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ASYMPTOTIC BEHAVIOUR OF THE REGULAR SOLUTION OF THREE-PARTICLE INTEGRODIFFERENTIAL EQUATIONS IN THE VICINITY OF THE TRIPLE COLLISION POINT

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Let us consider a system of three spinless particles interacting via central potentials representable as absolutely convergening power series expansions

$$V_i(x) = \sum_{n=-1}^{\infty} v_{in}\, x^n \quad , \quad i = 1, 2, 3 \tag{1}$$

centred at $x = 0$. Eigenstates of a system like that are determined by the set $\mathcal{E} = (L, M, \sigma)$ of three quantum numbers, namely the total angular momentum L, its third component M and parity $\sigma = \pm 1$ with respect to inversion $(\vec{x}, \vec{y}) \to -(\vec{x}, \vec{y})$ of any Jacoby vector pair. Feddeev /1/ components $\psi_i^{\mathcal{E}}$ of the wave function $\psi^{\mathcal{E}} = \psi_1^{\mathcal{E}} + \psi_2^{\mathcal{E}} + \psi_3^{\mathcal{E}}$ will be written in the bispherical basis $\mathcal{Y}^{LM}_{(\lambda,\ell)}(\hat{y}, \hat{x})$ /2/ as follows

$$\psi_i^{\mathcal{E}}(\vec{x}_i, \vec{y}_i) = 2\,(\rho^2 \sin 2\varphi_i)^{-1} \sum_{\alpha} \Phi_i^{\alpha L \sigma}(\rho, \varphi_i)\, \mathcal{Y}_{\alpha}^{LM}(\hat{y}_i, \hat{x}_i). \tag{2}$$

Here i=1,2,3, $\rho = (x_i^2 + y_i^2)^{1/2}$ is a hyperradius, $\varphi_i = arctg\,(y_i/x_i)$ /3/ is a hyperspherical angle. The partial components Φ satisfy an infinite set of integrodifferential equations. In the notation of paper /4/ it has the form:

$$\left\{ \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\Delta^{\alpha}_{\varphi_i} + E - V_i(\rho\cos\varphi_i) \right\} \Phi_i^{\alpha}(\rho, \varphi_i) =$$
$$V_i(\rho\cos\varphi_i) \sum_{j \neq i} \sum_{\beta} \langle \varphi_i \,|\, h_{\alpha\beta} \,|\, \Phi_j^{\beta}(\rho, \varphi_j) \rangle. \tag{3}$$

Here and further, where it is possible, we omit the subscripts L and σ. Indices $\alpha = (\lambda, \ell)$ and $\beta = (\lambda', \ell')$ in (2), (3) satisfy the triangle condition /2/ $\vec{\lambda} + \vec{\ell} = \vec{L} = \vec{\lambda}' + \vec{\ell}'$ and their sums $\lambda + \ell$ and $\lambda' + \ell'$ are even (odd) numbers for $\sigma = 1(-1)$. Solutions of set (3) are searched in the C^2-class of functions, which vanish at the triple collision point $\rho = 0$ and on the rays $\varphi = 0, \pi/2$. Those functions also satisfy certain /3/ boundary conditions as $\rho \to \infty$.

In that class of regular solutions the functions

$$W_{\alpha\rho}(\varphi) = N_{\alpha\rho}(\sin\varphi)^{\lambda+1}(\cos\varphi)^{\ell+1}P_n^{(\lambda+1/2,\,\ell+1/2)}(\cos 2\varphi) \tag{4}$$

form an agular basis. Here $\alpha = (\lambda, \ell)$, $\rho = 2n+\lambda+\ell$ is a hypermomentum, n=0,1..., $N_{\alpha\rho}$ is a normalization constant, $P_n^{(a,b)}$ is the Jacobi polynomial/5/. The functions (4) are eigenfunctions/4/ of nonlocal operators $h_{\alpha\beta}$ and operators

$$\Delta_\varphi^\alpha = \partial_\varphi^2 - \lambda(\lambda+1)\operatorname{cosec}^2\varphi - \ell(\ell+1)\sec^2\varphi .$$

In the spectral representation of these operators

$$\Delta_\varphi^\alpha = -\sum_{\rho=\lambda+\ell}^{\infty} |W_{\alpha\rho}\rangle(\rho+2)^2\langle W_{\alpha\rho}| , \tag{5}$$

$$h_{\alpha\beta}^{L} = \sum_{\rho=\lambda+\ell}^{\infty} |W_{\alpha\rho}\rangle\, \langle\alpha|\beta\rangle_{\rho L}^{ij}\, \langle \bar{W}_{\beta\rho}| \tag{6}$$

the hypermomentum ρ is by definition an even (odd) number for $\sigma = 1(-1)$. In (6) symbol $\langle\alpha|\beta\rangle_{\rho L}^{ij}$ stands for the Raynal--Revai/6/ coefficient which is equal to overlap the integral of two polyspherical hyperharmonics/5/.

$$Y_{\alpha\rho}^{LM}(w) = 2\operatorname{cosec}2\varphi\, W_{\alpha\rho}(\varphi)\, \mathcal{Y}_\alpha^{LM}(\hat{y},\hat{x}) \tag{7}$$

written in different $(i \neq j)$ sets of hyperspherical angles $w = (\varphi, \hat{y}, \hat{x})$.

Let us construct a fundamental system of the regular solutions (FSS) of eq.(3) around the point $\rho = 0$. Using the representation

$$\Phi_i^\alpha = \sum_{n=0}^{\infty} f_{in}^\alpha(\rho)\, W_{\alpha\rho}(\varphi_i)\, \delta_{\rho,\,2n+\lambda+\ell} \tag{8}$$

/4/ we may reduce set (3) to that of ordinary second-order equations for unknown radial functions f_{in}^α . The sequence of expansions (2), (8) is equivalent to the expansion of the Faddeev components over functions (7), therefore this construction means that the hyperharmonics method is applied to the Faddeev splitting of Schrödinger equation into a set of three equations for component Ψ_i^ε . The method of construction of FSS for ordinary differential equations is well known/7/. It was applied in papers/8,9/ where the asymptotical behaviour of atomic wave functions was investigated to find FSS for the Schrödinger equation preliminarly written in the hyperharmonics basis.

Using the technique of these works one can obtain FSS for equations for functions f_{in}^α , then calculate the sums (8) and find the FSS of set (3). But that construction is based on the matrix representation of all operators of system (3) in the basis (4) and therefore is not practical. We perform an equivalent construction directly attacking set (3).

Due to eq. (1), the leading term of the asymptotics for $\rho \to 0$ of each solution $\Phi_i^{\alpha K}$ is defined by the characteristic equation/10/ for the corresponding operator

$$\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\Delta_{\varphi_i}^\alpha .$$

Therefore using (5) we have

$$\Phi_i^{\alpha K} \xrightarrow[\rho\to 0]{} \rho^{K+2}\, C_{i0}^{\alpha K}\, W_{\alpha K}(\varphi_i), \tag{9}$$

where $C_{i0}^{\alpha K}$ are arbitrary constants, $\alpha = (\lambda, \ell)$, $K = 2m+\lambda+\ell$, $m = 0, ..$ All solutions with asymptotics (8) at a fixed value of K will be called the K-series solutions. The indices $\alpha = (\lambda, \ell)$ of the leading asymptotic term (9) obey the condition $\lambda + \ell \leqslant K$. Following paper/9/ we introduce a new variable $s = \ln\rho$ and look for the K -series solutions in the following form

$$\Phi_i^{\alpha K} = \rho^{K+2}\sum_{n=0}^{\infty} \exp(ns)\, U_{in}^{\alpha K}(s, \varphi_i)\, , \quad i = 1, 2, 3. \tag{10}$$

Here we assume that the exponential and unknown $U_{in}^{\alpha K}$-functions are linearly independent. The first function according to (9) is independent of s and equals

$$U_{i0}^{\alpha K}(\varphi) = C_{i0}^{\alpha K}\, W_{\alpha K}(\varphi) . \tag{11}$$

Now we insert the expansions (1) and (10) into eq.(3) and thus obtain the following recurrent equations

$$\begin{aligned} D_{in}^{\alpha K} U_{in}^{\alpha K} = {} & -E\, U_{i,n-2}^{\alpha K}(s, \varphi_i) + \\ & \sum_{m=0}^{n-1} v_{i,n-m-2}\,(\cos\varphi_i)^{n-m-2}\Big\{ U_{im}^{\alpha K}(s,\varphi_i) + \\ & \sum_{j\neq i}\sum_\beta \langle \varphi_i | h_{\alpha\beta} | U_{jm}^{\beta K}(s,\varphi_j)\rangle \Big\} . \end{aligned} \tag{12}$$

Here we put $U_{im}^{\alpha K} = 0$, $m < 0$ and denote

$$D_{in}^{\alpha K} = (\partial_s + n + K + 2)^2 + \Delta_{\varphi_i}^\alpha$$

In eq. (10) and set (12) indices $\alpha=(\lambda,\ell)$ do not satisfy the inequality $\lambda+\ell\leqslant K$ in the general case. Equation (12) with a fixed n will be called an n-block. Each n-block (n=1,2,...) consists of equations uncoupled with respect to indices i and α for $U^{\alpha K}_{in}$ -functions. Its right-hand side (RHS) $R^{\alpha K}_{in}$ contains only solutions of previous blocks. The $U^{\alpha K}_{in}$ functions are sums of a partial solution of n-block (12) and a general solution of the corresponding homogeneous equations. The latter is the sum

$$Z^{\alpha P}_{in}=\{C_+\exp(p_+s)+C_-\exp(p_-s)\}\,W_{\alpha P}(\varphi_i),$$

where $C_\pm$ are arbitrary constants, $P=2m+\lambda+\ell,\ m=0,1,\dots,p_\pm=\pm P-n-K$. The term with $p_\pm<0$ is irregular, another term with $p_\pm>0$ belongs to another P-series of regular solutions. We construct a regular FSS. Therefore without loss of generality we put $Z=0$ and search only for a partial solution of inhomogeneous set (12). The number P in sum (5) and K in set (12) are simultaneously even at $\sigma=1$ or odd at $\sigma=-1$. Therefore $(P-K)$ is always an even number. The kernel (Ker) of $D^{\alpha K}_{in}$ -operator in the basis (4) is equal to [10] $W_{\alpha,K+n}$ if n is even and vanishes for odd n. Thus [10] if the RHS of an n-block is a polynomial in s of degree m, then its solution is also a polynomial in s of degree m $(m+1)$ if n is an odd (even) number. The RHS of a 1-block of set (12) contains functions (11) and is independent of s. Therefore its solution and RHS of a 2-block are also independent of this variable. The solution of the 2-block is

$$U^{\alpha K}_{i2}=C^{\alpha K}_{i2}\,s\,W_{\alpha,K+2}(\varphi_i)+G^{\alpha K}_{i2}(\varphi_i).$$

Solutions of each equation for the $G^{\alpha K}_{i2}$-functions

$$D^{\alpha K}_{i2}\,G^{\alpha K}_{i2}(\varphi_i)=-2(K+4)\,C^{\alpha K}_{i2}\,W_{\alpha,K+2}(\varphi_i)+R^{\alpha K}_{i2}(\varphi_i)$$

exist if and only if its RHS is orthogonal to the corresponding function $W_{\alpha,K+2}$. This condition uniquely determines coefficients

$$C^{\alpha K}_{i2}=\frac{1}{2(K+4)}\langle W_{\alpha,K+2}(\varphi)\,|\,R^{\alpha K}_{i2}(\varphi)\rangle\equiv\frac{1}{2(K+4)}\int_0^{\pi/2}d\varphi\ W_{\alpha,K+2}(\varphi)\,R^{\alpha K}_{i2}(\varphi)\qquad(13)$$

as a functions of arbitrary constants $C^{\alpha K}_{i0}$. Further analysing set (12) one can show that $R^{\alpha K}_{in}$ is a polynomial of s at degree $[\frac{n-1}{2}]$. The solutions of an n-block are polynomials of s of the same degree for odd n and of degree $[\frac{n}{2}]$ for even n. Consequently, the representation (10) is identical with the generalized [9] Fock [11] expansion

$$\Phi^{\alpha K}_i=\rho^{K+2}\sum_{n=0}^{\infty}\rho^n\sum_{m=0}^{\bar m(n)}s^m\,F^{\alpha K}_{inm}(\varphi_i),\qquad(14)$$

where $\bar m(n)=[n/2]$ and

$$F^{\alpha K}_{in0}=U^{\alpha K}_{in},\quad n=0,1,\quad F^{\alpha K}_{i20}=G^{\alpha K}_{i2},\quad F^{\alpha K}_{i21}=C^{\alpha K}_{i2}\,W_{\alpha,K+2}.$$

Inserting (14) into the set (3) we obtain for another F-functions the following set of recurrence integrodifferential equations

$$\{(n+K+2)^2+\Delta^{\alpha}_{\varphi_i}\}F^{\alpha K}_{inm}(\varphi_i)+2(m+1)(n+K+2)F^{\alpha K}_{in,m+1}(\varphi_i)+$$

$$(m+1)(m+2)F^{\alpha K}_{in,m+2}(\varphi_i)+E\,F^{\alpha K}_{i,n-2,m}(\varphi_i)=$$

$$\sum_{p=0}^{n-1}v_{i,n-p-2}(\cos\varphi_i)^{n-p-2}\sum_{j\neq i}\sum_{\beta}\langle\varphi_i\,|\,h_{\alpha\beta}\,|\,F^{\beta K}_{ipm}(\varphi_j)\rangle.\qquad(15)$$

The order of solutions of equations of the type (15) is well-known [11]. A general regular solution of set (3) may be written as a sum of all K-series (10) or (14):

$$\Phi^{\alpha}_i=\sum_{K=\lambda+\ell}^{\infty}\rho^{K+2}\sum_{n=0}^{\infty}\exp(ns)\,U^{\alpha K}_{in}(s,\varphi_i)=\qquad(16a)$$

$$\sum_{K=\lambda+\ell}^{\infty}\rho^{K+2}\sum_{n=0}^{\infty}\rho^n\sum_{m=0}^{\bar m(n)}s^m\,F^{\alpha K}_{inm}(\varphi_i),\ \bar m(n)=[\tfrac{n}{2}].\ (16b)$$

If potentials are not central, then parity σ for $L>0$ is not a good quantum number. Therefore sums of indices $\lambda+\ell$ in pairs $\alpha=(\lambda,\ell)$ and also numbers P in eq.(4-6) and K in set (12) may be odd and even. The kernels of operators $D^{\alpha K}_{in}$ do not vanish in the basis (4) for arbitrary n. The expansions of K-series and partial components are represented as sums (14) and (16b) resp. with $\bar m(n)=n$. Now we find three more slowly decreasing terms of sum (16) for $\rho\to 0$. If $v_{i,-1}\neq 0$, $i=1,2,3$ those terms are the first terms of sum (10) for the K-series with a minimal value of $K=\lambda+\ell$. We solve eq. (12) with $n=1,2$ using (6) and (11). Further we subsitute the obtained $U_{in}^{\alpha K}$-function into (16a) and get

$$\Phi_i^{\alpha} = \rho^{K+2}\{ C_{i0}^{\alpha K} W_{\alpha K}(\varphi_i) + \rho\, U_{i1}^{\alpha K}(\varphi_i) + \rho^2 s\, C_{i2}^{\alpha K} W_{\alpha,K+2}(\varphi_i) + O(\rho^2)\} . \qquad (17a)$$

Here $\alpha=(\lambda;\ell)$, $K=\lambda+\ell$,

$$U_{i1}^{\alpha K} = v_{i,-1} C_{i1}^{\alpha K} \sum_{P=\lambda+\ell}^{\infty} W_{\alpha P}(\varphi_i)\langle W_{\alpha P}(\varphi)|\sec\varphi\, W_{\alpha K}(\varphi)\rangle/[(K+3)^2-(P+2)^2]$$

$$C_{i1}^{\alpha K} = C_{i0}^{\alpha K} + \sum_{j\neq i}\sum_{\beta}\langle\alpha|\beta\rangle_{KL}^{ij} C_{j0}^{\beta K}$$

and according to eq.(13)

$$C_{i2}^{\alpha K} = [\, v_{i,-1}/2(K+4)]\, \{\langle W_{\alpha,K+2}(\varphi)|\sec\varphi\, U_{i1}^{\alpha K}(\varphi)\rangle + \sum_{j\neq i}\sum_{\beta}\langle W_{\alpha,K+2}(\varphi_i)\sec\varphi_i\,|\,h_{\alpha\beta}\,|\,U_{j1}^{\beta K}(\varphi_j)\rangle .$$

If $v_{i,-1}=0$, $i=1,2,3$, then $U_{i1}^{\alpha K}$, $C_{i2}^{\alpha K}$ in eq.(17) vanish. Therefore the first three terms of asymptotics (16) for $\rho\to 0$ are determined by the first terms of sum (10) for the K-series with $K=\lambda+\ell$ and by the leading term (9) of the next (K+2)-series. Finding the solution of set (12) corresponding to these terms we then obtian

$$\Phi_i^{\alpha} = \rho^{K+2}\{ C_{i0}^{\alpha K} W_{\alpha K}(\varphi_i) + \rho^2 \sum_{P=K}^{K+2} A_i^{\alpha P} W_{\alpha P}(\varphi_i) + \rho^3 U_{i3}^{\alpha K}(\varphi_i) + O(\rho^4 s)\},$$

where (17b)

$$A_i^{\alpha P} = \frac{1}{4(K+3)}(v_{i0} C_{i1}^{\alpha K} - E\, C_{i0}^{\alpha K})\delta_{P,K} + C_{i0}^{\alpha,K+2}\delta_{P,K+2} ,$$

$$U_{i3}^{\alpha K} = v_{i1} C_{i1}^{\alpha K}\sum_{P=\lambda+\ell}^{\infty} W_{\alpha P}(\varphi_i)\langle W_{\alpha P}(\varphi)|\cos\varphi\, W_{\alpha K}(\varphi)\rangle/[(K+5)^2-(P+2)^2] .$$

It is clear that the method of paper [12] is more convenient for determining the asymptotic behaviour of the Faddeev components and wave function as $\rho\to 0$. This method does not require the calculation of infinite sums (2) over partial components written in the form (16). Therefore we find only the first three terms of the asymptotic expansion of components (2). They are determined by the most slowly decreasing partial components when $\rho\to 0$. As L and σ are fixed the sum of indices in any pair $\alpha=(\lambda,\ell)$ satisfies the inequality

$$\lambda+\ell \geqslant N_{L\sigma} \equiv L + \delta_{\sigma,(-1)^{L+1}} .$$

Keeping in the sum (2) partial components with indices $\alpha=(\lambda,\ell)$ such that $\lambda+\ell=N_{L\sigma}$ and using eq.(17), we obtain

$$\Psi_i^{\varepsilon}(\vec{x}_i,\vec{y}_i) = \rho^{N_{L\sigma}}\{ \Psi_{i00}^{\varepsilon}(\omega_i) + \rho\, \Psi_{i10}^{\varepsilon}(\omega_i) + \rho^2 s\, \Psi_{i21}^{\varepsilon}(\omega_i) + O(\rho^2)\} \qquad (18a)$$

if expansion (1) contains a Coulomb singularity term, and

$$\Psi_i^{\varepsilon}(\vec{x}_i,\vec{y}_i) = \rho^{N_{L\sigma}}\{ \Psi_{i00}^{\varepsilon}(\omega_i) + \rho^2\, \Psi_{i20}^{\varepsilon}(\omega_i) + \rho^3\, \Psi_{i30}^{\varepsilon}(\omega_i) + O(\rho^4 s)\} \qquad (18b)$$

otherwise. Functions of hyperspherical angles

$$\Psi_{inm}^{\varepsilon} = \sum_{\alpha} C_{in}^{\alpha K}\, Y_{\alpha,K+2\delta_{n,1}}^{LM} , \quad (n,m)=(0,0),(2,1) ,$$

$$\Psi_{i20}^{\varepsilon} = \sum_{\alpha}\sum_{P=K}^{K+2} A_i^{\alpha P}\, Y_{\alpha P}^{LM} ,$$

$$\Psi_{in0}^{\varepsilon} = v_{i,n-2}\sum_{\alpha} C_{i1}^{\alpha K}\sum_{P=\lambda+\ell}[(n+K+2)^2-(P+2)^2]^{-1} \langle W_{\alpha P}(\varphi)|(\cos\varphi)^{n-2}\, W_{\alpha K}(\varphi)\rangle\, Y_{\alpha P}^{LM} , \quad n=1,3,$$

where $\alpha=(\lambda,\ell)$, $\lambda+\ell=N_{L\sigma}=K$, are the sums of hyperharmonics (7).

In conclusion, we summarize the results. The regular FSS of eq. (3) has for small ρ the form of K-series (10) or (14). The F-function in sum (14) satisfies the recurrence chain of eqs. (15). These equations are useful both for the analytical work and for numerical solution. Asymptotic expansions of Faddeev partial and Faddeev components of the three-particle wave-function are resp. of form (17) and (18).

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Пупышев В.В. E5-87-902

Асимптотики регулярных решений трехчастичных интегродифференциальных уравнений в окрестности точки тройного столкновения

Показано, что асимптотическое разложение каждого из линейно независимых регулярных решений трехчастичных интегродифференциальных уравнений имеет в окрестности точки тройного столкновения вид двойной суммы, содержащей степени гиперрадиуса, его логарифма и неизвестные функции одного гиперсферического угла. Для этих функций в случае сферически-симметричных потенциалов получена рекуррентная система интегродифференциальных уравнений.

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Asymptotic Behaviour of the Regular Solution of Three-Particle Integrodifferential Equations in the Vicinity of the Triple Collision Point

It is shown that any regular solution of three-particle integrodifferential equations may be represented in the vicinity of the triple collision point as a double sum containing the hyperradius powers, its logarithm powers and unknown functions of a single hyperspherical angle. The recurrent integrodifferential equations are obtained for those functions in the case of spherically symmetric potentials. The asymptotic behaviour of Faddeev and partial Faddeev components of the wave function in the vicinity of the triple collision point is explored.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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