H}$ is self-dual.
(ii) The unit ball of $\mathcal{H}$ is complete with respect to the topology τ_1, i.e. $\mathcal{H} = [\mathcal{H}]_\tau^-$.
(iii) The unit ball of $\mathcal{H}$ is complete with respect to the topology τ_2.

Corollary 1.3.: [4,Cor.11]
If A is a W^*-algebra and $\mathcal{H}$ is a self-dual Hilbert A-module the linear span of the range of the A-valued inner product on $\mathcal{H}$ becomes both a W^*-subalgebra and an ideal in A.

Theorem 1.4.: [7,Prop.3.10.]
Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. Then, the set $End_A(\mathcal{H})$ of all bounded A-linear operators on $\mathcal{H}$ is a W^*-algebra.

These facts make clear that in the case of $\mathcal{H}$ being a self-dual Hilbert W^*-module, the spectral theorem ([10,Th.1.11.3.]) is valid for each self-adjoint element of $End_A(\mathcal{H})$. Moreover, there exists a polar decomposition for each element of $End_A(\mathcal{H})$ in $End_A(\mathcal{H})$. This is of importance throughout this paper.
Now let A be a commutative W^*-algebra, $\mathcal{H}$ be a self-dual Hilbert A-module. §2 deals with real self-dual Hilbert A_h-submodules $\mathcal{K}$ of $\mathcal{H}$ and with some bounded operators arising from them. In particular, a conjugate-A-linear involution J on $\mathcal{H}$ with respect to $\mathcal{K}$ is defined. §3 investigates the strongly continuous one-parameter group $\{\Delta^{it} ; t \in \mathbb{R}\}$ defined for each $\mathcal{K} \subseteq \mathcal{H}$ on $\mathcal{H}$. The relation of this group to a generalized K.M.S. condition is established. We remark that the main definitions of §2 and §3 are formulated without the restriction to A to be commutative.
§4 gives an interpretation of the results of the former paragraphs in terms of locally trivial Hilbert bundles over hyperstonian compact spaces. In §5 some aspects of the general non commutative case are discussed. Other applications can be found in [5].

§2 On some real subspaces of self-dual Hilbert W^*-moduli and related operators

Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. We suppose without loss of generality that the linear span of the range of the A-valued inner product on $\mathcal{H}$ is identical with A, (cf. Cor.1.3.). Let $\mathcal{K}$ be a norm-closed real subspace of $\mathcal{H}$ being invariant under the action of $\mathfrak{Z}(A)_h$ (the selfadjoint part of the centre of A) and having the following properties :

(i) $\mathcal{K} \cap i\mathcal{K} = \{\bar{0}\}$.
(ii) $\mathcal{K} + i\mathcal{K}$ is norm-dense in $\mathcal{H}$.
(iii) $\mathcal{K} = [\mathcal{K}]_\tau^-$.

As an example one can take $A = \mathcal{H}$ and $A_h = \mathcal{K}$. The following assertions make clear why the third condition above is necessary.

Definition 2.1.:
$$\mathcal{K}^\perp = \{\bar{x} \in \mathcal{H} : \langle \bar{x},\bar{y}\rangle + \langle \bar{y},\bar{x}\rangle = 0 \text{ for any } \bar{y} \in \mathcal{K}\}.$$

Proposition 2.2.: $\mathcal{K}^\perp$ is a norm-closed real subspace of $\mathcal{H}$ being invariant under the action of $\mathfrak{Z}(A)_h$ and satisfying the conditions $\mathcal{K} \cap \mathcal{K}^\perp = \{\bar{0}\}$, $\mathcal{K}^\perp = [\mathcal{K}^\perp]_\tau^-$.

Proof: The connection $\mathcal{K} \cap \mathcal{K}^\perp = \{\bar{0}\}$ follows from the non degeneracy of the A-valued inner product on $\mathcal{H}$. The equality
$$\begin{aligned} \langle \bar{y},a\bar{x}\rangle + \langle a\bar{x},\bar{y}\rangle &= \langle \bar{y},\bar{x}\rangle a + a\langle \bar{x},\bar{y}\rangle \\ &= a\langle \bar{y},\bar{x}\rangle + \langle \bar{x},\bar{y}\rangle a \\ &= \langle a\bar{y},\bar{x}\rangle + \langle \bar{x},a\bar{y}\rangle \\ &= 0 \end{aligned}$$
($a \in \mathfrak{Z}(A)_h$, $\bar{x} \in \mathcal{K}^\perp$, $\bar{y} \in \mathcal{K}$) shows that $\mathcal{K}^\perp$ is a real subspace of $\mathcal{H}$ being invariant under the action of $\mathfrak{Z}(A)_h$. If we consider a bounded set $\{\bar{x}_\alpha : \bar{x}_\alpha \in \mathcal{K}^\perp, \alpha \in I, \tau_2\text{-}\lim \bar{x}_\alpha = \bar{x} \in \mathcal{H}\}$ we get
$$\begin{aligned} 0 &= \lim_{\alpha \in I} f(\langle \bar{x}_\alpha,\bar{y}\rangle + \langle \bar{y},\bar{x}_\alpha\rangle) \\ &= f(\langle \bar{x},\bar{y}\rangle + \langle \bar{y},\bar{x}\rangle) \end{aligned}$$
for any $\bar{y} \in \mathcal{K}$, any normal state f on A. I.e., $\bar{x} \in \mathcal{K}^\perp$ and, thus, by Theorem 1.2. $\mathcal{K}^\perp = [\mathcal{K}^\perp]_\tau^-$. The norm-completeness is obvious now.

Corollary 2.3.: $(\mathcal{K}^\perp)^\perp = \mathcal{K}$.

Therefore, we have shown the necessity of the condition (iii) above. We remark that this condition is satisfied automatically if A is finite dimensional,(i.e.,particularly, if $\mathcal{H}$ is a Hilbert space). Now we pass on to considerations about the properties of such real subspaces $\mathcal{K}$ of $\mathcal{H}$ supposing A to be commutative.

<u>Proposition 2.4.:</u> There exist two A_h-linear projections $P:\mathcal{H} \longrightarrow \mathcal{K}$ and $Q:\mathcal{H} \longrightarrow i\mathcal{K}$, the operator norms of which are equal to one. They fulfil the equalities

$$(1) \quad \begin{aligned} \langle P(\bar{x}),\bar{y}\rangle + \langle \bar{y},P(\bar{x})\rangle &= \langle \bar{x},P(\bar{y})\rangle + \langle P(\bar{y}),\bar{x}\rangle \\ &= \langle P(\bar{x}),P(\bar{y})\rangle + \langle P(\bar{y}),P(\bar{x})\rangle \ , \\ \langle Q(\bar{x}),\bar{y}\rangle + \langle \bar{y},Q(\bar{x})\rangle &= \langle \bar{x},Q(\bar{y})\rangle + \langle Q(\bar{y}),\bar{x}\rangle \\ &= \langle Q(\bar{x}),Q(\bar{y})\rangle + \langle Q(\bar{y}),Q(\bar{x})\rangle \end{aligned}$$

for any $\bar{x},\bar{y} \in \mathcal{H}$.

<u>Proof:</u> We can consider the real self-dual Hilbert A_h-module $\mathcal{H}_* = \{\mathcal{H}, \langle\cdot,\cdot\rangle + \langle\cdot,\cdot\rangle^*\}$ since A_h is a real W^*-algebra with identical involution. The set $\mathcal{K}_* = \{\mathcal{K}, \langle\cdot,\cdot\rangle + \langle\cdot,\cdot\rangle^*\}$ is a real self-dual Hilbert A_h-submodule of $\mathcal{H}_*$. From [4,Th.9,Proof] we draw that $\mathcal{H}_* = \mathcal{K}_* + \mathcal{K}_*^{\perp}$. Thus, there exists a projection $P:\mathcal{H} \longrightarrow \mathcal{K}$ defined by the rule $P:\mathcal{H}_* \longrightarrow \mathcal{K}_*$. The projection satisfies the condition (1) by its definition and it is A_h-linear. Keeping in mind that P acts on $\mathcal{K}$ as the identical map the inequality

$$(2) \quad \begin{aligned} \langle \bar{x},\bar{x}\rangle &= \langle (P+(1-P))(\bar{x}),(P+(1-P))(\bar{x})\rangle \\ &= \langle P(\bar{x}),P(\bar{x})\rangle + \langle (1-P)(\bar{x}),(1-P)(\bar{x})\rangle \\ &\geq \langle P(\bar{x}),P(\bar{x})\rangle \ , \end{aligned}$$

being valid for any $\bar{x} \in \mathcal{H}$, shows $\|P\| = 1$. The proof is analogous for Q.

The projections P and Q do not commute, in general. We define $R := P+Q$ and $JT := P-Q$, where T is a positive selfadjoint operator and J is a partial isometry.

<u>Lemma 2.5.:</u> R is an injective A-linear operator on $\mathcal{H}$. The connection $0 \leq R \leq 2$ is valid. The same is true for the operator (2-R).

<u>Proof:</u> Denoting by i the square root from (-1) we can state the equalities $iP = Qi$ and $iQ = Pi$. Taking an element $c=c_1+c_2 i \in A$, $(c_1,c_2 \in A_h)$, and $\bar{x} \in \mathcal{H}$ arbitrarily the equality

$$\begin{aligned} R(c\bar{x}) &= P(c\bar{x}) + Q(c\bar{x}) \\ &= c_1\cdot P(\bar{x}) + c_2\cdot P(i\bar{x}) + c_1\cdot Q(\bar{x}) + c_2\cdot Q(i\bar{x}) \\ &= c_1\cdot P(\bar{x}) + c_2 i\cdot Q(\bar{x}) + c_1\cdot Q(\bar{x}) + c_2 i\cdot P(\bar{x}) \\ &= c\cdot(P(\bar{x}) + Q(\bar{x})) \\ &= c\cdot R(\bar{x}) \end{aligned}$$

is satisfied, i.e. the operator R is A-linear. If now $R(\bar{x})=\bar{0}$ for a certain $\bar{x} \in \mathcal{H}$, we obtain

$$(3) \quad \begin{aligned} 0 &= \langle R(\bar{x}),\bar{x}\rangle + \langle \bar{x},R(\bar{x})\rangle \\ &= \langle P(\bar{x}),\bar{x}\rangle + \langle \bar{x},P(\bar{x})\rangle + \langle Q(\bar{x}),\bar{x}\rangle + \langle \bar{x},Q(\bar{x})\rangle \\ &= 2\cdot(\langle P(\bar{x}),P(\bar{x})\rangle + \langle Q(\bar{x}),Q(\bar{x})\rangle), \quad \text{cf. (1).} \end{aligned}$$

Consequently, $P(\bar{x})=Q(\bar{x})=\bar{0}$. Since $\mathcal{K}+i\mathcal{K}$ is dense in $\mathcal{H}$ there follows $\bar{x}=\bar{0}$. Thus, R is an injective operator. Furthermore, since R has a conjugate operator R^* in $\mathrm{End}_A(\mathcal{H})$ the equality below is valid:

$$(4) \quad \begin{aligned} &= \langle R^2(\bar{x}),\bar{y}\rangle + \langle \bar{y},R^2(\bar{x})\rangle \\ &= \langle (P+PQ+QP+Q)(\bar{x}),\bar{y}\rangle + \langle \bar{y},(P+PQ+QP+Q)(\bar{x})\rangle \\ &= \langle (P+Q)(\bar{x}),(P+Q)(\bar{y})\rangle + \langle (P+Q)(\bar{y}),(P+Q)(\bar{x})\rangle \quad \text{, cf. (1)} \\ &= \langle R(\bar{x}),R(\bar{y})\rangle + \langle R(\bar{y}),R(\bar{x})\rangle \\ &= \langle R^*R(\bar{x}),\bar{y}\rangle + \langle \bar{y},R^*R(\bar{x})\rangle \end{aligned}$$

for any $\bar{x},\bar{y} \in \mathcal{H}$. Therefore, $(R-R^*)R=0$. Because of the injectivity of R we draw $R=R^*$. Using (3) we get

$$\begin{aligned} \langle R(\bar{x}),\bar{x}\rangle &= 1/2\cdot(\langle R(\bar{x}),\bar{x}\rangle + \langle \bar{x},R(\bar{x})\rangle) \\ &= 1/2\cdot(\langle P(\bar{x}),P(\bar{x})\rangle + \langle Q(\bar{x}),Q(\bar{x})\rangle) \\ &\geq 0 \end{aligned}$$

for any $\bar{x} \in \mathcal{H}$. On the other hand

$$\langle R(\bar{x}),\bar{x}\rangle \leq 2\cdot\langle \bar{x},\bar{x}\rangle$$

for any $\bar{x} \in \mathcal{H}$ by (2). Consequently, $0 \leq R \leq 2$. The proof for (2-R) is analogous changing P,Q to (1-P),(1-Q).

<u>Lemma 2.6.:</u> (P-Q), T and J are injective operators. (P-Q) and J are conjugate-A-linear, whereas $(P-Q)^2$ and T are A-linear. The equalities $J^2=id_{\mathcal{H}}$ and $T=R^{1/2}(2-R)^{1/2}$ are valid, and the operators J and T commute.

<u>Proof:</u> The following equality is satisfied for any $a=a_1+a_2 i \in A$, $(a_1,a_2 \in A_h)$, and any $\bar{x} \in \mathcal{H}$:

$$\begin{aligned} (P-Q)(a\bar{x}) &= a_1\cdot P(\bar{x}) + a_2\cdot P(i\bar{x}) - a_1\cdot Q(\bar{x}) - a_2\cdot Q(i\bar{x}) \\ &= a_1\cdot P(\bar{x}) + a_2 i\cdot Q(\bar{x}) - a_1\cdot Q(\bar{x}) - a_2 i\cdot P(\bar{x}) \\ &= (a_1-a_2 i)(P-Q)(\bar{x}). \end{aligned}$$

I.e., (P-Q) is conjugate-A-linear and, correspondingly, $(P-Q)^2$ is A-linear. Moreover, $(P-Q)^2=P-PQ-QP+Q=(2-R)R$, and thus, $(P-Q)^2$ is a selfadjoint positive, injective operator on $\mathcal{H}$. We define the operator $T:\mathcal{H} \longrightarrow \mathcal{H}$ by the formula

$$T = \left[(P-Q)^2\right]^{1/2} = (2-R)^{1/2}R^{1/2}.$$

Thus, T is a A-linear, bounded, injective, selfadjoint positive operator on $\mathcal{H}$. The sets $[T(\mathcal{H})]^-_\tau$ and $[T^2(\mathcal{H})]^-_\tau$ are both equal to $\mathcal{H}$. There exists a map J for which the equality $JT = (P-Q)$ is satisfied on $\mathcal{H}$ since the operators T and $(P-Q)$ are injective and $T^2 = (P-Q)^2$. J maps $T(\mathcal{H})$ into $\mathcal{H}$. Out of the equality $(P-Q)^2 = JTJT = T^2$ we draw $T = JTJ$ since T is injective. Hence, J can be extended to a map defined on $\mathcal{H}$ being bounded, conjugate-A-linear and injective. Furthermore, the existence of the inverse operator J^{-1} defined on $\mathcal{H}$ being bounded, conjugate-A-linear and injective is guaranteed. We obtain $JT = TJ^{-1}$ and $T^2 = JTJT = JT^2J^{-1}$, i.e. $JT^2 = T^2J$. Consequently, J commutes with T^2 and therefore, with T. Moreover, $J = J^{-1}$ on $\mathcal{H}$ and $J^2 = id_{\mathcal{H}}$.

Lemma 2.7.: T commutes with P,Q and R. The equalities $JP = (1-Q)J$, $JQ = (1-P)J$ and $JR = (2-R)J$ are valid.

Proof: (cf. [8,Proof of Prop.2.2.])
The equality $T^2P = (P-Q)^2P = P(P-Q)^2 = PT^2$ shows that P commutes with T^2 and hence, with T. The proof for Q and R is analogous. The following equality is valid:

$$TJP = JTP = (P-Q)P = (1-Q)(P-Q) = (1-Q)JT = T(1-Q)J \ .$$

Since T is injective we get $JP = (1-Q)J$. The equality $JQ = (1-P)J$ can be proved by analogous computations. For R we obtain the sought equation adding the first two.

Lemma 2.8.:

(i) $\langle J(\bar{x}),\bar{y}\rangle = \langle J(\bar{y}),\bar{x}\rangle$ for any $\bar{x},\bar{y}\in\mathcal{H}$.

(ii) $\langle J(\bar{x}),\bar{x}\rangle \geqq 0$ for any $\bar{x}\in\mathcal{K}$.
$\langle J(\bar{x}),\bar{x}\rangle \leqq 0$ for any $x\in i\mathcal{K}$.

Proof: There yields $PJP = P(1-Q)J = P(P-Q)J = PTJ^2 = PT$. Thus, $\langle J(\bar{x}),\bar{x}\rangle + \langle\bar{x},J(\bar{x})\rangle \geqq 0$ for any $\bar{x}\in\mathcal{K}$ since T is positive and P and T commute, (cf. (1)). On the other hand

$$\begin{aligned} 0 &= \langle J(\bar{x}),i\bar{x}\rangle + \langle i\bar{x},J(\bar{x})\rangle \\ &= i\cdot(\langle\bar{x},J(\bar{x})\rangle - \langle J(\bar{x}),\bar{x}\rangle) \end{aligned}$$

for any $\bar{x}\in\mathcal{K}$, (cf. Lemma 2.7.). Consequently, $\langle J(\bar{x}),\bar{x}\rangle = \langle\bar{x},J(\bar{x})\rangle$, $\langle J(\bar{x}),\bar{x}\rangle \geqq 0$ for any $\bar{x}\in\mathcal{K}$ and $\langle J(\bar{y}),\bar{y}\rangle \leqq 0$ for any $\bar{y}\in i\mathcal{K}$. Furthermore, the equality

$$\begin{aligned} 0 &= i\cdot(\langle i\bar{y},J(\bar{x})\rangle + \langle J(\bar{x}),\bar{y}i\rangle) \\ &= \langle J(\bar{x}),\bar{y}\rangle - \langle\bar{y},J(\bar{x})\rangle \end{aligned}$$

holds for any $\bar{x},\bar{y}\in\mathcal{K}$ (or respectively, $\bar{x},\bar{y}\in i\mathcal{K}$), (cf. Lemma

2.6.). Thus,

(5) $\langle J(\bar{x}),\bar{y}\rangle = \langle\bar{y},J(\bar{x})\rangle \in A_h$ for any $\bar{x},\bar{y}\in\mathcal{K}$ (or, respectively, $\bar{x},\bar{y}\in i\mathcal{K}$).

If we consider now $\bar{x}=\bar{x}_1+\bar{x}_2$, $\bar{y}=\bar{y}_1+\bar{y}_2$ ($\bar{x}_1,\bar{y}_1\in\mathcal{K}$, $\bar{x}_2,\bar{y}_2\in i\mathcal{K}$) there yields

$$\begin{aligned} &= \langle J(\bar{x}),\bar{y}\rangle \\ &= \langle J(\bar{x}_1),\bar{y}_1\rangle + \langle J(\bar{x}_2),\bar{y}_2\rangle + i\cdot(\langle J(\bar{x}_1),i\bar{y}_2\rangle + \langle J(i\bar{x}_2),\bar{y}_1\rangle) \\ &= \langle J(\bar{y}_1),\bar{x}_1\rangle + \langle J(\bar{y}_2),\bar{x}_2\rangle - i\cdot(\langle J(i\bar{y}_2),\bar{x}_1\rangle + \langle J(\bar{y}_1),i\bar{x}_2\rangle) \\ &= \langle J(\bar{y}),\bar{x}\rangle \ . \end{aligned}$$

(cf. Lemma 2.7. and (5)). Since $\mathcal{K}+i\mathcal{K}$ is norm-dense in $\mathcal{H}$ the equality $\langle J(\bar{x}),\bar{y}\rangle = \langle J(\bar{y}),\bar{x}\rangle$ is satisfied for any $\bar{x},\bar{y}\in\mathcal{H}$.

Corollary 2.9.: $J(\mathcal{K}) = (i\mathcal{K})^\perp$, $J(i\mathcal{K}) = \mathcal{K}^\perp$.

This follows from Lemma 2.7.. Finally, we can formulate the following

Proposition 2.10.:

(i) R and (2-R) are A-linear, injective operators with the property $0\leqq R\leqq 2$, $0\leqq 2-R\leqq 2$.

(ii) T is injective and A-linear. $T = R^{1/2}(2-R)^{1/2}$.

(iii) J is conjugate-A-linear and injective. $J^2 = id_{\mathcal{H}}$.
For any $\bar{x},\bar{y}\in\mathcal{H}$ yields $\langle J(\bar{x}),\bar{y}\rangle = \langle J(\bar{y}),\bar{x}\rangle$.

(iv) T commutes with P,Q,R,J.

(v) $JP = (1-Q)J$, $JQ = (1-P)J$, $JR = (2-R)J$.

For a better geometrical characterization of the operator J we show

Proposition 2.11.: The operator J defined above is the unique conjugate-A-linear partial isometry with the two properties:

(i) $J(\mathcal{K}) = i\mathcal{K}^\perp$, $J(i\mathcal{K}) = \mathcal{K}^\perp$.

(ii) $\langle J(\bar{x}),\bar{x}\rangle \geqq 0$ for any $\bar{x}\in\mathcal{K}$, $\langle J(\bar{x}),\bar{x}\rangle \leqq 0$ for any $\bar{x}\in i\mathcal{K}$.

Proof: (cf. [8,Prop.2.3.])
The operator J has the properties (i) and (ii) as we have shown at Lemmata 2.7. and 2.8.. Let K be another conjugate-A-linear partial isometry satisfying the conditions (i) and (ii) above. We get $(JK)P = J(1-Q)K = P(JK)$, i.e., JK commutes with P. Similarly it can be proved that JK commutes with Q and, hence, with R and T. Consequently,

(6) $$\begin{aligned} (JK)(RT) &= (RT)(JK) = ((P+Q)(P-Q)J)(JK) = (P(1-Q)-Q(1-P))K \\ &= PKP - QKQ \end{aligned}$$

$= K((1-Q)P-(1-P)Q) = (KJ)(J(P+Q)(P-Q))$
$= (KJ)(RT)$.

Since RT is injective, JK = KJ. Furthermore, from (6) and (ii) we draw $(RT)(JK) \geqq 0$ on $\mathcal{H}$, (cf. Lemma 2.8.). Because of the uniqueness of the polar decomposition and since $RT \geqq 0$ the equality $JK = id_{\mathcal{H}}$ must be satisfied. Hence, J = K.

Definition 2.12.: We take $\mathcal{H}$, $\mathcal{K}$ and $i\mathcal{K}$ as at the beginning of this paragraph. There exists an operator S defined on $D(S)=\mathcal{K}+i\mathcal{K}$ by the formula

$$S(\bar{x}+\bar{y}) := \bar{x}-\bar{y} \ , \ \bar{x}\in\mathcal{K} \ , \ \bar{y}\in i\mathcal{K} .$$

The operator S is unbounded, in general. Since $i\mathcal{K}^{\perp}$ and $\mathcal{K}^{\perp}$ fulfil the conditions "$i\mathcal{K}^{\perp}\cap\mathcal{K}^{\perp}=\{\bar{0}\}$" and "$(i\mathcal{K}^{\perp}+\mathcal{K}^{\perp})$ is dense in $\mathcal{H}$" , also there exists an operator F defined on $D(F)=i\mathcal{K}^{\perp}+\mathcal{K}^{\perp}$ by the formula

$$F(\bar{x}+\bar{y}) := \bar{x}-\bar{y} \ , \ \bar{x}\in i\mathcal{K}^{\perp}, \ \bar{y}\in\mathcal{K}^{\perp}.$$

It is also unbounded, in general. F and S are closed operators.

Proposition 2.13.:
(i) $F = JSJ$, $F = S^*$.
(ii) $((2-R)S)(\bar{x}) = (JT)(\bar{x})$ for any $\bar{x}\in D(S)$.
(iii) Taking $\Delta := (2-R)R^{-1}$ the polar decomposition is described by $S = J\Delta^{1/2}$, $F = J\Delta^{-1/2}$.

Proof: (cf. [8,§6,Prop.])
The equality F=JSJ follows from Corollary 2.10.. Taking $\bar{x}\in\mathcal{K}$, $\bar{y}\in i\mathcal{K}$, $z\in i\mathcal{K}^{\perp}$, $t\in\mathcal{K}^{\perp}$ we get

$$\begin{aligned}\langle S(\bar{x}+\bar{y}),\bar{z}+\bar{t}\rangle &= \langle\bar{x}-\bar{y},\bar{z}+\bar{t}\rangle \\ &= \langle\bar{x},\bar{z}\rangle - \langle\bar{y},\bar{t}\rangle \\ &= \langle\bar{x}+\bar{y},\bar{z}-\bar{t}\rangle \\ &= \langle\bar{x}+\bar{y},F(\bar{z}+\bar{t})\rangle \ .\end{aligned}$$

Therefore, $F\subseteq S^*$. On the other hand the equality

$$\langle\bar{x}-\bar{y},\bar{z}\rangle = \langle S(\bar{x}+\bar{y}),\bar{z}\rangle = \langle\bar{x}+\bar{y},S^*(\bar{z})\rangle$$

is satisfied for any $\bar{z}\in D(S^*)$, $x\in\mathcal{K}$, $y\in i\mathcal{K}$. If $\bar{y}=\bar{0}$ there holds $(\bar{z}-S^*(\bar{z}))\in\mathcal{K}^{\perp}$. Taking $\bar{x}=\bar{0}$ we get $(\bar{z}+S^*(\bar{z}))\in i\mathcal{K}^{\perp}$. Consequently, $\bar{z}=1/2\cdot((\bar{z}+S^*(\bar{z}))+(\bar{z}-S^*(\bar{z})))\in D(F)$. Thus, $F=S^*$. Furthermore,

$$\begin{aligned}((2-R)S)(\bar{x}+\bar{y}) &= (2-P-Q)(\bar{x}-\bar{y}) = (1-P)(\bar{x}-\bar{y}) + (1-Q)(\bar{x}-\bar{y}) \\ &= (1-Q)(\bar{x}) - (1-P)(\bar{y}) = (P-Q)(\bar{x}+\bar{y}) \\ &= JT(\bar{x}+\bar{y})\end{aligned}$$

for any $\bar{x}\in\mathcal{K}$, $\bar{y}\in i\mathcal{K}$. There follows $(2-R)S=JT$ on $D(S)\subseteq\mathcal{H}$.
From (i) we have $(JS)^*=S^*J=FJ=JS$ on D(S), (cf. (5)). Hence, JS is selfadjoint. On the other hand $(RJS)(\bar{x})=T(\bar{x})$ for any $\bar{x}\in D(S)$, (cf. (ii) above and Lemma 2.7.). Consequently, $JS(\bar{x})=((2-R)^{1/2}R^{-1/2})(\bar{x})$ and $\bar{x}\in D(\Delta^{1/2})$. Therefore, $JS\subseteq\Delta^{1/2}$ and since selfadjoint operators are maximal $JS=\Delta^{1/2}$. The second equality can be proved in a similar way.

Corollary 2.14.: $\Delta^{1/4}\mathcal{K} = \Delta^{-1/4}(i\mathcal{K}^{\perp})$, $\Delta^{1/4}(i\mathcal{K}) = \Delta^{-1/4}\mathcal{K}^{\perp}$.

Remark 2.15.: The naturally arising question is why we did not treat the subject of §2 in the following way:
First, suppose A being a σ-finite W^*-algebra. Then, there exists a normal faithful state f on A and $\mathcal{H}$ can be completed to a Hilbert space $\mathcal{H}_f$ with the inner product $f(\langle\cdot,\cdot\rangle)$. Investigating $\mathcal{K}_f$, the norm-completion of $\mathcal{K}$ in $\mathcal{H}_f$, we would get operators P_f, Q_f, R_f, J_f, T_f and Δ_f on $\mathcal{H}_f$ as done at [8,§§1,2]. There would "remain" only to restrict all these operators to $\mathcal{H}\subseteq\mathcal{H}_f$.
Secondly, we would generalize these results for non-σ-finite W^*-algebras A using the construction of [1,p.164].
But, unfortunately, all these operators, in general, do not leave $\mathcal{H}\subseteq\mathcal{H}_f$ invariant and/or are not $\mathfrak{Z}(A)_h$-linear, at least if A is not commutative (cf. §5 of this paper).
Therefore, we can make use of this method only if the existence of the appropriate module operators is already known. So, we will do in the following paragraph.
Moreover, it seems to be possible to generalize the results of this paragraph if A is assumed only to be any commutative C^*-algebra. The condition (iii) for $\mathcal{K}$ reformulates then as follows:
(iii)' $\mathcal{K}$ is a real self-dual Hilbert A_h-submodule of $\mathcal{H}$.
For suggestions in this direction look at §4 of the present paper.

§3 One-parameter groups and the generalized K.M.S. condition

In this paragraph we consider strongly continuous unitary one-parameter groups on self-dual Hilbert W^*-modules $\mathcal{H}$ satisfying a generalized K.M.S. condition with respect to real subspaces $\mathcal{K}$, as they were defined at §2 of this paper. Especially, supposing A to be commutative, we define the group $\{\Delta^{it}: t\in\mathbb{R}\}$ in terms of the operator R related to $\mathcal{K}\subseteq\mathcal{H}$. This group is obtained to be characterized by the generalized K.M.S. condition. We remark the close relations of our results to the results of M.A.Rieffel,

A. van Daele [8,§3] and to the subject of F.Combes,H.Zettl [2,§3] .

Suppose A is a commutative W^*-algebra. According to Theorem 2.10., (1.), R and (2-R) are both injective, A-linear, selfadjoint positive operators for which $0 \leqq R \leqq 2$, $0 \leqq (2-R) \leqq 2$. From [10,Th.1.11.3.] we draw the spectral representation of R and (2-R). The spectral measure, therefore, is concentrated on the open interval (0,2). Now we can define the one-parameter groups R^{it} and $(2-R)^{it}$, $t \in \mathbb{R}$, since the map $\lambda \longrightarrow \lambda^{it}$ is correctly defined, bounded and continuous on (0,2) for any $t \in \mathbb{R}$. These groups commute and are strongly continuous on $\mathcal{H}$. Moreover, the equality

(7) $JR^{it}J = (2-R)^{-it}$

is satisfied for any $t \in \mathbb{R}$, (cf. Prop.2.10.,(5.)), where the minus sign in the right exponent is caused by the conjugate-A-linearity of J.

Definition 3.1.: Let $\Delta^{it} := (2-R)^{it}R^{-it}$, $t \in \mathbb{R}$, so that $\{\Delta^{it} : t \in \mathbb{R}\}$ is a strongly continuous unitary one-parameter group defined on $\mathcal{H}$.

Proposition 3.2.: For any $t \in \mathbb{R}$ there holds :

(i) $J\Delta^{it} = \Delta^{it}J$,

(ii) $\Delta^{it}(\mathcal{K}) = \mathcal{K}$, $\Delta^{it}(\mathcal{K}^\perp) = \mathcal{K}^\perp$,

(iii) $\Delta^{it}[\Delta^{1/4}(\mathcal{K})]^-_\tau = [\Delta^{1/4}(\mathcal{K})]^-_\tau = [\Delta^{-1/4}(i\mathcal{K}^\perp)]^-_\tau = \Delta^{it}[\Delta^{-1/4}(i\mathcal{K}^\perp)]^-_\tau$.

Proof: (cf. [8,Proof of Prop.3.3.])
According to (7) we get

$$\begin{aligned} J\Delta^{it} &= J(2-R)^{it}R^{-it} \\ &= R^{-it}JR^{-it} \\ &= R^{-it}(2-R)^{it}J \\ &= \Delta^{it}J \end{aligned}$$

for any $t \in \mathbb{R}$ since R^{it} and $(2-R)^{it}$ commute, i.e., the group $\{\Delta^{it} : t \in \mathbb{R}\}$ commutes with J. Since this group is a function of R it commutes with T and R. Thus, it commutes with P and Q. Consequently, $\Delta^{it}(\mathcal{K}) \subseteq \mathcal{K}$ and also $\Delta^{-it}(\mathcal{K}) \subseteq \mathcal{K}$ since $t \in \mathbb{R}$ is arbitrarily chosen, i.e., $\Delta^{it}(\mathcal{K}) = \mathcal{K}$. The other equality can be proved in the same way. The connection (iii) follows from Corollary 2.14..

Suppose now A is a W^*-algebra, non commutative in general. We remark that an A-valued function g defined on $D(g) \subseteq \mathbb{C}$ is called to be analytic if g is strong analytic in the sense of [1,Prop.2.5.21.] .

Definition 3.3.: (the generalized K.M.S. condition)
Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. A strongly continuous unitary one-parameter group $\{U_t : t \in \mathbb{R}\}$ defined on $\mathcal{H}$ is said to satisfy the generalized K.M.S. condition with respect to the real norm-complete subspace $\mathcal{K}$ of $\mathcal{H}$, $\mathcal{K}$ being invariant under the action of $\mathfrak{Z}(A)_h$ and satisfying the conditions (i) - (iii) at the beginning of §2, if for any $\bar{x},\bar{y} \in \mathcal{K}$ there exists a map $g: \mathbb{C} \longrightarrow A$ defined, bounded and continuous on $\{z: -1 \leqq \mathrm{Im}(z) \leqq 0\}$ and analytic in the interior of this strip, with boundary values given by

$$\begin{aligned} g(t) &= \langle U_t(\bar{x}),\bar{y}\rangle \\ g(t-i) &= \langle \bar{y},U_t(\bar{x})\rangle \end{aligned}$$

for any $t \in \mathbb{R}$.

Proposition 3.4.: Such a function g as it is defined by Definition 3.3. is unique.

Proof: Suppose there are two functions g and h with the same given properties. Furthermore, first suppose A being σ-finite. Then , there exists a normal faithful state f on A and the complex-valued function $f((g-h)(g-h)^*)$ is defined, bounded and continuous on $\{z: -1 \leqq \mathrm{Im}(z) \leqq 0\}$, analytic in the interior of this strip with trivial boundary conditions. By [8,p.195] this function must be equal to zero on the given strip, (cf. Remark 2.14.). Consequently, g=h on $\{z: -1 \leqq \mathrm{Im}(z) \leqq 0\}$.
If A is not σ-finite by [1,p.164] there exists an increasing directed net $\{p_\alpha : \alpha \in I\}$ of projections, $p_\alpha \in A$, such that $p_\alpha A p_\alpha$ is σ-finite for any $\alpha \in I$ and $w^*\text{-}\lim p_\alpha = 1_A$. Investigating the Hilbert $p_\alpha A p_\alpha$-module $\mathcal{H}_\alpha = \{p_\alpha \mathcal{H}, p_\alpha \langle .,. \rangle p_\alpha\}$, $\mathcal{K}_\alpha = p_\alpha \mathcal{K}$ and the functions $p_\alpha g p_\alpha$, $p_\alpha h p_\alpha$ for any $\alpha \in I$ we get $p_\alpha g p_\alpha = p_\alpha h p_\alpha$ on the strip $\{z: -1 \leqq \mathrm{Im}(z) \leqq 0\}$. Hence, g=h on this strip.

Proposition 3.5.: Suppose the situation given in Definition 3.3.. A strongly continuous unitary one-parameter group $\{U_t : t \in \mathbb{R}\}$ on $\mathcal{H}$ satisfies the generalized K.M.S. condition with respect to $\mathcal{K}$ if and only if for any $\bar{x},\bar{y} \in \mathcal{K}$ there exists a function $g: \mathbb{C} \longrightarrow A$ defined, bounded and continuous on $\{z: -1/2 \leqq \mathrm{Im}(z) \leqq 0\}$, analytic in the interior of this strip with boundary conditions

$$\begin{aligned} g(t) &= \langle U_t(\bar{x}),\bar{y}\rangle \text{ for any } t \in \mathbb{R}, \\ g(t-i/2) &\in A_h \text{ for any } t \in \mathbb{R}. \end{aligned}$$

Proof: If $\bar{x},\bar{y}\in\mathcal{K}$ are given we define $g(a-bi):=g(a-(1-b)i)^*$ for any number $a\in\mathbb{R}$, $b\in[1/2,1]$. This function is defined, bounded and continuous on $\{z: -1\leqq \operatorname{Im}(z)\leqq 0\}$ and has boundary values

$$g(t) = \langle U_t(\bar{x}),\bar{y}\rangle ,$$
$$g(t-i) = \langle \bar{y},U_t(\bar{x})\rangle$$

for any $t\in\mathbb{R}$. If f is an arbitrary normal state on A the construction above gives us the Schwar z reflection principle applied to the analytic function $f(g(z))$ along the line $\operatorname{Im}(z)=-1/2$, i.e., $f(g(z))$ is analytic on $\{z: -1\leqq \operatorname{Im}(z)\leqq 0\}$ for any normal state f on A. Therefore, g is analytic on this strip.

Conversely, suppose the generalized K.M.S. condition is valid and g is the K.M.S. function. Defining another function $h: \mathbb{C}\longrightarrow A$ by the formula $h(z):=g(z-i)^*$ we have two functions g,h with the same properties and boundary values. Since the K.M.S. function is unique (Prop. 3.4.), g=h and $g(t-i/2)=h(t-i/2)=g(t-i/2)^*$ for any $t\in\mathbb{R}$. Consequently, $g(t-i/2)\in A_h$ for any $t\in\mathbb{R}$.

Now we return to consider the group $\{\Delta^{it}: t\in\mathbb{R}\}$ in the case of A being commutative.

Lemma 3.6.: For any complex number z with $\operatorname{Im}(z)\leqq 0$ we can define the bounded operator R^{iz}. The map $z\longrightarrow R^{iz}$ is strongly operator continuous on $\{z: \operatorname{Im}(z)\leqq 0\}$ and analytic in the interior of this set. It is uniformly bounded on horizontal strips of finite width. The same results are true for $(2-R)^{iz}$.

Proof: (cf. [8,Proof of Lemma 3.6.])
The spectral measure of R is supported on the open interval (0,2). For any complex number z with $\operatorname{Im}(z)\leqq 0$ the function $\lambda\longrightarrow\lambda^{iz}$ is bounded and continuous on (0,2). Thus, we can define R^{iz}. Since $|\lambda^{iz}|\leqq 2^{-\operatorname{Im}(z)}$ if $\lambda\in(0,2)$ and since $\operatorname{Im}(z)\leqq 0$ by supposition there follows that R^{iz} is uniformly bounded on horizontal strips of finite width. Now let $\bar{x}\in\mathcal{H}$ and $\{E_\varepsilon: \varepsilon>0\}$ be the spectral resolution of R. Then, the restriction of R to $(1-E_\varepsilon)\mathcal{H}$ has its spectrum in $(\varepsilon,2)$ and so has bounded logarithm. Consequently, the function $R^{iz}(1-E_\varepsilon)\bar{x}$ is analytic in the entire complex plane. Since R is injective, $(1-E_\varepsilon)\bar{x}$ converges to $\bar{x}$ as ε goes to zero. Thus, $R^{iz}(1-E_\varepsilon)\bar{x}$ converges to $R^{iz}(\bar{x})$, in fact uniformly on horizontal strips of finite width where $\operatorname{Im}(z)\leqq 0$ since R^{iz} is uniformly bounded there. Therefore, the map $z\longrightarrow R^{iz}\bar{x}$ is continuous on $\{z: \operatorname{Im}(z)\leqq 0\}$ and analytic in the interior of this set.

Proposition 3.7.: The one-parameter group Δ^{it} defined in Definition 3.1. satisfies the generalized K.M.S. condition with respect to $\mathcal{K}$.

Proof: (cf. [8,Proof of Prop.3.7.])
Let $\bar{x},\bar{y}\in\mathcal{K}$. We can not set $g(z)=\langle\Delta^{iz}(\bar{x}),\bar{y}\rangle$ since this is undefined, in general, on $\{z: -1/2\leqq \operatorname{Im}(z)\leqq 0\}$. But we obtain

$$2\bar{x} = 2P(\bar{x}) = (R+TJ)(\bar{x}) = (R^{1/2}(R^{1/2}+(2-R)^{1/2}J))(\bar{x})$$

and, therefore,

(8) $\quad \bar{x} = R^{1/2}(\bar{z})$ where $\bar{z} = 1/2\cdot(R^{1/2}+(2-R)^{1/2}J)(\bar{x})$.

Then

(9) $\quad \Delta^{it}(\bar{x}) = \Delta^{it}R^{1/2}(\bar{z}) = (2-R)^{it}R^{1/2-it}(\bar{z})$

and we can define a function

$$g(z):= \langle(2-R)^{iz}R^{1/2-iz}(\bar{z}),\bar{y}\rangle$$

on $\{z: -1/2\leqq \operatorname{Im}(z)\leqq 0\}$. This function is defined, bounded and continuous on $\{z: -1/2\leqq \operatorname{Im}(z)\leqq 0\}$ and analytic in the interior of this set, since Lemma 3.6. is valid and the multiplication of operators is strongly continuous on bounded sets. Using (8) we get

$$g(t) = \langle\Delta^{it}(\bar{x}),\bar{y}\rangle \text{ for any } t\in\mathbb{R}.$$

There remains to show $g(t-i/2)\in A_h$ for any $t\in\mathbb{R}$. The equality

$$\begin{aligned} g(t-i/2) &= \langle((2-R)^{it}(2-R)^{1/2}R^{it})(\bar{z}),\bar{y}\rangle \quad \text{, cf. (9)}\\ &= \langle\Delta^{it}(2-R)^{1/2}(\bar{z}),\bar{y}\rangle\\ &= \langle(2-R)^{1/2}(\bar{z}),\Delta^{-it}(\bar{y})\rangle \end{aligned}$$

is satisfied where

$$\begin{aligned} 2(2-R)^{1/2}(\bar{z}) &= (2-R)^{1/2}(R^{1/2}+(2-R)^{1/2}J)(\bar{x}) \quad \text{, cf. (9)}\\ &= (T+JR)(\bar{x}) \quad \text{, cf. Prop. 2.10.,(5.)}\\ &= J\,(TJ+R)(\bar{x})\\ &= 2JP(\bar{x})\\ &= 2J(\bar{x}). \end{aligned}$$

Consequently, $g(t-i/2)=\langle J(\bar{x}),\Delta^{-it}(\bar{y})\rangle\in A_h$ for any $t\in\mathbb{R}$ because of Corollary 2.9. and Proposition 3.2., (ii).

We want to prove

Proposition 3.8.: The group $\{\Delta^{it}: t\in\mathbb{R}\}$ is the unique strongly continuous one-parameter group of unitary operators on $\mathcal{H}$ mapping $\mathcal{K}$ onto $\mathcal{K}$ and satisfying the generalized K.M.S. condition with respect to $\mathcal{K}$.

For this end we show the more general

Theorem 3.9.: Let A be any W^*-algebra and let $\mathcal{H}, \mathcal{K}$ as in paragraph two. Let $\{V_t : t\in\mathbb{R}\}$ be a strongly continuous unitary one-parameter group on $\mathcal{H}$, mapping $\mathcal{K}$ onto $\mathcal{K}$ and satisfying the generalized K.M.S. condition with respect to $\mathcal{K}$. Then, this group is the unique strongly continuous unitary one-parameter group with these properties.

Proof: Let $\{U_t : t\in\mathbb{R}\}$ be another such group on $\mathcal{H}$. We want to show $U_t=V_t$ for any $t\in\mathbb{R}$.
First, suppose A to be σ-finite and let us denote a normal faithful state on A by f. We extend the both one-parameter groups from $\mathcal{H}$ to the Hilbert space $\mathcal{H}_f$, $\mathcal{H}_f$ being the completion of $\mathcal{H}$ with respect to the norm $f(\langle\cdot,\cdot\rangle)^{1/2}$, (cf. [7,Prop.2.6.]). The extended one-parameter groups are strongly continuous and unitary on $\mathcal{H}_f$. They map $\mathcal{K}_f$ onto $\mathcal{K}_f$, where $\mathcal{K}_f$ denotes the norm-completion of $\mathcal{K}$ in $\mathcal{H}_f$. Also they satisfy the K.M.S. condition with respect to $\mathcal{K}_f$ what can be seen applying f to the generalized K.M.S. condition for $\{U_t : t\in\mathbb{R}\}$ and $\{V_t : t\in\mathbb{R}\}$, respectively. Consequently, from [8,Th.3.8.,Th.3.9.] we derive $V_t=U_t$, $t\in\mathbb{R}$, on $\mathcal{H}_f$ and by construction also on $\mathcal{H}$.
Secondly, if now A is supposed not to be σ-finite there exists a directed increasing net $\{p_\alpha : \alpha\in I\}$ of projections of A satisfying $w^*\text{-}\lim p_\alpha=1_A$ such that $p_\alpha A p_\alpha$ is a σ-finite W^*-algebra for each $\alpha\in I$. Looking for the Hilbert $p_\alpha A p_\alpha$-module $\mathcal{H}_\alpha:=\{p_\alpha\mathcal{H}, p_\alpha\langle\cdot,\cdot\rangle p_\alpha\}$, $\mathcal{K}_\alpha:=p_\alpha\mathcal{K}_\alpha$ and the groups $\{p_\alpha U_t : t\in\mathbb{R}\}$, $\{p_\alpha V_t : t\in\mathbb{R}\}$ we get $p_\alpha U_t=p_\alpha V_t$ on $p_\alpha\mathcal{H}$ for each $\alpha\in I$, for any $t\in\mathbb{R}$. Therefore, $U_t=V_t$ for any $t\in\mathbb{R}$ on $\mathcal{H}$. So we are done.

We remark that for Corollary 3.8. an appropriate generalization of the proof at [8,Th.3.8.] can be realized. But it will be larger than the present one.

Theorem 3.10.: Let A be any W^*-algebra and let $\mathcal{H}, \mathcal{K}$ as in paragraph two. Let $\{U_t : t\in\mathbb{R}\}$ be a strongly continuous one-parameter group of unitaries on $\mathcal{H}$. Suppose that $\mathcal{K}_o$ is a real submanifold of $\mathcal{H}$ being invariant under the action of $\mathfrak{Z}(A)_h$ and such that $\{U_t : t\in\mathbb{R}\}$ satisfies the generalized K.M.S. condition for $\mathcal{K}_o$. Then $\{U_t : t\in\mathbb{R}\}$ also satisfies the generalized K.M.S. condition with respect to the smallest real norm-closed subspace, $\mathcal{K}$, invariant under $\{U_t : t\in\mathbb{R}\}$, containing $\mathcal{K}_o$ and satisfying $\mathcal{K}=[\mathcal{K}]_\tau^-$. Furthermore, $\mathcal{K}\cap i\mathcal{K}=\{0\}$, so that if we let $\mathcal{H}_1$ denote the norm-closure of $\mathcal{K}+i\mathcal{K}$, then we can define $\{\Delta^{it} : t\in\mathbb{R}\}$ on $\mathcal{H}_1$ using $\mathcal{K}$. Then $\mathcal{H}_1$ is invariant under $\{U_t : t\in\mathbb{R}\}$ and $U_t=\Delta^{it}$ for any $t\in\mathbb{R}$ on $\mathcal{H}_1$.

Proof: (cf. [8,Th.3.9.])
The proof is exactly the same as for [8,Th.3.9.]. The only difficult point is to prove that $\mathcal{K}_1=[\mathcal{K}_1]_\tau^-$. Indeed, let $\bar{y}\in\mathcal{K}_o$, let $\{\bar{x}_\alpha : \bar{x}_\alpha\in\mathcal{K}_1, \alpha\in I\}$ be a bounded net and let g_α be the generalized K.M.S. function (on $\{z: -1\leqq \mathrm{Im}(z)\leqq 0\}$) for the pair $(\bar{x}_\alpha,\bar{y})$. Assume that $\{\bar{x}_\alpha : \alpha\in I\}$ converges to $\bar{x}\in\mathcal{H}$ with respect to the τ_1-topology. Furthermore, first assume A to be σ-finite, and let f be an arbitrary normal faithful state on A. Then, applying the maximum modulus principle on the strip $\{z: -1\leqq\mathrm{Im}(z)\leqq 0\}$ to the function $f(g_\alpha-g_\beta)$ we obtain

$$|f(g_\alpha-g_\beta)(z)|\leqq f(\langle\bar{x}_\alpha-\bar{x}_\beta,\bar{x}_\alpha-\bar{x}_\beta\rangle)^{1/2}\cdot f(\langle\bar{y},\bar{y}\rangle)^{1/2}$$

for any $\alpha,\beta\in I$ and, hence, $\{f(g_\alpha) : \alpha\in I\}$ is a uniform Cauchy net of complex-valued functions being bounded and continuous on the given strip and analytic on its interior. Moreover,

$$\lim_{\alpha\in I} f(g_\alpha(t)) = \lim_{\alpha\in I} f(\langle U_t(\bar{x}_\alpha),\bar{y}\rangle) = f(\langle U_t(\bar{x}),\bar{y}\rangle)$$

so that this Cauchy net converges uniformly to a function f_g being defined, bounded and continuous on the given strip and analytic on its interior. Moreover, there holds $f_g(t-i)^*=f_g(t)=f(\langle U_t(\bar{x}),\bar{y}\rangle)$. Since f was arbitrarily chosen, by [1,Prop.2.5.21.] there has to exist an A-valued function g which is defined, bounded and continuous on $\{z: -1\leqq\mathrm{Im}(z)\leqq 0\}$ and analytic on the interior of this strip, and which satisfies the boundary conditions

$$g(t-i)^* = g(t) = \langle U_t(\bar{x}),\bar{y}\rangle \quad \text{for any } t\in\mathbb{R}.$$

Therefore, if A is σ-finite there holds $\mathcal{K}_1=[\mathcal{K}_1]_\tau^-$.
Suppose now A to be non-σ-finite. Then, there exists a net $\{p_\alpha : p_\alpha\in A, \alpha\in I\}$ of projections such that $p_\alpha A p_\alpha$ is σ-finite for any $\alpha\in I$ and $w^*\text{-}\lim p_\alpha = 1_A$. Consequently, $p_\alpha\mathcal{K}_1=[p_\alpha\mathcal{K}_1]_\tau^-$ for any $\alpha\in I$ and, hence, $\mathcal{K}_1=[\mathcal{K}_1]_\tau^-$. The desired statement is proved.

§4 An interpretation on locally trivial Hilbert bundles

In two papers of R.G.Swan [11] and J.Dixmier,A.Douady [3] it was stated that for every locally trivial Hilbert bundle $\xi=(E,X,p,H)$ with compact basis X the set of all continuous sections $\Gamma(\xi)$ is a Hilbert C(X)-module. Later A.O.Takahashi [12] has proved that for every compact space X the category of Hilbert C(X)-moduli $\mathcal{H}$ is equivalent to the category of locally trivial Hilbert bundles $\xi=(E,X,p,H)$. The equivalence is realized by the map $\mathcal{H} \longleftrightarrow \Gamma(\xi)$, (cf. [6,Th.8.11.]). Thus, a C(X)- linear operator on $\mathcal{H}$ can be interpreted as an operator on $\Gamma(\xi)$ leaving the fibres invariant.

Now let X be hyperstonian. Let ξ be a Hilbert bundle over X with the fibre H. For our purpose the Hilbert bundle ξ must be "full" in that sense that $\mathcal{H}=\Gamma(\xi)$ has to be self-dual, cf. Theorem 1.2.. Let $\mathcal{K} \subseteq \mathcal{H}=\Gamma(\xi)$ be as in §2 of the present paper. If we fix some $x \in X$ and regard $\mathcal{H}_x=\{\bar{x}(x): \bar{x} \in \mathcal{H}\} \subseteq H$, $\mathcal{K}_x=\{\bar{x}(x): \bar{x} \in \mathcal{K}\} \subseteq H$, the classical Tomita-Takesaki theory on H obtains the bounded operators $P_x, Q_x, R_x, T_x, J_x, \Delta_x^{it}$ $(t \in \mathbb{R})$ for the pair $(\mathcal{H}_x, \mathcal{K}_x)$. All these operators can be continued to bounded operators on H. For the global operators defined by $(\mathcal{H},\mathcal{K})$ we get the following property:

(10) $(U(\bar{x}))(x) = U_x(\bar{x}(x))$ for any $x \in X$, $\bar{x} \in \mathcal{H} = \Gamma(\xi)$,
for $U= P,Q,R,T,J,\Delta^{it}$ $(t \in \mathbb{R})$.

Therefore, in our setting we could define these global bounded operators for the pair $(\mathcal{H},\mathcal{K})$ by the formula (10) only splitting the **appropriate** bounded operators for the pairs $(\mathcal{H}_x,\mathcal{K}_x)$, $x \in X$, on X.

The natural question arising from this observation is whether or not this could be done if we merely suppose that X is any compact space, $\mathcal{H}=\Gamma(\xi)$ is the set of continuous sections of any locally trivial Hilbert bundle ξ over X and $\mathcal{K}$ is a Hilbert $C(X)_h$-module out of $\mathcal{H}$, satisfying the conditions (i) and (ii) at the beginning of §2 ?

If both $\mathcal{H}$ and $\mathcal{K}$ are self-dual the global operators P,Q,R,T,J can be defined in this way, (10), by the appropriate local operators. For the strongly continuous unitary one-parameter groups $\{\Delta_x^{it}: x \in X, t \in \mathbb{R}\}$ the splitted group $\{\Delta^{it}: t \in \mathbb{R}\}$ maps $\mathcal{H}=\Gamma(\xi)$ into a larger class of sections of the Hilbert bundle ξ, in general. Therefore, this question must remain to be unanswered.

§5 Remarks on the non commutative case

Let A be a noncommutative W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. Let $\mathcal{K}$ be as defined in §2 of the present paper. The following example shows the difficulties arising if we try to repeat our conception for the noncommutative case.

Example 5.1.: Let $A=End_{\mathbb{C}}(l_2)$ where l_2 is the countably generated

Hilbert space. Let $\mathcal{H}=l_2(A)'$ and $\mathcal{K}=l_2(A_h)'$, (cf. [4]). We want to define the $\mathfrak{Z}(A)_h$-linear projections $P:\mathcal{H} \longrightarrow \mathcal{K}$, $Q:\mathcal{H} \longrightarrow i\mathcal{K}$. If $\{a_i: i=1,\dots,n,\ a_i \in A\} \in \mathcal{H}$ is a sequence of finite length and if $a_i=a_{1i}+a_{2i}i$, $(a_{1i},a_{2i} \in A_h)$, we would define

$$P:\ \{a_i: i \in \mathbb{N}\} \longrightarrow \{a_{1i}: i \in \mathbb{N}\}\ ,$$
$$Q:\ \{a_i: i \in \mathbb{N}\} \longrightarrow \{a_{2i}: i \in \mathbb{N}\}$$

by analogy to the (unique) decomposition of each element $a_i \in A$. Now we take the element $\bar{b}=\{b_i: b_i \in A,\ i \in \mathbb{N}\} \in l_2(A)'$ where

$$b_i:\ \bar{e}_1 \longrightarrow \bar{e}_i\ ,\quad \bar{e}_j \longrightarrow \bar{0}\ ,\ j \neq 1,$$
$$b_i^*:\ \bar{e}_i \longrightarrow \bar{e}_1\ ,\quad \bar{e}_j \longrightarrow \bar{0}\ ,\ j \neq i.$$

(if $\{\bar{e}_i: i \in \mathbb{N}\}$ denotes the standard orthonormal basis of $l_2(A)$). Therefore,

$$P:\ \{b_i: i \in \mathbb{N}\} \longrightarrow \{1/2\cdot(b_i+b_i^*): i \in \mathbb{N}\}$$

by definition. But the sequence $\{1/2\cdot(b_i+b_i^*): i \in \mathbb{N}\}$ does not belong to $l_2(A)'$. To see this we look at

$$1/4\cdot\left[\sum_{i=1}^{N}(b_i+b_i^*)^2\right](\bar{e}_1) = (N+3)/4.\bar{e}_1,\ N \in \mathbb{N}.$$

Obviously the sequence above diverges and, hence, $P(\bar{b})$ does not belong to $l_2(A)'$. The condition being equivalent to $P(\bar{a}) \in l_2(A)'$ for a certain $\bar{a} \in l_2(A)'$ is

$$\{a_{1i}: i \in \mathbb{N},\ a_i=a_{1i}+a_{2i}i,\ a_{1i},a_{2i} \in A_h\} \in l_2(A)'\ .$$

The set D(P) of all such elements $\bar{a} \in l_2(A)'$ yields $[D(P)]_\tau^- = \mathcal{H}$, but D(P) is not norm-dense in $\mathcal{H}$.

If we sum up our results we obtain that P and Q are unbounded $\mathfrak{Z}(A)_h$-linear operators on $\mathcal{H}$ with not norm-dense, different ranges of definition D(P), D(Q) for which $\mathcal{H}=[D(P)]_\tau^- = [D(Q)]_\tau^- = [D(P) \cap D(Q)]_\tau^-$. The operator R=P+Q is the identity operator on $\mathcal{H}$ being A-linear. The operator J is unbounded conjugate-A-linear and it is defined on $D(P) \cap D(Q)$. J acts there by the formula

$$J:\ \bar{a}=\{a_i: i \in \mathbb{N}\} \longrightarrow \{a_i^*: i \in \mathbb{N}\}\ .$$

I.e., if A is **an infinite-dimensional** noncommutative W^*-algebra we can not repeat the bounded operator approach of this paper! The main reason is the unequivalence of the convergence of the series

$$\sum_{i=1}^{\infty} a_i a_i^* \quad \text{and} \quad \sum_{i=1}^{\infty} a_i^* a_i$$

for sequences $\{a_i : a_i \in A\} \in l_2(A)$ in A, in general.

Now we discuss the case of A being noncommutative and finite-dimensional. In this case A can be assumed to be a complex matrix algebra of finite dimension. The main phenomena will be shown by the following example.

Example 5.2.: Let A be a finite-dimensional C^*-algebra and let $\mathcal{H}=A$ with the A-valued inner product $\langle a,b\rangle = ab^*$, $a,b \in \mathcal{H}$. Let $\mathcal{K}=A_h$. Since A is finite-dimensional the norm induced on $A=\mathcal{H}$ by the complex-valued inner product $f(\langle\cdot,\cdot\rangle)$ is equivalent to the given Hilbert norm on $\mathcal{H}$ for any faithful positive (normal) state f on A. By [4,Th.2] for the positive faithful state f, there exists a bounded linear operator C on $\mathcal{H}$ such that

(i) $f(\langle a,b\rangle)=tr(\langle a,C(b)\rangle)$ for any $a,b \in \mathcal{H}$.

(ii) $0 \leqq C = C^*$ respectively to $tr(\langle\cdot,\cdot\rangle)$ and to $f(\langle\cdot,\cdot\rangle)$.

(iii) There exists C^{-1} being bounded and linear.

By the equality

$$\begin{aligned} tr(\langle a,C(\mathbf{1}_A)b\rangle) &= tr(\langle ab,C(\mathbf{1}_A)\rangle) = f(\langle ab,\mathbf{1}_A\rangle) \\ &= f(\langle a,b\rangle) = tr(\langle a,C(b)\rangle) \end{aligned}$$

for any $a \in \mathcal{H}$, $b \in \mathcal{K}$ there turns out that

$C(b) = C(\mathbf{1}_A)b$ for any $b \in \mathcal{K}$ (and by complex linearity for any $b \in \mathcal{H}$).

Therefore,

$$f(\langle a,b\rangle) = tr(\langle a,b\rangle \cdot C(\mathbf{1}_A)) \quad \text{for any } a,b \in \mathcal{H}$$

and $C(\mathbf{1}_A)$ is a selfadjoint positive element of $\mathcal{H}=A$, the eigenvalues of which are all greater than zero.

Since we can not regard $\mathcal{K}$ as a real (self-dual) Hilbert submodule of $\mathcal{H}$ and since the Hilbert norm is equivalent to the norm $f(\langle\cdot,\cdot\rangle)^{1/2}$ for any faithful positive state f on A, we would like to define the projection $P:\mathcal{H} \longrightarrow \mathcal{K}$ as the projection P_f mapping the real Hilbert space $\{\mathcal{H}, \mathrm{Re}\, f(\langle\cdot,\cdot\rangle)\}$ onto the real Hilbert subspace $\{\mathcal{K}, \mathrm{Re}\, f(\langle\cdot,\cdot\rangle)\}$. But, unfortunately, such a projection P_f depends on the choice of f ! If f=tr we get $P_{tr}: a \in \mathcal{H} \longrightarrow 1/2(a+a^*) \in \mathcal{K}$. This projection is $\mathcal{Z}(A)_h$-linear and bounded by one. If f is now an arbitrary faithful positive state on A and if we suppose $P_f=P_{tr}$ by (i) we obtain the equality (if $a=a_1+a_2i \in \mathcal{H}$, $a_1,a_2 \in A_h$)

$$\begin{aligned} tr((a_1b + ba_1)\cdot C_f(\mathbf{1}_A)) &= f(a_1b + ba_1) \\ &= f(ab + ba) \\ &= tr((a_1b+ba_1+i(a_2b-ba_2))\cdot C_f(\mathbf{1}_A)) \end{aligned}$$

for any $a \in \mathcal{H}$, $b \in \mathcal{K}$, (cf. Proposition 2.4.(1)). Therefore,

$$\begin{aligned} 0 &= tr((a_2b-ba_2)\cdot C_f(\mathbf{1}_A)) \\ &= tr(b\cdot C_f(\mathbf{1}_A)\cdot a_2 - a_2\cdot C_f(\mathbf{1}_A)\cdot b) \end{aligned}$$

for any $a_2,b \in \mathcal{K}=A_h$. This is true if and only if $C_f(\mathbf{1}_A)$ is a diagonal matrix. But, since f is arbitrarily chosen, the matrix $C_f(1_A)$ can be any **selfadjoint** positive matrix with strictly positive eigenvalues.

This is a contradiction to our supposition $P_f=P_{tr}$.

Therefore, $P_f=P_{tr}$ if and only if $C_f(\mathbf{1}_A)$ is a selfadjoint positive diagonal matrix with strictly positive elements.

Summing up we have to ask whether for each finite C^*-algebra A and for any given $\mathcal{H}$ and $\mathcal{K}$ exists a faithful positive state f on A such that the induced projection $P_f:\mathcal{H} \longrightarrow \mathcal{K}$ is $\mathcal{Z}(A)_h$-linear and bounded by one with respect to the Hilbert norm on $\mathcal{H}$. Unfortunately, we are not able to answer this question at present.

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Франк М. E5-87-95

Однопараметрические группы, возникающие в вещественном подпространстве автодуальных гильбертовых модулей

Обобщаются результаты М.А.Рифеля и А. фан Дэля для гильбертовых C*-модулей над коммутативными W*-алгебрами. Изучаются некоторые специальные вещественные подпространства таких гильбертовых W*-модулей и относящиеся к ним операторы. В частности, установлено соотношение между сильнонепрерывными унитарными однопараметрическими группами операторов, связанными с ними, и обобщенным условием КМШ. Все главные определения сформулированы без предпосылки коммутативности подлежащей W*-алгебры. Дается интерпретация этих результатов для множеств непрерывных сечений "автодуальных" локально тривиальных гильбертовых расслоений над компактными пространствами.

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One-Parameter Groups Arising from Real Subspaces of Self-Dual Hilbert W*-Moduli

The paper generalizes the results of M.A.Rieffel and A. van Daele for Hilbert C*-moduli over commutative W*-algebras. Some special real subspaces of such Hilbert W*-moduli and the related operators are investigated. Particularly, the relation is established between strongly continuous unitary one-parameter groups of operators arising from them and the generalized K.M.S. condition. All key definitions are formulated without any commutativity supposition for the underlying W*-algebra. The interpretation of these results is given for sets of continuous sections of "self-dual" locally trivial Hilbert bundles over compact spaces.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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