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VON NEUMANN REPRESENTATIONS ON SELF-DUAL HILBERT W*-MODULI

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§1 Introduction

A (left) pre-Hilbert A-module over a certain C^*-algebra A is an A-module $\mathcal{H}$ equipped with an A-valued inner product, i.e., an A-valued nondegenerate sesquilinear mapping $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow A$, $\langle\cdot,\cdot\rangle$ being A-linear at the first argument. The pre-Hilbert A-module $\mathcal{H}$ is Hilbert if it is complete with respect to the norm $\|\cdot\| = \|\langle\cdot,\cdot\rangle\|_A^{1/2}$. We suppose always that the linear structures on A and on $\mathcal{H}$ are compatible. For further basic facts concerning Hilbert C^*-moduli we refer to [6]. A Hilbert A-module $\mathcal{H}$ over a C^*-algebra A is called self-dual if every bounded module map $r:\mathcal{H} \longrightarrow A$ is of the form $\langle\cdot,\bar{a}\rangle$ for some $\bar{a}\in\mathcal{H}$. In this paper we restrict our attention mainly to Hilbert W^*-moduli. For them some more facts are known. We need the following ones:

Definition 1.1.: [3, Def.7]

Let A be a W^*-algebra, $\mathcal{H}$ be a pre-Hilbert A-module and P be the set of all normal states on A. The topology induced on $\mathcal{H}$ by the seminorms

$$f(\langle\cdot,\cdot\rangle)^{1/2}\ ,\ f\in P,$$

is denoted by τ_1. The topology induced on $\mathcal{H}$ by the linear functionals

$$f(\langle\cdot,\bar{a}\rangle)\quad ,\ f\in P,\ \bar{a}\in\mathcal{H},$$

is denoted by τ_2.

Throughout this paper we use the following notation. If $\mathcal{K}$ is a subset of the Hilbert W^*-module $\mathcal{H}$, $[\mathcal{K}]_\tau$ denotes the set $\{\lambda\cdot\bar{x} : \lambda\in\mathbb{R}_+,\ \bar{x}\in\mathcal{K}_0\}$ where $\mathcal{K}_0$ is the τ_1-completion of the set $\{\bar{x}\in\mathcal{K} : \|\bar{x}\|\leq 1\}$.

Theorem 1.2.: [3,Th.9]
Let A be a W^*-algebra and $\mathcal{H}$ be a Hilbert A-module. The following conditions are equivalent:

(i) $\mathcal{H}$ is self-dual.

(ii) The unit ball of $\mathcal{H}$ is complete with respect to the topology τ_1, i.e., $\mathcal{H} = [\mathcal{H}]^-_\tau$.

(iii) The unit ball of $\mathcal{H}$ is complete with respect to the topology τ_2.

Corollary 1.3.: [3,Cor.11]
If A is a W^*-algebra and $\mathcal{H}$ is a self-dual Hilbert A-module the linear span of the range of the A-valued inner product on $\mathcal{H}$ becomes both a W^*-subalgebra and an ideal in A.

Theorem 1.4.: [6,Prop.3.10.]
Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. Then, the set $End_A(\mathcal{H})$ of all bounded A-linear operators on $\mathcal{H}$ is a W^*-algebra.

These facts make clear that in the case of $\mathcal{H}$ being a self-dual Hilbert W^*-module the spectral theorem ([9,Th.1.11.3.] , [5,Th.7,8,9]) is valid for each self-adjoint element of $End_A(\mathcal{H})$. Moreover, there exists a polar decomposition for each element of $End_A(\mathcal{H})$ in $End_A(\mathcal{H})$. This is of importance for the existence of certain operators arising from some special real subspaces $\mathcal{K}$ of self-dual Hilbert W^*-moduli $\mathcal{H}$, as they were treated in [4].
We remark that for a given W^*-algebra A any considered self-dual Hilbert A-module $\mathcal{H}$ can be assumed to have A as the linear span of the range of its A-valued inner product, (cf. Corollary 1.3.). Otherwise we would change A to this linear span of the range, $B \subset A$, and we would consider $\mathcal{H}$ as a Hilbert B-module.
Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. §2 of this paper investigates (generalized) von Neumann algebras M on $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}$ in $\mathcal{H}$. The definition of a cyclic element for M is modified in a nonobvious way. We show the relation between von Neumann algebras on $\mathcal{H}$ possessing a cyclic-separating element and such special real subspaces $\mathcal{K}$ of $\mathcal{H}$ as they were investigated in [4]. In addition, a generalization of Kaplansky's density theorem for $*$-algebras of bounded operators on $\mathcal{H}$ is stated. Under the supposition that A is commutative, we consider the partial conjugate-linear involution J and the strongly continuous unitary

one-parameter group $\{\Delta^{it} : t \in \mathbb{R}\}$ arising from that real subspace of $\mathcal{H}$ which corresponds to the pair $(M,\bar{x})$ in §3. They would fulfil the conditions of Tomita-Takesaki's theorem for M and M'.
§4 deals with the natural cone $\mathcal{P} \subseteq \mathcal{H}$ connected with the pair $(M,\bar{x})$. We investigate the main properties of this cone.

§2 Von Neumann algebras on self-dual Hilbert W^*-moduli

Let A be an arbitrary W^*-algebra. We would like to show that von Neumann algebras of bounded A-linear operators on a self-dual Hilbert A-module $\mathcal{H}$, which possess a cyclic-separating element $\bar{x} \in \mathcal{H}$, are closely related to certain special real subspaces $\mathcal{K}$ of $\mathcal{H}$, as they were treated in [4]. The key point of this paragraph is Definition 2.5. modifying the notion of a cyclic element for a von Neumann algebra over a self-dual Hilbert W^*-module. Thus, new aspects are produced in the discussions. At the end of this paragraph we state a generalization of Kaplansky's density theorem for $*$-algebras of bounded A-linear operators on self-dual Hilbert W^*-moduli over A.

Definition 2.1.: Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. A C^*-subalgebra M of $End_A(\mathcal{H})$, (the set of all bounded A-linear operators on $\mathcal{H}$), coinciding with its bicommutant M'' is called a "generalized" von Neumann algebra.

Corollary 2.2.: Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. Any "generalized" von Neumann algebra on $\mathcal{H}$ is a W^*-algebra, i.e., the word "generalized" is superfluous in Definition 2.1..

Proof: Since $End_A(\mathcal{H})$ is a W^*-algebra [6,Prop.3.10.] there exists a normal faithful representation π of $End_A(\mathcal{H})$, i.e., $\pi(End_A(\mathcal{H}))= = \pi(End_A(\mathcal{H}))''$. Moreover, $\pi(M)' \subseteq \pi(End_A(\mathcal{H}))'' = \pi(End_A(\mathcal{H}))$ since $\pi(End_A(\mathcal{H}))' \subseteq \pi(M)'$ holds for the commutants. By supposition $M = M''$, and hence, $\pi(M) = \pi(M)''$. Consequently, the representation π is faithful and normal for M, too.

We state now an important topological property of these von Neumann algebras.

Definition 2.3.: (cf. [3, Def.15])
Let A be a W^*-algebra, $\mathcal{H}$ be a Hilbert A-module and P be the set of all normal states on A. The bounded net $\{B_\alpha : B_\alpha \in End_A(\mathcal{H}), \alpha \in I\}$ converges to $B \in End_A(\mathcal{H})$ relative to the topology $\mathcal{T}_3$ if there exists

$$w^*\text{-}\lim_{\alpha\in I} \langle B_\alpha(\bar a),\bar b\rangle = \langle B(\bar a),\bar b\rangle$$

for any $\bar a,\bar b \in \mathcal{H}$. (For other topologies, cf. [3]).

<u>Proposition 2.4.:</u> Let A be a W^*-algebra and $\mathcal{H}$ be a self-dual Hilbert A-module. Let M be a C^*-subalgebra of $End_A(\mathcal{H})$. Then M is a von Neumann algebra if and only if the unit ball of M is complete with respect to the topology $\mathcal{T}_3$.

<u>Proof:</u> We take a bounded net with the following properties:

(i) $\{B_\alpha : B_\alpha \in M, \alpha \in I\}$ converges to $B\in End_A(\mathcal{H})$ relative to the topology $\mathcal{T}_3$.

(ii) $B_\alpha C = CB_\alpha$ for any $C\in M'$, $\alpha\in I$.

Therefore,

$$\begin{aligned} 0 &= w^*\text{-}\lim_{\alpha\in I} \langle (B_\alpha C - CB_\alpha)(\bar x),\bar y\rangle \\ &= w^*\text{-}\lim_{\alpha\in I} (\langle B_\alpha(C(\bar x)),\bar y\rangle - \langle B_\alpha(\bar x), C^*(\bar y)\rangle) \\ &= \langle (BC-CB)(\bar x),\bar y\rangle \end{aligned}$$

for any $\bar x,\bar y \in \mathcal{H}$. Thus, $B\in M''=M$.
On the other hand, suppose that the unit ball of M is complete with respect to the topology $\mathcal{T}_3$. We consider a faithful normal representation π of $End_A(\mathcal{H})$, (cf., Theorem 1.4.). It is also a faithful representation for M. By [6, Remark 3.9.] the topology $\mathcal{T}_3$ coincides with the weak topology on $End_A(\mathcal{H})$. Therefore, the unit ball of $\pi(M)$ is complete with respect to the weak* topology, and hence, by [1, Th.2.4.11.] $\pi(M)=\pi(M)''$. Consequently, $M=M''$ since π is a normal representation of $End_A(\mathcal{H})$.

By [6, Remark 3.9.], [3, Th.9.] we conclude that the topology $\mathcal{T}_3$ coincides with the weak* topology on the unit ball of M.
In further discussions we are especially interested in von Neumann algebras M on self-dual Hilbert W^*-moduli $\mathcal{H}$ possessing somewhat like a cyclic-separating element $\bar x\in\mathcal{H}$ similarly as σ-finite von Neumann algebras possess one in certain Hilbert spaces.

<u>Definition 2.5.:</u> Let A be a W^*-algebra, $\mathcal{H}$ be a self-dual Hilbert A-module and M be a von Neumann algebra on $\mathcal{H}$. An element $\bar x\in\mathcal{H}$ is called to be cyclic for M iff $[M\bar x]^-_\tau = \mathcal{H}$. It is called to be separating for M iff $B(\bar x)=\bar 0$ for some $B\in M$ implies $B=0$.

<u>Proposition 2.6.:</u> Let A be a W^*-Algebra, $\mathcal{H}$ be a self-dual Hilbert A-module and M be a von Neumann algebra on $\mathcal{H}$. If an element $\bar x\in\mathcal{H}$ is cyclic for M, it is separating for M'. If an element $\bar x\in\mathcal{H}$ is separating for M' and $A[M\bar x]^-_\tau \subseteq [M\bar x]^-_\tau$, it is cyclic for M.

<u>Proof:</u> First, we consider an element $B'\in M'$ such that $B'(\bar x)=\bar 0$. Then: $B'(B(\bar x))=(B'B)(\bar x)=(BB')(\bar x)=B(B'(\bar x))=\bar 0$ for any $B\in M$. If $\bar x$ is cyclic for M the element $B'\in M'$ must be equal to zero.
Secondly, there exists a projection $P'\in End_A(\mathcal{H})$, $P':\mathcal{H}\longrightarrow [M\bar x]^-_\tau$, (cf. Theorem 1.2.). The projection P' is contained in M'. Therefore, $(id_{\mathcal{H}}-P')(\bar x)=\bar 0$, and hence, $P'=id_{\mathcal{H}}$ since $\bar x\in\mathcal{H}$ is supposed to be separating for M'.

We know that a von Neumann algebra M possesses a cyclic-separating element in a certain Hilbert space if and only if M is σ-finite, [1, Prop.2.5.6.]. In our setting this statement is not in all cases true. However, we can state the following

<u>Lemma 2.7.:</u> Let A be a W^*-algebra, $\mathcal{H}$ be a self-dual Hilbert A-module and M be a von Neumann algebra on $\mathcal{H}$ possessing a cyclic-separating element in $\mathcal{H}$. Then:

(i) If A is σ-finite, M is σ-finite.

(ii) If M is σ-finite, the centre $\mathfrak{Z}(A)$ of A is σ-finite.

(iii) The von Neumann algebra M is not necessarily σ-finite.

<u>Proof:</u> If $\bar x\in\mathcal{H}$ is a cyclic-separating element for M and if $\{E_\alpha : E_\alpha\in M, \alpha\in I\}$ is a set of mutually orthogonal nonzero projections we obtain

$$\begin{aligned} \langle \bar x,\bar x\rangle &\geq \sum_{\alpha,\beta\in I} \langle E_\alpha(\bar x), E_\beta(\bar x)\rangle \\ &= \sum_{\alpha\in I} \langle E_\alpha(\bar x), E_\alpha(\bar x)\rangle \geq 0. \end{aligned}$$

Since A is σ-finite by assumption, there exists a faithful normal state f on A. Therefore,

$$+\infty > f(\langle \bar x,\bar x\rangle) \geq \sum_{\alpha\in I} f(\langle E_\alpha(\bar x), E_\alpha(\bar x)\rangle) > 0$$

and $+\infty > f(\langle E_\alpha(\bar x),E_\alpha(\bar x)\rangle) > 0$ for any $\alpha\in I$. Thus, the set I must be countable and M is σ-finite. This proves (i).
To show (ii) we remark that

$\{a\cdot id_{\mathcal{H}} : a\in \mathcal{Z}(A)\} = \mathcal{Z}(End_A(\mathcal{H})) \subseteq \mathcal{Z}(M)$

(cf. [7,Cor.7.10.,Prop.8.1.]). Consequently, if M is σ-finite, $\mathcal{Z}(A)$ has to be σ-finite, too.
The third statement can be shown considering a non-σ-finite W^*-algebra A, first, as a self-dual Hilbert A-module $\mathcal{H}$ with A-valued inner product $\langle a,b\rangle_A := ab^*$, $a,b\in A$, and secondly, as a von Neumann algebra M on itself ($A=\mathcal{H}$) with cyclic-separating element $1_A\in A$, where the elements of M are defined as multiplications of A with elements of A from the right. So , the Proposition is proved.

Now we describe the relation between von Neumann algebras M on self-dual Hilbert W^*-moduli $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$ and those special real subspaces $\mathcal{K}$ of $\mathcal{H}$ as they were treated in [4] under the supposition, that A is commutative.

<u>Proposition 2.8.:</u> Let A be a commutative W^*-algebra, $\mathcal{H}$ be a self-dual Hilbert A-module and M be a von Neumann algebra on $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$. Let $\mathcal{K} := [M_h\bar{x}]^-_\tau$. Then $\mathcal{K}$ is a real subspace of $\mathcal{H}$ being invariant under the action of A satisfying the conditions:
(i) $\mathcal{K}\cap i\mathcal{K}=\{\bar{0}\}$,
(ii) $\mathcal{K}+i\mathcal{K}$ is norm-dense in $\mathcal{H}$.
Moreover, $[M'_h\bar{x}]^-_\tau \subseteq i\mathcal{K}^\perp$.

<u>Proof:</u> For any $B\in M_h$, $B'\in M'_h$,the following two equalities are valid:

$$\langle B(\bar{x}),B'(\bar{x})\rangle = \langle B'(\bar{x}),B(\bar{x})\rangle \quad ,$$
$$0= \langle iB(\bar{x}),B'(\bar{x})\rangle + \langle B'(\bar{x}),iB(\bar{x})\rangle \quad .$$

Therefore, $B'(\bar{x})\in i\mathcal{K}^\perp$ for any $B'\in M'_h$. Now we obtain the relation $[M'_h\bar{x}]^-_\tau\subseteq i\mathcal{K}^\perp$ by obvious computations, (cf. [4, Prop.2.2.]). From $M'\bar{x}\subseteq i\mathcal{K}^\perp+\mathcal{K}^\perp\subseteq(\mathcal{K}\cap i\mathcal{K})^\perp$ we derive $(\mathcal{K}\cap i\mathcal{K})^\perp=\mathcal{H}$ since $\bar{x}\in\mathcal{H}$ is cyclic for M′by Proposition 2.6. and since $[(\mathcal{K}\cap i\mathcal{K})^\perp]^-_\tau=(\mathcal{K}\ i\mathcal{K})^\perp$ by [4,Proof of Prop.2.2.]. Consequently, $\mathcal{K}\cap i\mathcal{K}=\{\bar{0}\}$. Moreover, $M\bar{x}\subseteq\mathcal{K}+i\mathcal{K}\subseteq\mathcal{H}$ and we get that $\mathcal{K}+i\mathcal{K}$ has to be norm-dense in $\mathcal{H}$ since $\bar{x}\in\mathcal{H}$ is cyclic for M. The proposition is proved.

If A is a commutative W^*-algebra,we can apply the results of [4] to $\mathcal{K}=[M_h\bar{x}]^-_\tau$. We will make use of this in §3 of this paper.
At the end of the second paragraph we prove a generalization of Kaplansky's density theorem we need later.

<u>Definition 2.9.:</u> Let A be a W^*-algebra and $\{\mathcal{H},\langle\cdot,\cdot\rangle\}$ be a Hilbert A-module. We say that a bounded net $\{B_\alpha : \alpha\in I,\ B_\alpha\in End^*_A(\mathcal{H})\}$ converges to $B\in End^*_A(\mathcal{H})$ relative to the topology σ-$\mathcal{T}^*_2$ iff there exists

$$w^*\text{-}\lim_{\alpha\in I}\sum_{n=1}^{\infty}\left(\langle B_\alpha(\bar{x}_n),B_\alpha(\bar{x}_n)\rangle + \langle B^*_\alpha(\bar{x}_n),B^*_\alpha(\bar{x}_n)\rangle\right) = \sum_{n=1}^{\infty}\left(\langle B(\bar{x}_n),B(\bar{x}_n)\rangle + \langle B^*(\bar{x}_n),B^*(\bar{x}_n)\rangle\right)$$

for any sequence $\{\bar{x}_n : \bar{x}_n\in\mathcal{H},\ n\in\mathbb{N},\ \sum_{n=1}^{\infty}\|\bar{x}_n\|^2<+\infty\}$.

<u>Theorem 2.10.:</u> Let A be a W^*-algebra and $\{\mathcal{H},\langle\cdot,\cdot\rangle\}$ be a self-dual Hilbert A-module. Let N be a self-adjoint algebra of bounded, A-linear operators on $\mathcal{H}$, and let M be the von Neumann algebra arising as the linear hull of the $\mathcal{T}_3$-completion of the unit ball of N. Then,the unit ball of N is σ-$\mathcal{T}^*_2$-dense in the unit ball of M.

<u>Proof:</u> First, suppose A to be σ-finite. Let f be a normal faithful state on A. Considering the extensions of the operator algebras N and M from $End_A(\mathcal{H})$ to $End_{\mathbb{C}}(\mathcal{H}_f)$, cf. [6,Th.2.8.],(where $\mathcal{H}_f$ denotes the closure of $\mathcal{H}$ with respect to the norm $f(\langle\cdot,\cdot\rangle)^{1/2}$), we can apply Kaplansky's density theorem to $N\subseteq End_{\mathbb{C}}(\mathcal{H}_f)$. Therefore, the unit ball of $N\subseteq End_{\mathbb{C}}(\mathcal{H}_f)$ is σ-strong*-dense in the unit ball of $M''\subseteq End_{\mathbb{C}}(\mathcal{H}_f)$, the bicommutant of M in $End_{\mathbb{C}}(\mathcal{H}_f)$. Consequently, the same is true if we change M′′ to $M\subseteq M''$. Since the state f is faithful and normal we get the desired statement in the case of A being σ-finite.
Secondly, let A be non-σ-finite. Then,by [1,p.164] there exists a directed increasing net $\{p_\alpha : \alpha\in I\}$ of projections of A such that $p_\alpha Ap_\alpha$ is σ-finite for any $\alpha\in I$ and that there exists $w^*\text{-}\lim p_\alpha=1_A$. As it has been shown,the unit ball of $p_\alpha N$ is σ-$\mathcal{T}^*_2$-dense in the unit ball of $p_\alpha M$ for any $\alpha\in I$. Hence, the sought statement can be derived easily since $\alpha\in I$ is arbitrary and $w^*\text{-}\lim p_\alpha=1_A$.

<u>§3 A Tomita-Takesaki type theorem</u>

Throughout this paragraph A is assumed to be a commutative W^*-algebra. We want to show that the Tomita-Takesaki theorem is valid for von Neumann algebras M on self-dual Hilbert A-moduli $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$,for its commutants and for the derived from the pair $(M,\bar{x})$ modular operators J and $\{\Delta^{it} : t\in\mathbb{R}\}$ on

$\mathcal{H}$, (cf. Prop.2.9., [4,Prop.2.10.,Def.3.1.]). We remark that this theorem was stated in [2,§3] for the special case $M=\mathrm{End}_A(\mathcal{H})$.

<u>Theorem 3.1.:</u> Let A be a commutative W^*-algebra, $\mathcal{H}$ be a self-dual Hilbert A-module and M be a von Neumann algebra on $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$. If $\mathcal{K}$ is defined in terms of the pair $(M,\bar{x})$ as at Proposition 2.8., and if J and $\{\Delta^{it}: t\in\mathbb{R}\}$ are defined by $\mathcal{K}$, ([4,Prop.2.10.,Def.3.1.]), there hold:

(i) $JMJ = M'$,

(ii) $\Delta^{it}M\Delta^{-it} = M$ for any $t\in\mathbb{R}$.

First, we prove the theorem under the assumption that A is σ-finite. There are two ways to do that. The first one is to use the constructions from [7,§4] looking always for modifications needed in the proofs since, in general, norm-completeness of $\mathcal{K}$ is changed to τ_1-completeness of the unit ball of $\mathcal{K}$. This can be done, but we will not do so in this paper. From the proof in [7,§4] we give only the generalization of the key Lemma 4.3. at the end of this paragraph because it seems to be of more general interest. The second way of proving Theorem 3.1. bases on the following lemma:

<u>Lemma 3.2.:</u> Let A be a commutative σ-finite W^*-algebra and let $\mathcal{H}$ be a self-dual Hilbert A-module. Let $M\subseteq\mathrm{End}_A(\mathcal{H})$ be a von Neumann algebra possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$. Let f be a faithful normal state on A and denote by $\mathcal{H}_f$ the norm-closure of $\mathcal{H}$ with respect to $f(\langle\cdot,\cdot\rangle)^{1/2}$. Then, any bounded A-linear operator on $\mathcal{H}$ can be continued to a (unique) bounded linear operator on $\mathcal{H}_f$ with the same norm, and moreover,

(i) $M'\subseteq\mathrm{End}_A(\mathcal{H})$ is identical with $M'\subseteq\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$,

(ii) $M=M''$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$.

<u>Proof:</u> Because of [6,Th.2.8.] we have only to prove (i) and (ii). For a fixed faithful normal state f on A we consider a projection $Z\in\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$ commuting with any $B\in M\subseteq\mathrm{End}_A(\mathcal{H})\subseteq\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$. Then, we get

(1) $ZB(\bar{x}) = BZ(\bar{x})$ for any $B\in M$.

If $\bar{x}=\bar{x}_1+\bar{x}_2$ is the unique decomposition of $\bar{x}$ in $\mathcal{H}_f$ with respect to the subspaces $Z\mathcal{H}_f$, $\bar{x}_1\in Z\mathcal{H}_f$, and $(\mathrm{id}-Z)\mathcal{H}_f$ we draw from (1)

$$B(\bar{x}_1) = Z(B(\bar{x}_1) + B(\bar{x}_2)) \quad \text{for any } B\in M,$$
$$\text{i.e., } B(\bar{x}_1)\in Z\mathcal{H}_f \text{ and } B(\bar{x}_2)\in(\mathrm{id}-Z)\mathcal{H}_f \text{ for any } B\in M.$$

Since Z is the identical operator on $(Z\mathcal{H}_f\cap\mathcal{H})$ and since Z commutes by assumption with any $\{a\cdot\mathrm{id}_{\mathcal{H}}: a\in A\}\subseteq M$, the projection $Z\in\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$ is the extension of a (unique) projection $Z\in M'\subseteq\mathrm{End}_A(\mathcal{H})$. Thus, $M'\subseteq\mathrm{End}_A(\mathcal{H})$ is identical with $M'\subseteq\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$. The second statement above is now obvious.

<u>Lemma 3.3.:</u> If A is σ-finite the statement of Theorem 3.1. is true.

<u>Proof:</u> If f is a faithful normal state on A we consider $M\subseteq\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$ and $\bar{x}\in\mathcal{H}\subseteq\mathcal{H}_f$, (cf. Lemma 3.2.). The appropriate operators J_f and $\{\Delta_f^{it}: t\in\mathbb{R}\}$ are the extensions of the operators J and $\{\Delta^{it}: t\in\mathbb{R}\}$ from $\mathcal{H}$ to $\mathcal{H}_f$, (cf. [4, Prop.2.11.,3.8.]). By [7,Th.4.2.] we get

$$J_fMJ_f = M' \ , \ \Delta_f^{it}M\Delta_f^{-it} = M \text{ for any } t\in\mathbb{R} \text{ on } \mathcal{H}_f$$

for $M,M'\subseteq\mathrm{End}_{\mathbb{C}}(\mathcal{H}_f)$. Applying Lemma 3.2. the desired lemma yields.

Now there remains to prove Theorem 3.1. in the case of A being non-σ-finite.

<u>Lemma 3.4.:</u> If A is not σ-finite the statement of Theorem 3.1. is true, too.

<u>Proof:</u> Let $\mathcal{K}$ be defined in terms of $(M,\bar{x})$ as at Proposition 2.8.. The operators J and $\{\Delta^{it}: t\in\mathbb{R}\}$ are defined for $\mathcal{K}$ as in [4,§§2,3]. By [1,p.164] there exists an increasing directed net $\{p_\alpha: \alpha\in I\}$ of projections of A such that $p_\alpha Ap_\alpha$ is a σ-finite W^*-algebra for any $\alpha\in I$ and that $w^*\text{-}\lim p_\alpha=1_A$. Investigating for a fixed $\alpha\in I$ the linear spaces $\mathcal{H}_\alpha=\{p_\alpha\mathcal{H},\ p_\alpha\langle\cdot,\cdot\rangle p_\alpha\}$ and $\mathcal{K}_\alpha=\{p_\alpha\mathcal{K},\ p_\alpha\langle\cdot,\cdot\rangle p_\alpha\}$, we obtain operators J_α, Δ_α^{it} $(t\in\mathbb{R})$ on $\mathcal{H}_\alpha$. The bounded operator J_α satisfies the conditions of [4,Cor.2.9.] with respect to $p_\alpha\mathcal{K}$. The same conditions are valid for the bounded operator $p_\alpha J$. Consequently, by [4,Prop. 2.11.]

$$p_\alpha J = J_\alpha \quad \text{on } \mathcal{H}_\alpha \text{ for any } \alpha\in I.$$

Similarly, the operators $p_\alpha\Delta^{it}$ and Δ_α^{it} $(t\in\mathbb{R})$ both satisfy the generalized K.M.S. condition with respect to $\mathcal{K}_\alpha= p_\alpha\mathcal{K}$ and moreover, $p_\alpha\Delta^{it}(p_\alpha\mathcal{K})\equiv\Delta_\alpha^{it}(p_\alpha\mathcal{K})\equiv p_\alpha\mathcal{K}$ for any $t\in\mathbb{R}$. Therefore, by [4,Prop.3.8.]

$$p_\alpha\Delta^{it} = \Delta_\alpha^{it} \quad \text{on } \mathcal{H}_\alpha \text{ for any } \alpha\in I, \text{ any } t\in\mathbb{R}.$$

Since $\Delta_\alpha^{it}B\Delta_\alpha^{-it}\in\mathrm{End}_A(\mathcal{H})$ for any $B\in M$, any $t\in\mathbb{R}$, since the unit ball of M is $\mathcal{T}_3$-complete by Proposition 2.4. and since for any $B\in M$, any $\bar{y}\in\mathcal{H}$ there exists

$$\tau_1\text{-}\lim_{\alpha\in I} \Delta_\alpha^{it} B \Delta_\alpha^{-it} (p_\alpha \bar{y}) = \Delta^{it} B \Delta^{-it}(\bar{y}) \ , \ t\in\mathbb{R},$$

we obtain

(2) $\mathcal{T}_3\text{-}\lim_{\alpha\in I} \Delta_\alpha^{it} B \Delta_\alpha^{-it} = \Delta^{it} B \Delta^{-it} \in M, \ t\in\mathbb{R},$

for any $B\in M$. Therefore, $\Delta^{it} M \Delta^{-it} \equiv M$, $t\in\mathbb{R}$, since $t\in\mathbb{R}$ can be changed to $(-t)\in\mathbb{R}$ in the relation (2).
Similarly, we prove the relation $JMJ=M'$ using the $\mathcal{T}_3$-completeness of the unit ball of M and M', respectively.

The Theorem 3.1. is proved.

<u>Corollary 3.5.:</u> $[M_h' \bar{x}]_\tau^- = i\mathcal{K}^\perp$.

<u>Proof:</u> From Theorem 3.1. we draw $M_h'(\bar{x}) = JM_hJ(\bar{x}) = JM_h(\bar{x})$. By [4,Cor. 2.9.] the relation $J(\mathcal{K}) = i\mathcal{K}^\perp$ holds. Since $[M_h\bar{x}]_\tau^- = \mathcal{K}$ by definition and since J is injective, $[M_h' x]_\tau^- = i\mathcal{K}^\perp$.

In addition we like to state the lemma [7,Lemma 4.3.] in a generalized form.

<u>Lemma 3.6.:</u> Suppose the situation given at Theorem 3.1.. Let $B'\in M_h'$. Then for any $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ there exists a unique $B\in M_h$ such that

$$2\cdot\langle C(\bar{x}), B'(\bar{x})\rangle = \lambda\cdot\langle C(\bar{x}), B(\bar{x})\rangle + \bar{\lambda}\cdot\langle B(\bar{x}), C(\bar{x})\rangle$$

for any $C\in M_h$.

<u>Proof:</u> First, let A be $\mathcal{B}$-finite. We assume $\mathrm{Re}(\lambda)=1$ and $0 \leqq B' \leqq 1$ since the sought equality is real linear. Let f be a fixed faithful normal positive state on A existing by [1,Prop.2.5.6.]. Define two normal functionals r and r_B on M (for any given $B\in M_h$) by the formula

$$r(C) = f(2\cdot\langle C(\bar{x}), B'(\bar{x})\rangle)$$
$$r_B(C) = f(\lambda\cdot\langle C(\bar{x}), B(\bar{x})\rangle + \bar{\lambda}\cdot\langle B(\bar{x}), C(\bar{x})\rangle), \text{ for } C\in M_h .$$

Because of [6,Remark 3.9.], [3,Cor.17], Proposition 2.4. and Corollary 2.2., the functionals r and r_B are in the self-adjoint part of the predual of M. Let

$$V := \{ r_B : B\in M_h, \|B\| \leqq 1 \}.$$

This set is obviously convex. Since the map $B \longrightarrow r_B$ is continuous with respect to both the w^*-topology on M and the weak topology on the predual of M, the set V is weakly compact.

Assume $r\notin V$. Then by the Hahn-Banach separation theorem there exists an element $D\in M_h$ such that

(3) $r_B(D) < r(D)$ for any $B\in M_h$, $\|B\| \leqq 1$.

Let $D=U\cdot|D|$ be the polar decomposition of $D\in M_h$ existing since $\mathcal{H}$ is self-dual, (cf. Prop. 1.4.). U is self-adjoint and commutes with $|D|$ since D is self-adjoint. Taking $B=U$ we draw from (3):

$$\begin{aligned}
1/2\cdot f(\lambda\cdot\langle D(\bar{x}), U(\bar{x})\rangle + \bar{\lambda}\cdot\langle U(\bar{x}), D(\bar{x})\rangle) &< \\
< f(\langle D(\bar{x}), B'(\bar{x})\rangle) & \\
= f(\langle D(B')^{1/2}(\bar{x}), (B')^{1/2}(\bar{x})\rangle) & \\
\leqq f(\langle |D|(B')^{1/2}(\bar{x}), (B')^{1/2}(\bar{x})\rangle) & \\
= f(\langle |D| B'(\bar{x}), \bar{x}\rangle) & \\
\leqq f(\langle |D|(\bar{x}), \bar{x}\rangle) & \\
= f(\langle D(\bar{x}), U(\bar{x})\rangle) & \\
= 1/2\cdot f(\lambda\cdot\langle D(\bar{x}), U(\bar{x})\rangle + \bar{\lambda}\cdot\langle U(\bar{x}), D(\bar{x})\rangle). &
\end{aligned}$$

This is a contradiction. Consequently, $r\in V$, i.e., there exists an element $B_f\in M_h$, $\|B_f\| \leqq 1$, for which

(4) $2\cdot f(\langle C(\bar{x}), B'(\bar{x})\rangle) = f(\lambda\cdot\langle C(\bar{x}), B_f(\bar{x})\rangle + \bar{\lambda}\cdot\langle B_f(\bar{x}), C(\bar{x})\rangle)$

for any $C\in M_h$. We define

$$D := (2\cdot\langle C(\bar{x}), B'(\bar{x})\rangle - \lambda\cdot\langle C(\bar{x}), B_f(\bar{x})\rangle - \bar{\lambda}\cdot\langle B_f(\bar{x}), C(\bar{x})\rangle)\cdot C.$$

If $C\in M_h$, the element D belongs to M_h. Putting D to (4) we obtain

(5) $f(|2\cdot\langle C(\bar{x}), B'(\bar{x})\rangle - \lambda\cdot\langle C(\bar{x}), B_f(\bar{x})\rangle - \bar{\lambda}\cdot\langle B_f(\bar{x}), C(\bar{x})\rangle|^2) = 0$

for any $C\in M_h$. Since f is faithful and positive we get the desired equality in the case of A being $\mathcal{B}$-finite. By the way we have shown that $B_f\in M_h$ does not depend on f.
We remark that the only place is (4) where the $\mathcal{B}$-finiteness of A is needed, because we did not know whether or not B_f depends on f.
Now the restriction on A to be $\mathcal{B}$-finite can be dropped in an obvious way using [1,p.164]. The lemma is proved.

<u>§4 Natural positive cones in self-dual Hilbert W^*-moduli</u>

Throughout this paragraph A is assumed to be commutative. Let $\mathcal{H}$ be a self-dual Hilbert W^*-module over A. We consider von Neumann algebras M on $\mathcal{H}$ possessing a cyclic-separating element $\bar{x}\in\mathcal{H}$. The modular involution, the modular operator and the appropriate modular group of automorphisms associated with the pair $(M,\bar{x})$ are denoted by J, Δ and $\{\Delta^{it} : t\in\mathbb{R}\}$, respectively. For details see [4] and Proposition 2.8.. The aim of this paragraph is to show the geometrical properties of the natural positive cone $\mathcal{P} \subseteq \mathcal{H}$ arising from the pair $(M,\bar{x})$ in some sense similarly as in the case $A=\mathbb{C}$, cf. [1].

<u>Definition 4.1.:</u> (cf. [1,Def.2.5.25.])
For a pair $(M,\bar{x})$ we consider the set $\mathcal{P}_0$, the τ_1-completion of the set $\{BJBJ(\bar{x}) : B\in M, \|BJBJ(\bar{x})\| \leq 1\}$. Then, the natural positive cone $\mathcal{P}$ associated with the pair $(M,\bar{x})$ is defined as $\mathcal{P} := [\mathcal{P}_0]_\tau^-$.

<u>Proposition 4.2.:</u> The closed set $\mathcal{P} \subseteq \mathcal{H}$ has the following properties:

(i) $\mathcal{P} = [\Delta^{1/4} M_+ \bar{x}]_\tau^- = [\Delta^{1/4}[M_+\bar{x}]_\tau^-]_\tau^-$
$= [\Delta^{-1/4} M'_+ \bar{x}]_\tau^- = [\Delta^{-1/4}[M'_+\bar{x}]_\tau^-]_\tau^-$,
where M_+ and M'_+ are the self-adjoint positive parts of M and M', respectively. Therefore, $\mathcal{P}$ is a convex cone.

(ii) $\Delta^{it}\mathcal{P} = \mathcal{P}$ for any $t\in\mathbb{R}$.

(iii) For any $\bar{y}\in\mathcal{P}$ the equality $J(\bar{y})=\bar{y}$ holds.

(iv) If $B\in M$ we get $BJBJ(\mathcal{P}) \subseteq \mathcal{P}$.

<u>Proof:</u> (cf. [1,Prop.2.5.26.])
Let $M_0 \subseteq M$ be the $*$-algebra of the entire analytic elements of the group $\{\Delta^{it} : t\in\mathbb{R}\}$. For any $B\in M_0$ there holds

$$
\begin{aligned}
(6)\qquad \Delta^{1/4}BB^*(\bar{x}) &= \Delta^{1/4}B\Delta^{-1/4}\Delta^{1/4}B^*\Delta^{-1/4}(\bar{x}) \\
&= \Delta^{1/4}B\Delta^{-1/4}(\Delta^{-1/4}B\Delta^{1/4})^*(\bar{x}) \\
&= \Delta^{1/4}B\Delta^{-1/4}J\Delta^{1/2}(\Delta^{-1/4}B\Delta^{1/4})(\bar{x}) \\
&= (\Delta^{1/4}B\Delta^{-1/4})\, J\, (\Delta^{1/4}B\Delta^{-1/4})\, J\, (\bar{x}).
\end{aligned}
$$

Since $\Delta^{1/4}M_0\Delta^{-1/4}=M_0$ and because of Theorem 2.10., the relation (6) yields

$$\mathcal{P} \subseteq [\Delta^{1/4}M_+\bar{x}]_\tau^- \subseteq [\Delta^{1/4}[M_+\bar{x}]_\tau^-]_\tau^- \quad .$$

On the other hand, $[M_{0+}\bar{x}]_\tau^- = [M_+\bar{x}]_\tau^-$ by Theorem 2.10. For an arbitrary $\bar{y}\in[M_+\bar{x}]_\tau^-$ we take such a bounded net $\{B_\alpha\bar{x} : \alpha\in I,\ B_\alpha\in M_{0+}\}$ that $\tau_1\text{-}\lim B_\alpha\bar{x}=\bar{y}$. Then, the equality (6) implies $\Delta^{1/4}B_\alpha\bar{x}\in\mathcal{P}$. But

$$
\begin{aligned}
&= \tau_1\text{-}\lim_{\alpha\in I}\ J\Delta^{1/2}B_\alpha\bar{x} \\
&= \tau_1\text{-}\lim_{\alpha\in I}\ B_\alpha\bar{x} \\
&= \bar{y} \;=\; J\Delta^{1/2}\bar{y}.
\end{aligned}
$$

Therefore,

$$\langle \Delta^{1/4}(\bar{y}-B_\alpha\bar{x}), \Delta^{1/4}(\bar{y}-B_\alpha\bar{x})\rangle = \langle J(\bar{y}-B_\alpha\bar{x}), J\Delta^{1/2}(\bar{y}-B_\alpha\bar{x})\rangle$$

and $\Delta^{1/4}\bar{y}\in\mathcal{P}$, i.e., $[\Delta^{1/4}[M_+\bar{x}]_\tau^-]_\tau^- \subseteq \mathcal{P}$. Finally,

$$\mathcal{P} = [\Delta^{1/4}M_+\bar{x}]_\tau^- = [\Delta^{1/4}[M_+\bar{x}]_\tau^-]_\tau^- \quad .$$

If $\mathcal{P}'$ denotes the natural cone associated with $(M',\bar{x})$, there holds

$$B'JB'J(\bar{x}) = J(JB'J)J(JB'J)(\bar{x}) = (JB'J)J(JB'J)J(\bar{x})$$

for any $B'\in M'$ since $JB'J\in M$, i.e., $\mathcal{P}'=\mathcal{P}$. Since Δ^{-1} is the modular operator with respect to the pair $(M',\bar{x})$, we obtain

$$\mathcal{P} = \mathcal{P}' = [\Delta^{-1/4}M'_+\bar{x}]_\tau^- = [\Delta^{-1/4}[M'_+\bar{x}]_\tau^-]_\tau^- \quad .$$

The first statement is proved.
To prove the second one we keep in mind that

$$\Delta^{it}\Delta^{1/4}M_+\bar{x} = \Delta^{1/4}\Delta^{it}M_+\bar{x} = \Delta^{1/4}\Delta^{it}M_+\Delta^{-it}\bar{x} = \Delta^{1/4}M_+\bar{x}.$$

If $B,C\in M$ the third statement follows from the equality

$$J(BJBJ(\bar{x})) = (JBJ)BJ(\bar{x}) = B(JBJ)J(\bar{x}) = BJBJ(\bar{x}),$$

whereas (iv) can be derived from

$$(BJBJ)(CJCJ)(\bar{x}) = BC(JBJ)(JCJ)(\bar{x}) = BCJ(BC)J(\bar{x}).$$

So , the proposition is proved.

<u>Proposition 4.3.:</u> The following relations are valid:

$$[M_+\bar{x}]_\tau^- = \{\bar{y}\in\mathcal{H} : \langle\bar{y},\bar{z}\rangle \geq 0 \text{ for any } \bar{z}\in[M'_+\bar{x}]_\tau^-\} ,$$
$$[M'_+\bar{x}]_\tau^- = \{\bar{y}\in\mathcal{H} : \langle\bar{y},\bar{z}\rangle \geq 0 \text{ for any } \bar{z}\in[M_+\bar{x}]_\tau^-\} .$$

<u>Proof:</u> First, suppose A to be σ-finite and let f be any normal faithful state on A. We denote by $[M_+\bar{x}]_f$ the closure of $[M_+\bar{x}]_\tau^-$ with respect to the norm $f(\langle\cdot,\cdot\rangle)^{1/2}$ on $\mathcal{H}$. The set $[M'_+\bar{x}]_f$ is defined analogously. Then, by [1,Prop.2.5.27.] the following set identities are valid on the Hilbert space $\mathcal{H}_f$, the closure of $\mathcal{H}$ with respect to the norm $f(\langle\cdot,\cdot\rangle)^{1/2}$:

$$[M_+\bar{x}]_f = \{\bar{y}\in\mathcal{H}_f : f(\langle\bar{y},\bar{z}\rangle)\geq 0 \text{ for any } \bar{z}\in[M'_+\bar{x}]_f\},$$
$$[M'_+\bar{x}]_f = \{\bar{y}\in\mathcal{H}_f : f(\langle\bar{y},\bar{z}\rangle)\geq 0 \text{ for any } \bar{z}\in[M_+\bar{x}]_f\}.$$

Since f is an arbitrary normal faithful state, we obtain

$$[M_+\bar{x}]_\tau^- = \{\bar{y}\in\mathcal{H} : f(\langle\bar{y},\bar{z}\rangle)\geq 0 \text{ for any } \bar{z}\in[M'_+\bar{x}]_\tau^-,\ \text{any } f\in P\},$$
$$[M'_+\bar{x}]_\tau^- = \{\bar{y}\in\mathcal{H} : f(\langle\bar{y},\bar{z}\rangle)\geq 0 \text{ for any } \bar{z}\in[M_+\bar{x}]_\tau^-,\ \text{any } f\in P\}.$$

Thus, we get the sought relations above.
Secondly, let A be non-σ-finite now. There exists a directed increasing net $\{p_\alpha : \alpha\in I\}$ of projections of A with the properties that

$p_\alpha A p_\alpha$ is σ-finite for any $\alpha \in I$ and that w^*-lim $p_\alpha = 1_A$. From the first part of the present proof ona derives

$$p_\alpha \cdot [M_+ \bar{x}]_\tau^- = \{\bar{y} \in p_\alpha \mathcal{H} \langle \bar{y}, \bar{z} \rangle \geqq 0 \text{ for any } \bar{z} \in p_\alpha \cdot [M'_+ \bar{x}]_\tau^-\},$$
$$p_\alpha \cdot [M'_+ \bar{x}]_\tau^- = \{\bar{y} \in p_\alpha \mathcal{H} \langle \bar{y}, \bar{z} \rangle \geqq 0 \text{ for any } \bar{z} \in p_\alpha \cdot [M_+ \bar{x}]_\tau^-\}.$$

for any $\alpha \in I$, if $p_\alpha M$ on $p_\alpha \mathcal{H}$ is considered. Since $\alpha \in I$ is arbitrarily chosen we get the desired relations.

<u>Proposition 4.4.:</u>

(i) The cone $\mathcal{P}$ is self-adjoint, i.e.,
$\mathcal{P} = \mathcal{P}^\vee := \{\bar{y} \in \mathcal{H}: \langle \bar{y}, \bar{z} \rangle \geqq 0 \text{ for any } \bar{z} \in \mathcal{P}\}$.

(ii) $\mathcal{P} \cap (-\mathcal{P}) = \{\bar{0}\}$.

(iii) If $J(\bar{y}) = \bar{y}$ for a certain $\bar{y} \in \mathcal{H}$ there exists a unique decomposition $\bar{y} = \bar{y}_1 - \bar{y}_2$ with $\bar{y}_1, \bar{y}_2 \in \mathcal{P}$, and $\bar{y}_1 \perp \bar{y}_2$.

(iv) The linear hull of $\mathcal{P}$ is $\mathcal{H}$.

<u>Proof:</u> (i) First, suppose A to be σ-finite and let f be an arbitrary faithful normal state on A. If we consider the Hilbert space $\mathcal{H}_f$ and the cone $\mathcal{P}_f \subseteq \mathcal{H}_f$, (the closures of $\mathcal{H}$ and $\mathcal{P}$, respectively, with respect to the norm $f(\langle \cdot, \cdot \rangle)^{1/2}$), there turns out (by [1,2.3.28.]) that $\mathcal{P}_f = \mathcal{P}_f^\vee = \{\bar{y} \in \mathcal{H}_f: f(\langle \bar{y}, \bar{z} \rangle) \geqq 0 \text{ for any } \bar{z} \in \mathcal{P}_f\}$.

Since f is arbitrarily chosen, we obtain

$$\begin{aligned} \mathcal{P} &= \{\bar{y} \in \mathcal{H}: f(\langle \bar{y}, \bar{z} \rangle) \geqq 0 \text{ for any } \bar{z} \in \mathcal{P}, \text{ any } f \in P\} \\ &= \{\bar{y} \in \mathcal{H}: \langle \bar{y}, \bar{z} \rangle \geqq 0 \text{ for any } \bar{z} \in \mathcal{P}\} \\ &= \mathcal{P}^\vee \end{aligned}$$

as desired. Secondly, if A is non-σ-finite there exists a directed increasing net $\{p_\alpha: \alpha \in I\}$ such that $p_\alpha A p_\alpha$ is σ-finite for any $\alpha \in I$ and that w^*-lim $p_\alpha = 1_A$. Considering $p_\alpha \mathcal{P} \subseteq p_\alpha \mathcal{H}$ we get $p_\alpha \mathcal{P} = p_\alpha \mathcal{P}^\vee$ for any $\alpha \in I$. That implies $\mathcal{P} = \mathcal{P}^\vee$.

(ii) If $\bar{y} \in \mathcal{P} \cap (-\mathcal{P})$ we draw from (i) that $\langle \bar{y}, \bar{y} \rangle = 0$, and hence, $\bar{y} = \bar{0}$.

(iii) Assume $J(\bar{y}) = \bar{y}$. The cone $\mathcal{P}$ is a closed convex subset in the self-dual Hilbert A-module $\mathcal{H}$ having the property that the subset $\mathcal{P}_0 = \{\bar{y} \in \mathcal{P}: \|\bar{y}\| \leqq 1\}$ is τ_1-complete. Therefore, there exists a unique element $\bar{y}_1 \in \mathcal{P}$ such that

(7) $\langle \bar{y} - \bar{y}_1, \bar{y} - \bar{y}_1 \rangle = \inf\{\langle \bar{y} - \bar{z}, \bar{y} - \bar{z} \rangle : \bar{z} \in \mathcal{P}\}$.

Let us denote $\bar{y}_2 := \bar{y} - \bar{y}_1$. For any $\bar{z} \in \mathcal{P}$, $\lambda > 0$ the inequality

$$\langle \bar{y} - \bar{y}_1, \bar{y} - \bar{y}_1 \rangle \leqq \langle \bar{y} - (\bar{y}_1 + \lambda \bar{z}), \bar{y} - (\bar{y}_1 + \lambda \bar{z}) \rangle$$

holds since $(\bar{y}_1 + \lambda \bar{z}) \in \mathcal{P}$. Therefore,

$$0 \leqq \lambda \cdot (\langle \bar{y}_2, \bar{z} \rangle + \langle \bar{z}, \bar{y}_2 \rangle) + \lambda^2 \cdot \langle \bar{z}, \bar{z} \rangle .$$

Since $\lambda > 0$ is arbitrarily chosen the inequality $\langle \bar{y}_2, \bar{z} \rangle + \langle \bar{z}, \bar{y}_2 \rangle \geqq 0$ has to be valid. But, since $\bar{z} \in \mathcal{P}$ there holds $J(\bar{y}_2) = \bar{y}_2$, $J(\bar{z}) = \bar{z}$, and hence, $\langle \bar{y}_2, \bar{z} \rangle = \langle J(\bar{y}_2), J(\bar{z}) \rangle = \langle \bar{y}_2, \bar{z} \rangle^*$. Thus, we get $\bar{y}_2 \in \mathcal{P}$ since $\bar{z} \in \mathcal{P}$ was arbitrarily chosen and (i) is valid. Consequently, $\bar{y} = \bar{y}_1 - \bar{y}_2$ where $\bar{y}_1, \bar{y}_2 \in \mathcal{P}$.
Let us show that $\bar{y}_1 \perp \bar{y}_2$. Since $(1-\lambda)\bar{y}_1 \in \mathcal{P}$ for any $0 \leqq \lambda \leqq 1$ we get

$$\langle \bar{y}_2, \bar{y}_2 \rangle \leqq \langle \bar{y}_2 - \lambda \cdot \bar{y}_1, \bar{y}_2 - \lambda \cdot \bar{y}_1 \rangle$$

because of (7). This is equivalent to

$$\lambda^2 \cdot \langle \bar{y}_1, \bar{y}_1 \rangle - \lambda \cdot (\langle \bar{y}_1, \bar{y}_2 \rangle + \langle \bar{y}_2, \bar{y}_1 \rangle) \geqq 0.$$

Finally, $\langle \bar{y}_1, \bar{y}_2 \rangle \leqq 0$, and since $\bar{y}_1, \bar{y}_2 \in \mathcal{P} = \mathcal{P}^\vee$, we obtain $\langle \bar{y}_1, \bar{y}_2 \rangle = 0$.
To show the uniqueness of such a decomposition we assume the existence of two decompositions

$$\bar{y} = \bar{y}_1 - \bar{y}_2, \quad \bar{y}_1, \bar{y}_2 \in \mathcal{P}, \quad \bar{y}_1 \perp \bar{y}_2 ,$$
$$\bar{y} = \bar{z}_1 - \bar{z}_2, \quad \bar{z}_1, \bar{z}_2 \in \mathcal{P}, \quad \bar{z}_1 \perp \bar{z}_2 .$$

Then $\bar{y}_1 - \bar{z}_1 = \bar{y}_2 - \bar{z}_2$, and furthermore,

$$\begin{aligned} \langle \bar{y}_1 - \bar{z}_1, \bar{y}_1 - \bar{z}_1 \rangle &= \langle \bar{y}_1 - \bar{z}_1, \bar{y}_2 - \bar{z}_2 \rangle \\ &= -\langle \bar{z}_1, \bar{y}_2 \rangle - \langle \bar{y}_1, \bar{z}_2 \rangle \leqq 0. \end{aligned}$$

Therefore, $\bar{y}_1 = \bar{z}_1$, $\bar{y}_2 = \bar{z}_2$ and such a decomposition is unique.
(iv) If an element $\bar{y} \in \mathcal{H}$ is orthogonal to the linear hull of $\mathcal{P}$, there must be $\langle \bar{y}, \bar{y} \rangle = 0$, and hence, $\bar{y} = \bar{0}$. There only remains to remark that the unit ball of the linear hull of $\mathcal{P}$ is τ_1-complete by construction.

<u>Corollary 4.5.:</u>

(i) If $\bar{y} \in \mathcal{P}$, $\bar{y}$ is cyclic for M if and only if $\bar{y}$ is separating for M.

(ii) If $\bar{y} \in \mathcal{P}$ is cyclic (and hence, separating) for M, for the modular involution $J_{\bar{y}}$ and for the natural positive cone $\mathcal{P}_{\bar{y}}$ associated with the pair $(M, \bar{y})$ $J = J_{\bar{y}}$, $\mathcal{P} = \mathcal{P}_{\bar{y}}$ hold.

The proof of this corollary is analogous to that of [1,Prop.2.3.30.], and thus, will be omitted.

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Франк М. E5-87-94

Представления фон Неймана в автодуальных гильбертовых W*-модулях

Рассматриваются алгебры фон Неймана М-ограниченных операторов в автодуальных гильбертовых W*-модулях Н, имеющие циклично отделяющий элемент х в Н. Обнаружены тесные связи между ними и некоторыми специальными вещественными подпространствами в Н. При предположении, что лежащая в основе W*- алгебра коммутативна, теорема типа Томиты-Танесаки доказывается. Исследуется естественный конус в Н, связанный с парой /М, х/. Описаны его свойства.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Frank M. E5-87-94

Von Neumann Representations on Self-Dual Hilbert W*-Moduli

Von Neumann algebras M of bounded operators on self-dual Hilbert W*-moduli H possessing a cyclic-separating element $\bar{x}$ in H are considered. The close relation of them to certain real subspaces of H is established. Under the supposition that the underlying W*-algebra is commutative, a Tomita-Takesaki type theorem is stated. The natural cone in H arising from the pair $(M,\bar{x})$ is investigated and its properties are obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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