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TOPOLOGIES ON TENSOR PRODUCTS

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0. Introduction

The present paper is motivated by the investigations of axiomatic quantum field theory (QFT). In the framework of Gårding-Wightman axioms the (m-point) Wightman functions W_m are considered. The W_m are linear functionals on the m-fold (algebraic) tensor product $E^{\otimes m} = E \otimes \dots \otimes E$, m=1,2,..., of a certain linear space E. Usually, in QFT there are taken the Schwartz-spaces $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{D}(\mathbb{R}^d)$, $d \in \mathbb{N}$, and the Jaffe-spaces (/4/) for E. In order to prove statements on the continuity of the Wightman functions W_n one has to introduce topologies on $E^{\otimes n}$, n=2,3,... .

Historically seen, the problem of defining suitable locally convex (l.c.) topologies on tensor products was mainly treated by J.von Neumann, R.Schatten and A.Grothendieck (/7/, /12/, /3/). Starting with an l.c. space $E[t]$, the general theory of l.c. spaces provides the injective, projective, and inductive topology on $E^{\otimes n}$, n=2,3,... . The present paper is concerned with the question whether or not some of these topologies coincide. Finally, the results are discussed for $\mathcal{S}(\mathbb{R}^d)^{\otimes n}$ and $\mathcal{D}(\mathbb{R}^d)^{\otimes n}$. In a forthcoming paper these topologies will be used for defining suitable l.c. topologies on tensor algebras.

The pattern of this paper is as follows. In Section 1 there are introduced the injective, projective and inductive topologies on $E^{\otimes n}$, n=2,3,..., and some of their properties are listed. Section 2 is aimed at establishing sufficient conditions for the coincidence of the projective and inductive topology. In Section 3 there are given estimations between the norms defining respectively, the projective and injective topology on $(\mathcal{S}(\mathbb{R}^d))^{\otimes n}$. Furthermore, it is shown that the projective and inductive topologies coincide on $(\mathcal{S}(\mathbb{R}^d))^{\otimes n}$, and that they do not on $(\mathcal{D}(\mathbb{R}^d))^{\otimes n}$.

The notations and definitions used in the following are taken from /6/, /11/. Especially, for l.c. topologies τ, τ' defined on the same vector space F, let $\tau < \tau'$ and $\tau \lneq \tau'$ denote that τ is weaker (coarser) than τ', and that τ is strictly weaker then τ', respectively.

1. Definition of l.c. topologies on tensor products

Throughout this paper let $E[t]$, $F[t']$ be l.c. spaces over the field of complex numbers $\mathbb{C}$, and let $n \in \mathbb{N}$, $n \geq 2$, be fixed. Further, let us put

$$E^{\otimes n} = E \otimes \ldots \otimes E,$$

the n-fold (algebraic) tensor product of E by itself.

The theory of l.c. spaces provides the class of "topologies compatible with the tensor product", see /6;§44.1/. Among these topologies there are the injective, projective, β-, and inductive topology which will be given in the following. Further let us denote the injective (resp. projective, β-, inductive) topology on respectively, $E \otimes F$ and $E^{\otimes n}$ by $t \otimes_\varepsilon t'$ (resp. $t \otimes_\pi t'$, $t \otimes_\beta t'$, $t \otimes_\iota t'$) and ε_n (resp. π_n, β_n, ι_n).

Let $\mathcal{P}(t) = \{p_\alpha; \alpha \in A\}$ and $\mathcal{P}(t') = \{p'_\beta; \beta \in B\}$, A,B are directed sets of indices, be systems of semi-norms defining the topologies t and t' on E and F, respectively. Then the topologies $t \otimes_\varepsilon t'$, ε_n, $t \otimes_\pi t'$, π_n are defined by the following systems of semi-norms:

$$\mathcal{P}(t \otimes_\varepsilon t') = \{x \longrightarrow \varepsilon_{\alpha,\beta}(x);\ \alpha \in A,\ \beta \in B\},$$

$$\mathcal{P}(\varepsilon_n) = \{y \longrightarrow \varepsilon_{\alpha_1,\ldots,\alpha_n}(y);\ \alpha_i \in A,\ i=1,\ldots,n\},$$

$$\mathcal{P}(t \otimes_\pi t') = \{x \longrightarrow \pi_{\alpha,\beta}(x);\ \alpha \in A,\ \beta \in B\},$$

$$\mathcal{P}(\pi_n) = \{y \longrightarrow \pi_{\alpha_1,\ldots,\alpha_n}(y);\ \alpha_i \in A,\ i=1,\ldots,n\},$$

where

$$\varepsilon_{\alpha,\beta}(x) = \sup\{|\sum_{i=1}^{m} T(e^{(i)})\, S(f^{(i)})|;\ T \in U_\alpha^o,\ S \in U_\beta^o\},$$

$$\varepsilon_{\alpha_1,\ldots,\alpha_n}(y) = \sup\{|\sum_{i=1}^{r} \prod_{j=1}^{n} T^{(j)}(e^{(j,i)})|;\ T^{(j)} \in U_{\alpha_j}^o,\ j=1,\ldots n\},$$

$$\pi_{\alpha,\beta}(x) = \inf\{\sum_{i=1}^{m} p_\alpha(e^{(i)})\, p'_\beta(f^{(i)});\ x = \sum_{i=1}^{m} e^{(i)} \otimes f^{(i)}\},$$

$$\pi_{\alpha_1,\ldots,\alpha_n}(y) = \inf\{\sum_{i=1}^{r} \prod_{j=1}^{n} p_{\alpha_j}(e^{(j,i)});\ y = \sum_{i=1}^{r} e^{(1,i)} \otimes \ldots \otimes e^{(n,i)}\}$$

for $x = \sum_{i=1}^{m} e^{(i)} \otimes f^{(i)} \in E \otimes F$, $y = \sum_{i=1}^{r} e^{(1,i)} \otimes \ldots \otimes e^{(n,i)} \in E^{\otimes n}$, and the polar sets $U_\gamma^o = \{T \in E';\ |T(e)| \leq p_\gamma(e),\ e \in E\}$ $(\gamma = \alpha, \alpha_j)$, $U_\beta^o = \{S \in F';\ |T(f)| \leq p'(f),\ f \in F\}$.

Furthermore, if there are systems of semi-norms consisting only of Hilbertian semi-norms and defining t and t', respectively, then the so-called σ-topologies exist on $E^{\otimes n}$ and $E \otimes F$. They are given by

$$\mathcal{P}(\sigma_n) = \{y \longrightarrow \sigma_{\alpha_1,\ldots,\alpha_n}(y);\ \alpha_i \in A,\ i=1,2,\ldots,n\},$$

$$\mathcal{P}(t \otimes_\sigma t') = \{x \longrightarrow \sigma_{\alpha,\beta}(x);\ \alpha \in A,\ \beta \in B\},$$

where

$$\sigma_{\alpha,\beta}(x) = (\sum_{i,k=1}^{m} \langle e^{(i)}, e^{(k)}\rangle_\alpha \langle f^{(i)}, f^{(k)}\rangle_\beta)^{1/2},$$

$$\sigma_{\alpha_1,\ldots,\alpha_n}(y) = (\sum_{i,k=1}^{r} \prod_{j=1}^{n} \langle e^{(j,i)}, e^{(j,k)}\rangle_{\alpha_j})^{1/2},$$

$\langle\cdot,\cdot\rangle_\gamma = (p_\gamma(\cdot))^2$, $\gamma = \alpha, \alpha_j$, $\langle\cdot,\cdot\rangle_\beta = (p'_\beta(\cdot))^2$, and x,y are given as above. Let us remark that the σ-topologies exist, if $E[t]$ and $F[t']$ are nuclear l.c. spaces, /8/.

Let $\mathcal{B}(E^{\otimes n})$ (resp. $\mathcal{L}(E^{\otimes n})$, $\mathcal{X}(E^{\times n})$) denote the set of all continuous (resp. separately continuous, hypocontinuous) multilinear forms on $E \times \ldots \times E$, n-times. Further, let $\mathcal{B}(E \times F)$, $\mathcal{L}(E \times F)$ and $\mathcal{X}(E \times F)$ stand for the corresponding sets of bilinear forms on $E \times F$. Recall $t \otimes_\pi t'$ and π_n are the finest l.c. topologies on $E \otimes F$ and $E^{\otimes n}$ satisfying

$$(E \otimes F\ [t \otimes t'])' = \mathcal{B}(E \times F) \text{ and } (E^{\otimes n}[\pi_n])' = \mathcal{B}(E^{\otimes n}),$$

Finally, the β- and the inductive topology on $E \otimes F$ denoted by $t \otimes_\beta t'$ and $t \otimes_\iota t'$ are defined as the finest l.c. topology on $E \otimes F$ such that

$$(E \otimes F[t \otimes_\beta t'])' = \mathcal{X}(E \times F) \text{ and } (E \otimes F[t \otimes_\iota t'])' = \mathcal{L}(E \times F)$$

are satisfied, respectively. The β- and the inductive topology on $E^{\otimes n}$, which are denoted by β_n and ι_n, respectively, are analogously defined.

Using the introduced notations

$$\mathfrak{f}_{i+j} = \mathfrak{f}_i \otimes_\mathfrak{f} \mathfrak{f}_j, \tag{1}$$

$i,j=1,2,\ldots$, follow, where $\mathfrak{f} \in \{\varepsilon, \sigma, \pi, \beta, \iota\}$ and $\mathfrak{f}_1 = t$.

The following relations are valid on $E^{\otimes n}$:

$$\varepsilon_n\ (<\sigma_n) < \pi_n < \beta_n < \iota_n. \tag{2}$$

Further, let us remark that respectively, ε_n and ι_n are the weakest and finest topology within the class of compatible topologies on $E^{\otimes n}$.

2. On the equivalence of certain topologies on $E^{\otimes n}$

Concerning the equivalence of the injective and projective topology one has the following result proven in /8/, /5/.

Proposition 1 (Grothendieck, Pietsch, John):

a) If $E[t]$ is nuclear, then $\varepsilon_n = \pi_n$ holds true on $E^{\otimes n}$.

b) If there is a system of Hilbertian semi-norms defining t on E , and if further $\varepsilon_n = \pi_n$ is satisfied on $E^{\otimes n}$, then the nuclearity of $E[t]$ follows.

Remark to b):
G.Pisier constructed an example of an infinite dimensional Banach space B (hence B is not nuclear) such that $\varepsilon_2 = \pi_2$ is fulfilled on $B^{\otimes 2}$, /9/. Consequently, the assumption of b) concerning the existence of a t-defining system of Hilbertian semi-norms is not any redundance.

The aim of the following is to give sufficient conditions for the equivalence of π_n, β_n and ι_n, respectively.

Lemma 1:

a) If $E[t]$ is a metrizable barrelled space, then $E^{\otimes m}[\pi_m]$ are also metrizable barrelled spaces, m=2,3,4,... .

b) If $E[t]$ and $F[t']$ are barrelled spaces, then $E \otimes F[t \otimes_\iota t']$ is barrelled, too.

Proof:
a) The assertion under consideration follows inductively by the following.

(I): If $X[\xi]$ and $E[t]$ are metrizable barrelled spaces, then $X \otimes E\ [\xi \otimes_\pi t]$ is also a metrizable barrelled space.

Proof of (I): The metrizability of $X \otimes E\,[\xi \otimes_\pi t]$ is a consequence of /6 ;§41.2(7)/.

For proving the barrelledness,recall that the equicontinuous subsets of $(X \otimes E[\xi \otimes_\pi t])'$ are just the equicontinuous subsets of the continuous bilinear forms on $X \times E$, /6 ;§41.3(4)/. Hence we are done, if every concerning the duality $(X \otimes E, (X \otimes E[\xi \otimes_\pi t])')$ weakly bounded subset H of $(X \otimes E[\xi \otimes_\pi t])'$ represents an equicontinuous subset of bilinear forms on $X \times E$.

The weakly boundedness of H yields that for each $(x,e) \in X \times E$ there is a constant $c < \infty$ such that

$$\sup\{|b(x,e)|;\ b \in H\} < c.$$

Consequently, for each $x \in X$ the set

$$\tilde{H}(x) = \{b(x,.);\ b \in H\}$$

is a weakly bounded subset of E'. Since $E[t]$ is barrelled, $\tilde{H}(x)$ is an equicontinuous subset of E'. For each $e \in E$, it follows analogously

$$\tilde{\tilde{H}}(e) = \{b(.,e);\ b \in H\}$$

represents an equicontinuous subset of X'. Hence H is a separately equicontinuous subset and thus equicontinuous due to the continuity theorem of Bourbaki, /6 ;§40.2(2)/. This completes the proof of (I).

b) If $\chi : E \times F \longrightarrow E \otimes F$ denotes the canonical bilinear mapping, then $t \otimes_\iota t'$ represents the finest l.c. topology on $E \otimes F$ such that all the mappings

$$g_e : F[t'] \longrightarrow E \otimes F[t \otimes_\iota t'],\ e \in E,$$
$$g_f : E[t] \longrightarrow E \otimes F[t \otimes_\iota t'],\ f \in F,$$

are continuous, where $g_e(f) = \chi(e,f)$, $g_f(e) = \chi(e,f)$, $(e \in E,\ f \in F)$.
Hence, the barrelledness of $E[t]$, $F[t']$ implies the barrelledness of $E \otimes F[t \otimes_\iota t']$ by /10;II.7.2/.

Proposition 2:

a) If $E[t]$ is barrelled, then $\beta_n = \iota_n$ holds true on $E^{\otimes n}$.

b) If $E[t]$ is a (DF)-space, then $\pi_n = \beta_n$ is satisfied on $E^{\otimes n}$.

c) If $E[t]$ is a metrizable barrelled space, then $\pi_n = \iota_n$.

Proof:
a) Using (1) and Lemma 1b), the barrelledness of $E^{\otimes n}[\iota_n]$ follows. Hence, every separately continuous multi-linear form on $E \times \ldots \times E$, n-times, is also hypocontinuous due to /6 ;40.2(5)/. Thus $\beta_n = \iota_n$ follows.
b) Applying /6 ;§41.4(7)/ and (1), $E^{\otimes m}[\pi_m]$ are also (DF)-spaces, m=2,3,...,n. Using the continuity theorem for bilinear mappings, /6 ;§40.2(10)/, $\mathfrak{X}(E_n) = \mathcal{B}(E_n)$ follows. This proves b).
c) Using a continuity theorem from /6 ;§40.2(1)/, the assumptions of c) and Lemma 1a) imply that every separately continuous bilinear form on $E^{\otimes m}[\pi_m] \times E[t]$ is also continuous, m=1,2,... . Hence

$$\pi_m \otimes_\pi t = \pi_m \otimes_\iota t \qquad (3)$$

is satisfied on $E^{\otimes m+1}$. The further proof is given by induction: (3) yields $\pi_2 = t \otimes_\pi t = t \otimes_\iota t = \iota_2$. Now, assume that $\pi_k = \iota_k$ for some $k \in \mathbb{N}$, $2 \le k < n$. Then

$$\pi_{k+1} = \pi_k \otimes_\pi t = \pi_k \otimes_\iota t = \iota_k \otimes_\iota t = \iota_{k+1}$$

follow because of (1), (3). Thus the proof is completed.

3. Topologies on the tensor products of Schwartz-spaces

This section is aimed at an application of the results of the previous section to the Schwartz-spaces $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{D}(\mathbb{R}^d)$, $d \in \mathbb{N}$, /2/.

a) $\mathcal{S}(\mathbb{R}^d)$

Let us confirm the following abbreviations:

$(\tilde{m}) = (m_1,\dots,m_d) \in (\mathbb{N}^*)^d$, $(\tilde{m}+k) = (m_1+k,\dots,m_d+k)$, $k \in \mathbb{N}^*$,

$N_i = (1+(x_i)^2-(\partial/\partial x_i)^2)/2$, $i=1,2,\dots, nd$,

$N^{(\tilde{m})} = \prod_{i=1}^{d} (N_i)^{m_i}$, $(N_i)^o = I$ (identical mapping on $\mathcal{S}(\mathbb{R})$),

$x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $dx = dx_1 \dots dx_d$, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

Then

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d); \int |N^{(m)} f(x)|^2 dx < \infty,\ (\tilde{m}) \in (\mathbb{N}^*)^d\}.$$

Further, the Schwartz-space topology ρ_d on $\mathcal{S}(\mathbb{R}^d)$ is given by the following system of Hilbertian norms

$$\mathcal{P}(\rho_d) = \{f \longrightarrow \|f\|_{(\tilde{m})};\ (\tilde{m}) \in (\mathbb{N}^*)^d\},$$

$f \in \mathcal{S}(\mathbb{R}^d)$, $\|f\|_{(\tilde{m})} = (\int |N^{(m)} f(x)|^2 dx)^{1/2}$. Recall also

$$(\mathcal{S}(\mathbb{R}^d))^{\otimes n} \subset \mathcal{S}(\mathbb{R}^{nd}),$$

where $f_1 \otimes \dots \otimes f_n \in (\mathcal{S}(\mathbb{R}^d))^{\otimes n}$ is identified with the function $(x^1,\dots,x^n) \longrightarrow f_1(x^1)\dots f_n(x^n)$, $x^i \in \mathbb{R}^d$ $(i=1,2,\dots,n)$.

For $h \in \mathcal{S}(\mathbb{R}^{nd})$ let

$$\|h\|_{(\tilde{m}^1,\dots,\tilde{m}^n)} = (\int | N^{(\tilde{m}^1,\dots,\tilde{m}^n)} h(x_1,\dots,x_{nd})|^2 dx_1 \dots dx_{nd})^{1/2},$$

where $N^{(\tilde{m}^1,\dots,\tilde{m}^n)} = \prod_{i=1}^{d} \prod_{r=o}^{n-1} (N_{i+rd})^{m_i^{r+1}}$, $(\tilde{m}^1),\dots, (\tilde{m}^n) \in (\mathbb{N}^*)^d$.

Then $\{\|.\|_{(\tilde{m}^1,\dots,\tilde{m}^n)};\ (\tilde{m}^1),\dots,(\tilde{m}^n) \in (\mathbb{N}^*)^d\}$ defines the Schwartz-space topology ρ_{nd} on $\mathcal{S}(\mathbb{R}^{nd})$.

The following lemma proves estimations between semi-norms on $(\mathcal{S}(\mathbb{R}^d))^{\otimes 2}$.

<u>Lemma 2:</u>

Let $h = \sum_{i=1}^{k} f^{(i)} \otimes g^{(i)} \in (\mathcal{S}(\mathbb{R}^d))^{\otimes 2}$, $(\tilde{m})$, $(\tilde{r}) \in (\mathbb{N}^*)^d$ be given. Then,

a) $\|h\|_{(\tilde{m},\tilde{r})} = \sigma_{(\tilde{m}),(\tilde{r})}(h)$,

b) $\pi_{(\tilde{m}),(\tilde{r})}(h) \le (\pi^2/6)^d \min\{\varepsilon_{(\tilde{m}+2),(\tilde{r})}(h),\ \varepsilon_{(\tilde{m}),(\tilde{r}+2)}(h)\}$.

<u>Remark 1:</u>

a) Lemma 2b) and Proposition 1b) imply the nuclearity of $\mathcal{S}(\mathbb{R}^d)$. For a direct proof of nuclearity we refer to /2/.

b) Similar estimations to these of Lemma 2 are also valid for $(\mathcal{S}(\mathbb{R}^d))^{\otimes n}$ with arbitrary $n \in \mathbb{N}$, $n \geq 2$.

Proof of Lemma 2:

a) follows by the following equalities

$$\sigma_{(\tilde{m}),(\tilde{r})}(h) =$$

$$= (\sum_{i,j=1}^{k} (\int \overline{N^{(\tilde{m})} f^{(i)}(x)}\ N^{(\tilde{m})} f^{(j)}(x) dx)(\int \overline{N^{(\tilde{r})} g^{(i)}(x)}\ N^{(\tilde{r})} g^{(j)}(x) dx))^{1/2}$$

$$= (\int | \sum_{i=1}^{k} N^{(\tilde{m},\tilde{r})}\ f^{(i)}(x_1,\dots,x_d)\ g^{(i)}(x_{d+1},\dots,x_{2d})|^2 dx_1 \dots dx_{2d})^{1/2}$$

$$= \|h\|_{(\tilde{m},\tilde{r})}.$$

b) Let us put

$$\langle f,g \rangle = \int \overline{f(x)}\ g(x)\ dx,$$

$f,g \in \mathcal{S}(\mathbb{R}^d)$, and $(\tilde{s})$, $(\tilde{\mu})$, $(\tilde{r})$, $(\tilde{m}) \in (\mathbb{N}^*)^d$, $\tilde{s}^{\tilde{\mu}} = \prod_{i=1}^{d} (s_i+1)^{\mu_i}$.

Further, let be given the Hermite-functions

$$\phi^o = \pi^{-1/4}\ e^{-t^2/2},$$

$$\phi^k = (2^k k!)^{-1/2} (-1)^k\ \pi^{-1/4}\ e^{t^2/2}\ (d/dt)^k\ e^{-t^2},$$

$k=1,2,\dots,$ and

$$\phi^{(\tilde{s})} = \phi^{s_1} \otimes \dots \otimes \phi^{s_d}.$$

Using the theorem for the N-representation of $\mathcal{S}$ (/10; Theorem V.13/), the Hermite decomposition

$$f = \sum_{(\tilde{s})} \alpha_{(\tilde{s})}\ \phi^{(\tilde{s})}$$

converges for each $f \in \mathcal{S}(\mathbb{R}^d)$, where $\alpha_{(\tilde{s})} = \langle \phi^{(\tilde{s})}, f \rangle$. Recall also

$$N^{(\tilde{r})}\ \phi^{(\tilde{m})} = \prod_{i=1}^{d} (m_i+1)^{r_i}\ \phi^{(\tilde{m})}, \tag{4}$$

$$\|\phi^{(\tilde{m})}\|^2_{(\tilde{o})} = 1, \tag{4'}$$

$(\tilde{o}) = (0,\dots,0) \in (\mathbb{N}^*)^d$. Furthermore $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)'$ holds true, where $a \in \mathcal{S}(\mathbb{R}^d)$ is identified with the linear functional

$$\langle a,. \rangle \in \mathcal{S}(\mathbb{R}^d)'.$$

Then (4), (4') imply the equivalence

$$\lambda \phi^{(\tilde s)} \in U^o_{(\tilde\mu)} \text{ if and only if } |\lambda| \le \tilde s^{\tilde\mu}, \qquad (5)$$

$\lambda\in\mathbb{C}$, $U^o_{(\tilde\mu)} = \{T\in(\mathcal{S}(\mathbb{R}^d))'; \ |T(.)| \le \|\cdot\|_{(\tilde\mu)}\}$. Further,

$$\varepsilon_{(\tilde\mu),(\tilde r)}\Big(\sum_{i=1}^k f^{(i)}\otimes g^{(i)}\Big) = \sup\Big\{\Big|\sum_{i=1}^k T(f^{(i)})S(g^{(i)})\Big|; \ T\in U^o_{(\tilde\mu)}, S\in U^o_{(\tilde r)}\Big\}$$

$$\ge \sup\Big\{\Big|S\Big(\sum_{i=1}^k \tilde T(f^{(i)})\, g^{(i)}\Big)\Big| ; \ S\in U^o_{(\tilde r)}\Big\}$$

$$= \Big\|\sum_{i=1}^k \tilde T(f^{(i)})\, g^{(i)}\Big\|_{(\tilde r)} \qquad (6)$$

hold for all $\tilde T\in U^o_{(\tilde\mu)}$. Finally one gets

$$\pi_{(\tilde m),(\tilde r)}(h) = \pi_{(\tilde m),(\tilde r)}\Big(\sum_{i=1}^k\Big(\sum_{(\tilde s)}\langle\phi^{(\tilde s)},f^{(i)}\rangle\,\phi^{(\tilde s)}\Big)\otimes g^{(i)}\Big) \le$$

$$\overset{(*)}{\le} \sum_{(\tilde s)} \|\phi^{(\tilde s)}\|_{(\tilde m)} \Big\|\sum_{i=1}^k\langle\phi^{(\tilde s)},f^{(i)}\rangle g^{(i)}\Big\|_{(\tilde r)} \le$$

$$\overset{(**)}{\le} \sum_{(\tilde s)} \|\phi^{(\tilde s)}\|_{(\tilde m)}\, (s^{n+2})^{-1}\, \varepsilon_{(\tilde m+2),(\tilde r)}(h) =$$

$$\overset{(+)}{=} \Big(\sum_{(\tilde s)} \tilde s^{\tilde n}\,(\tilde s^{\tilde n+2})^{-1}\Big)\, \varepsilon_{(\tilde m+2),(\tilde r)}(h) =$$

$$= (\pi/6)^d\, \varepsilon_{(\tilde m+2),(\tilde r)}(h).$$

Similar estimations yield $\pi_{(\tilde m),(\tilde r)}(h) \le (\pi/6)^d\, \varepsilon_{(\tilde m),(\tilde r+2)}(h)$.
This completes the proof.
((*) is a consequence of the triangle inequality.
(**): (5) implies $\tilde s^{\tilde m+2}\phi^{(\tilde s)}\in U^o_{(\tilde m+2)}$. Considering (6) for $(\tilde\mu) = (\tilde m+2)$ and $\tilde T = \tilde s^{\tilde m+2}\phi^{(\tilde s)}$, inequality (**) follows.
(+) is a consequence of (4), (4').)

Concerning the topologies introduced in Section 1 the following is satisfied.

Proposition 3:

On $(\mathcal{S}(\mathbb{R}^d))^{\otimes n}$, $\varepsilon_n = \tau_n$ holds true.

Proof:

$\varepsilon_n = \pi_n$ follows by Lemma 2b) and Remark 1b). Since $\mathcal{S}(\mathbb{R}^d)$ is an (F)-space, the assumptions of Proposition 2c) are satisfied. Thus, $\pi_n = \tau_n$.

b) $\mathcal{D}(\mathbb{R}^d)$

The Schwartz-space $\mathcal{D}(\mathbb{R}^d)$ of test-functions with compact support is given by

$$\mathcal{D}(\mathbb{R}^d) = \bigcup_{i=1}^\infty \mathcal{D}^{(i)}(\mathbb{R}^d),$$

where $\mathcal{D}^{(a)}(\mathbb{R}^d) = \{f\in C^\infty(\mathbb{R}^d); \ supp(f)\subset\{x\in\mathbb{R}^d; |x_1|\le a,\dots,|x_d|\le a\}\}$, $a>0$. Further, let $\mathcal{D}^{(a)}(\mathbb{R}^d)$ be furnished with the l.c. topology $\delta^{(a)}$ which is defined by

$$\{f\longrightarrow \sup\{|D^{(m)} f(x)| ; \ x\in\mathbb{R}^d\}; \ (\tilde m)\subset(\mathbb{N}^*)^d\},$$

$f\in\mathcal{D}^{(a)}(\mathbb{R}^d)$, $D^{(\tilde m)} = \prod_{i=1}^d (\partial/\partial x_i)^{m_i}$, $(\tilde m)\in(\mathbb{N}^*)^d$. Then, let δ denote the topology of the strictly inductive limit of the l.c.spaces $\mathcal{D}^{(i)}(\mathbb{R}^d)$, i=1,2,..., i.e.,

$$\mathcal{D}(\mathbb{R}^d)[\delta] = \varinjlim \mathcal{D}^{(i)}(\mathbb{R}^d)[\delta^{(i)}].$$

Note that δ is defined by the following system of semi-norms:

$$\mathcal{P}(\delta) = \{f\longrightarrow Q_{(\gamma_i)(\nu_i)}(f); \ (\gamma_i)_{i=0}^\infty, (\nu_i)_{i=0}^\infty \in (\mathbb{N}^*)^{\mathbb{N}^*}\},$$

$f\in\mathcal{D}(\mathbb{R}^d)$, $Q_{(\gamma_i)(\nu_i)}(f) = \sum_{i=0}^\infty \gamma_i\, q_{i,\nu_i}(f)$,

$q_{i,s}(f) = \sup\{\max\{|D^r f(x)| ; \ r=0,1,\dots,s\}; \ i\le|x|<i+1\}$.

Further, noticing $\mathcal{D}^{(i)}(\mathbb{R}^d)[\delta^{(i)}]$ are non-normable (F)-spaces, i=1,2,..., it follows that $\mathcal{D}(\mathbb{R}^d)[\delta]$ is an (LF)-space which is not an (LB)-space.

For the sake of notational simplicity let us put d=1. The generalization to arbitrary $d\in\mathbb{N}$ is straightforward. Let us define the bilinear form $b: \mathcal{D}(\mathbb{R})\times\mathcal{D}(\mathbb{R})\longrightarrow\mathbb{C}$ by

$$b(f,g) = \sum_{m=0}^\infty D^m f(x)\big|_{x=0}\, g(m+\tfrac{1}{2}),$$

$f,g\in\mathcal{D}(\mathbb{R})$, $D = d/dx$.

Lemma 3:

The bilinear form b is separately continuous, but not continuous, i.e., $b\in\mathfrak{B}(\mathcal{D}(\mathbb{R})^{\otimes 2})\setminus\mathcal{B}(\mathcal{D}(\mathbb{R})^{\otimes 2})$.

Proof:

a) Let be given a fixed $g\in\mathcal{D}(\mathbb{R})$. Then there is an $m'\in\mathbb{N}$ such that $g\in\mathcal{D}^{(m')}(\mathbb{R})$. Hence the following hold:

$$|b(f,g)| = \Big|\sum_{m=0}^{m'} D^m f(x)\big|_{x=0}\, g(m+\tfrac{1}{2})\Big| \le \sum_{m=0}^{m'}|g(m+\tfrac{1}{2})|\, q_{0,m}(f). \qquad (7)$$

Further, for fixed $f \in \mathcal{D}(\mathbb{R})$ and $\gamma_m = |D^m f(x)\restriction_{x=0}|$, $m=0,1,2,\dots$, one gets

$$|b(f,g)| \le \sum_{m=0}^{\infty} \gamma_m \sup\{|g(x)|;\ m \le |x| < m+1\} = \sum_{m=0}^{\infty} \gamma_m q_{m,0}(g). \qquad (7')$$

The separate continuity of $b(.,.)$ is yielded by (7) and (7').

b) Let us assume that $b(.,.)$ is continuous. Then there have to be sequences $(\gamma_\mu)_{\mu=0}^{\infty}$, $(r_\mu)_{\mu=0}^{\infty} \in (\mathbb{N}^*)^{\mathbb{N}^*}$ such that

$$|b(f,g)| \le Q_{(\gamma_\mu)(r_\mu)}(f)\, Q_{(\gamma_\mu)(r_\mu)}(g), \qquad (8)$$

$f,\ g \in \mathcal{D}(\mathbb{R})$.

Now choose $g \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(g) \subset (-1/2,1/2)$, $g(0) = d > 0$, and put

$$g^{(m)}(x) = g(x-m-\tfrac{1}{2}),$$

$$c_m = \max\{\sup\{|D^s g(x)|;\ x \in \mathbb{R}\ ;\ s=0,1,\dots,r_m\}.$$

Note $\mathrm{supp}(g^{(m)}) \subset (m,m+1)$, and

$$Q_{(\gamma_\mu)(r_\mu)}(g^{(m)}) = \gamma_m c_m . \qquad (9)$$

Further choose a sequence $(h^{(\nu)})_{\nu=r_0+1}^{\infty}$, $h^{(\nu)} \in \mathcal{D}(\mathbb{R})$, such that

$$\mathrm{supp}(h^{(\nu)}) \subset (-1/3,1/3)$$

$$\sup\{|D^\nu h^{(\nu)}(x)|;\ x \in \mathbb{R}\} > \nu\gamma_0 \gamma_\nu c_\nu d^{-1} q_{0,r_0}(h^{(\nu)}). \qquad (10)$$

(Such a sequence $(h^{(\nu)})$ exists because of $q_{0,\nu}(h^{(\nu)}) = \sup\{|D^\nu h^{(\nu)}(x)|;\ x \in \mathbb{R}\}$, $\nu > r_0$, and the non-normability of $\mathcal{D}^{(1)}(\mathbb{R})$.)

Furthermore, there are $x_\nu \in (-1/3,1/3)$ such that

$$\sup\{|D^\nu h^{(\nu)}(x)\ ;\ x \in \mathbb{R}\} = |D^\nu h^{(\nu)}(x)\restriction_{x=x_\nu}| .$$

Putting $f^{(\nu)}(x) = h^{(\nu)}(x-x_\nu)$, one gets

$$Q_{(\gamma_\mu)(r_\mu)}(f^{(\nu)}) = \gamma_0 q_{0,r_0}(f^{(\nu)}) = \gamma_0 q_{0,r_0}(h^{(\nu)}), \qquad (11)$$

$$|D^\nu f^{(\nu)}(x)\restriction_{x=0}| = \sup\{|D^\nu h^{(\nu)}(x)|;\ x \in \mathbb{R}\}, \qquad (11')$$

$\nu = r_0+1,\ r_0+2, \dots$. Finally using (9), (10), (11), and (11'),

$$|b(f^{(\nu)},g^{(\nu)})| / (Q_{(\gamma_\mu)(r_\mu)}(f^{(\nu)})\, Q_{(\gamma_\mu)(r_\mu)}(g^{(\nu)})) =$$

$$= (|D^\nu f^{(\nu)}(x)\restriction_{x=0}|\, |g(\nu+\tfrac{1}{2})|) / (\gamma_0 q_{0,r_0}(f^{(\nu)})\, \gamma_\nu q_{\nu,r_\nu}(g^{(\nu)})) >$$

$$> \nu$$

follow for $\nu > r_0$. But this yields a contradiction to (8) for $\nu \longrightarrow \infty$. Hence the proof is completed.

Proposition 4:

On $(\mathcal{D}(\mathbb{R}^d))^{\otimes n}$, $\varepsilon_n = \pi_n \neq \beta_n = \iota_n$ hold.

Proof:

The nuclearity of $\mathcal{D}(\mathbb{R}^d)[\delta]$,/2/, implies $\varepsilon_n = \pi_n$ by Proposition 1a). Since $\mathcal{D}(\mathbb{R}^d)[\delta]$ is an (LF)-space, it is barrelled. Hence, $\beta_n = \iota_n$ follows by Proposition 2b). Finally, Lemma 3 yields $\pi_n \lneq \iota_n$.

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Хофман Г. E5-87-891

О топологиях над тензорными произведениями

Исследованы локально-выпуклые топологии на n-кратном тензорном произведении $E\otimes\ldots\otimes E$, обусловленные исследованиями аксиоматической квантовой теории поля. Даны достаточные критерии эквивалентности инъективной, проективной и индуктивной топологий. Эти результаты применяются к пространству Шварца $S(\mathcal{R}^d)$, $D(\mathcal{R}^d)$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Hofmann G. E5-87-891

Topologies on Tensor Products

Motivated by the investigations of axiomatic QFT, l.c. topologies on the n-fold tensor product $E\otimes\ldots\otimes E$ are studied. Sufficient criteria for the equivalence of respectively, the injective, projective and inductive topologies are given. The results are applied to the Schwartz-spaces $S(\mathcal{R}^d)$, $D(\mathcal{R}^d)$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987