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G.N.Afanasiev

CLOSED EXPRESSIONS FOR SOME USEFUL INTEGRALS INVOLVING LEGENDRE FUNCTIONS AND SUM RULES FOR ZEROES OF BESSEL FUNCTIONS

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§ 1. Introduction

In ref.[1] closed analytical expressions were obtained for a number of useful sums and integrals involving Legendre functions. The key point of the method used was in comparing of the magnetic vector-potential components for the toroidal solenoid derived by different methods (but for the same gauge and boundary conditions). Their coincidence stems from the well-known theorem (see, e.g.[2]) according to which a harmonic function (the difference of two solutions of the same Poisson equation is just function like that) equal to zero at infinity (the vector-potentials of ref.[1] satisfy this condition) is identically equal to zero. This trick (i.e. the construction of new relations between special functions by comparing the solutions of the same equation derived by different methods) is not altogether new. A lot of examples of the same type may be found in the well-known treatise on the Bessel functions[3].

The present treatment proceeds along the same lines as ref.[1] and may be viewed as its continuation. It is organized as follows. In §2 we consider three different integral representations for the same function. By comparing them we obtain the integrals involving Legendre functions in a closed form. In § 3 we study how the eigenvalues of the Schroedinger equation change when the vector-potential $\vec{A} \neq 0$ (but $\vec{H} = \operatorname{rot}\vec{A} = 0$) is switched on in the simply connected space. According to the theory (see, e.g. [4]), in the simply connected region the eigenvalues for $\vec{A} \neq 0$ should be the same as for $\vec{A} = 0$. Evaluating the second order terms of the perturbation theory (PT) explicitly and equating them to zero we get the sum rules for zeroes of the Bessel functions of the integer and semi-integer orders. The expressions obtained are lacking in mathematical handbooks, treatises and original publications (see, e.g., [3,5,6]).

§ 2. Closed Expressions for Some Integrals Involving Legendre Functions

In ref.[7] a function α was used which connected the vector-potential of the toroidal solenoid in different gauges. It is defined by the following double integral

$$\alpha(\rho,z) = \frac{1}{2}\int \frac{dx_1\,dy_1}{|\vec{r} - \vec{r}_1|} \tag{2.1}$$

The integration in (2.1) is performed inside a circle of the radius a lying in the $z_1 = 0$ plane: $z_1 = 0$, $0 \le \rho_1 \le a$, $0 \le \varphi_1 \le 2\pi$. For this function in /7/, the following three different integral representations

$$\alpha = \pi\left(\sqrt{z^2+a^2} - |z|\right) - \sqrt{a}\int_0^\rho \frac{dx}{\sqrt{x}}\, Q_{\frac{1}{2}}(t) \tag{2.2}$$

$$\alpha = \frac{1}{\sqrt{\rho}}\int_0^a \sqrt{x}\,dx \cdot Q_{-\frac{1}{2}}(y) \tag{2.3}$$

$$\alpha = \sqrt{\mathrm{ch}\mu - \cos\theta}\sum_{n=0}^{\infty} \alpha_n(\mu)\cdot\cos n\theta \tag{2.4}$$

$$\left(y = \frac{z^2+x^2+\rho^2}{2\rho x},\ t = \frac{z^2+x^2+a^2}{2ax}\right)$$

were obtained. The variables μ, θ entering into (2.4) are toroidal coordinates. They are connected with the cylindrical ones as follows:

$$\rho = a\,\frac{\mathrm{sh}\mu}{\mathrm{ch}\mu - \cos\theta},\quad z = a\,\frac{\sin\theta}{\mathrm{ch}\mu - \cos\theta}\quad (0 < \mu < \infty,\ -\pi < \theta < \pi). \tag{2.5}$$

Q_μ is the Legendre function of the 2-nd kind. The function $\alpha_n(\mu)$ is equal to

$$\alpha_n(\mu) = \frac{2a}{1+\delta_{n0}}\cdot(-1)^n\cdot\Big[Q_{n-\frac{1}{2}}(\mathrm{ch}\mu)\int_1^{\mathrm{ch}\mu} P_{n-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}} + \tag{2.6}$$

$$+ P_{n-\frac{1}{2}}(\mathrm{ch}\mu)\int_{\mathrm{ch}\mu}^{\infty} Q_{n-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}}\Big],$$

P_μ is the Legendre function of the 1st kind.

Now we try to express some integrals in a closed form. At first equate (2.2) and (2.3)

$$\pi\left(\sqrt{z^2+a^2} - |z|\right) - \sqrt{a}\int_0^\rho \frac{dx}{\sqrt{x}}\, Q_{\frac{1}{2}}(t) = \frac{1}{\sqrt{\rho}}\int_0^a \sqrt{x}\,dx \cdot Q_{-\frac{1}{2}}(y). \tag{2.7}$$

Then, taking the limits $\rho \to \infty$ and $z \to \infty$ and differentiating the expressions obtained we get explicit expressions for the following integrals

$$\int_0^\infty \frac{dx}{\sqrt{x}}\cdot Q_{\frac{1}{2}}(t) = \frac{\pi}{\sqrt{a}}\left(\sqrt{z^2+a^2} - |z|\right)$$

$$\int_0^\infty \sqrt{x}\,dx\cdot Q^1_{\frac{1}{2}}(t) = -\frac{\pi a^{5/2}}{|z|}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}}\,\frac{1}{\sqrt{t^2-1}}\,Q^1_{-\frac{1}{2}}(t) = -\frac{\pi a^{3/2}}{|z|}$$

$$\int_0^\infty \frac{dx}{x^{3/2}}\,\frac{1}{\sqrt{t^2-1}}\,Q^1_{\frac{1}{2}}(t) = \pi\sqrt{a}\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{|z|}\right).$$

The change of variables permits one to transform these integrals into the known ones (see, e.g. /5/).

Put $\rho = a$, $z = 0$ in (2.7). This gives

$$\pi = \int_0^1 dx\left[\frac{1}{\sqrt{x}}\,Q_{\frac{1}{2}}\left(\frac{1+x^2}{2x}\right) + \sqrt{x}\,Q_{-\frac{1}{2}}\left(\frac{1+x^2}{2x}\right)\right]. \tag{2.8}$$

Now return to Eqs. (2.2)-(2.4). Set in (2.4) $\mu = 0$. In accordance with (2.5) we should take $\rho = 0$ in (2.2). Equating (2.2) and (2.4) results in

$$\pi\left(\sqrt{z^2+a^2} - |z|\right) = \sqrt{1-\cos\theta}\sum_{n=0}^{\infty}\alpha_n(0)\cdot\cos n\theta. \tag{2.9}$$

where

$$\alpha_n(0) = \frac{2a\cdot(-1)^n}{1+\delta_{n0}}\int_0^\infty Q_{n-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}}$$

Substitute $z = \frac{a\sin\theta}{1-\cos\theta}$ into (2.9), divide both sides by $\sqrt{1-\cos\theta}$ and integrate over θ:

$$\int_1^\infty \frac{dx}{(1+x)^{3/2}}\cdot Q_{n-\frac{1}{2}}(x) = \frac{(-1)^n}{\sqrt{2}}\int_{-\pi/2}^{\pi/2}\frac{d\varphi}{1+\cos\varphi}\cdot\cos 2n\varphi \qquad \left(\varphi = \frac{\theta}{2}\right) \tag{2.10}$$

The integral in the r.h.s. of (2.10) is easily evaluated:

$$\int_{-\pi/2}^{\pi/2}\frac{d\varphi}{1+\cos\varphi}\cos 2n\varphi = 2 - 2\pi n - 2\left[1-(-1)^n\right] - 8n\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}$$

Thus

$$\int_1^\infty \frac{dx}{(1+x)^{3/2}}\,Q_{n-\frac{1}{2}}(x) = \sqrt{2}\left[1 - \pi n\cdot(-1)^n - 4n\cdot(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right]. \tag{2.II}$$

In particular cases:

$$\int_1^\infty Q_{-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}} = \sqrt{2}$$

$$\int_1^\infty Q_{\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}} = \sqrt{2}\,(\pi-3) \qquad (2.12)$$

$$\int_1^\infty Q_{\frac{3}{2}}(x)\,\frac{dx}{(1+x)^{3/2}} = \sqrt{2}\cdot\left(\frac{19}{3}-2\pi\right).$$

Using the Whipple relation between the Legendre functions one may transform (2.11) to the following form:

$$\int_0^\mu d\mu\cdot\exp\left(-\tfrac{3}{2}\mu\right)\cdot P^{n}_{-\frac{1}{2}}(\mathrm{ch}\,\mu)=\frac{2}{\sqrt{\pi}}\,\frac{1}{\Gamma\left(\frac{1}{2}-n\right)}\cdot\left[1-\pi n\cdot(-1)^n-4n\cdot(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right] \qquad (2.13)$$

($P^{\mu}_{\lambda}(x)$ is the associated Legendre function of the 1st kind). These expressions are lacking in the mathematical literature.

§ 3. The Sum Rules for the Zeroes of the Bessel functions

3.1. Consider an infinite cylinder C of the radius R. Let its axis coincides with the z axis:

$$\rho=\sqrt{x^2+y^2}=R\,, \quad -\infty<z<\infty$$

We are interested in the eigenvalues and eigenfunctions of the Schroedinger equation inside the cylinder. The following boundary condition is imposed on the eigenfunction: $\Psi=0$ for $\rho\geqslant R$ (this is equivalent to the solution of the Schroedinger equation with the potential $V(\rho)=0$ for $\rho<R$ and $V=\infty$ for $\rho>R$). The eigenfunctions and eigenvalues of the Schroedinger equation are equal (see, e.g. /8/) to:

$$\Psi^0_{ms}=C_{ms}\cdot J_m\left(\lambda_{ms}\frac{\rho}{R}\right)\cdot\exp(im\varphi)$$
$$E^0_{ms}=\frac{\hbar^2}{2\mu R^2}\lambda^2_{ms}. \qquad (3.1)$$

Here μ is the mass of a particle moving inside C, m is its angular momentum, $\hbar$ is the Planck constant, λ_{ms} is an s-th root of the equation

$$J_m(x)=0.$$

Finally, C_{ms} is the normalized constant

$$C_{ms}=\frac{1}{R\sqrt{\pi}}\left[-J_{m-1}(\lambda_{ms})\cdot J_{m+1}(\lambda_{ms})\right]^{1/2}.$$

For simplicity (and without loss of generality) we limited ourselves in (3.1) to the motion in the $z=0$ plane.

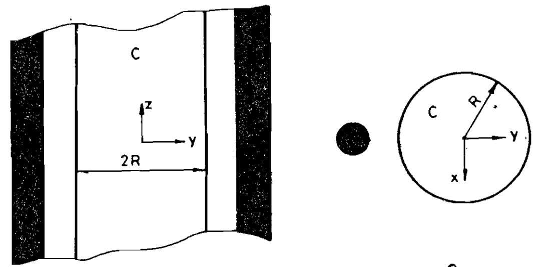


Fig. 1. Outside an infinite cylinder C there are two infinite cylindrical solenoids (darkened) which produce inside C a magnetic field with $\vec{A}\neq 0$, $\vec{H}=\mathrm{rot}\,\vec{A}=0$. According to theory, the magnetic field does not change the eigenfrequencies of the cylindrical cavity.

Now install, outside C, two infinite cylindrical solenoids of the radius a with opposite magnetic fluxes ($\Phi_1=-\Phi_2=\Phi$) (see fig. 1). Let their axes be parallel to the z axis and pass through the points $\pm d\,(>R+a)$ of the y axis. Outside both the solenoids the strength $\vec{H}$ of the magnetic field equals zero while the magnetic vector-potential $\vec{A}$ ($\vec{H}=\mathrm{rot}\,\vec{A}$) differs from zero:

$$A_\rho=\frac{\Phi d}{\pi}\cos\varphi\,\frac{\rho^2+d^2}{Z}\,,\quad A_\varphi=\frac{\Phi d}{\pi}\sin\varphi\,\frac{\rho^2-d^2}{Z}\,,\quad A_z=0 \qquad (3.2)$$

$$\left(Z=\rho^4+d^4+2d^2\rho^2\cos2\varphi\,,\quad \mathrm{div}\,\vec{A}=0\right).$$

Does the presence of a nonzero $\vec{A}$ inside C change the energy levels E^0_{ms}? We note that the space accessible for particles (the interior of C) is simply connected. Theory (see, e.g. /4/) says that the existence of curlless nonzero vector-potentials could not lead to the observable effects in a simply-connected space. This means, in particular, that eigenvalues of the Schroedinger equation

$$-\frac{\hbar^2}{2\mu}\left(\vec\nabla-\frac{ie}{\hbar c}\vec A\right)^2\Psi = E\Psi \qquad (3.3)$$

with a nonzero $\vec A$ given by Eqs.(3.2) should coincide with E^0_{ms} determined by Eqs. (3.1). The invariance of eigenvalues takes place for any value of the dimensionless parameter $\gamma=\frac{e\Phi}{\hbar c}$. This means that in the perturbation expansion in γ :

$$H = H_0 + H_1 + H_2$$

$$\Psi_{ms} = \Psi^0_{ms} + \Psi^1_{ms} + \Psi^2_{ms} + \dots$$

$$E_{ms} = E^0_{ms} + E^1_{ms} + E^2_{ms} + \dots$$

the corrections to E^0_{ms} should separately vanish in each order in γ. In the first-order PT one has:

$$E^1_{ms} = \langle\Psi^0_{ms}|H_1|\Psi^0_{ms}\rangle \qquad \left(H_1 = \frac{2ie}{\hbar c}\vec A\vec\nabla\right) \qquad (3.4)$$

From the expansion of $\vec A$ in the angular variable φ

$$A_\rho = -\frac{\Phi d}{\pi}\,\frac{\rho^2+d^2}{2d^2\rho^2}\left(1+\mathrm{th}\frac{\mu}{2}\right)\sum_{n=1}^{\infty}(-1)^n e^{-\mu n}\cos[(2n-1)\varphi],$$

$$A_\varphi = -\frac{\Phi d}{\pi}\,\frac{\rho^2-d^2}{2d^2\rho^2}\left(1+\mathrm{cth}\frac{\mu}{2}\right)\sum_{n=1}^{\infty}(-1)^n e^{-\mu n}\sin[(2n-1)\varphi], \quad \left(\mathrm{ch}\mu=\frac{\rho^4+d^4}{2d^2\rho^2}\right) \qquad (3.5)$$

it follows at once that the equation

$$E^1_{ms}=0$$

is satisfied automatically (due to the angular dependence of $\vec A\vec\nabla$). Nevertheless, the eigenfunctions are modified in the same order:

$$|\Psi_{ms}\rangle = |\Psi^0_{ms}\rangle + \sum_{\substack{ns'\\(n\neq m)}}|\Psi^0_{ns'}\rangle\frac{\langle\Psi^0_{ns'}|H_1|\Psi^0_{ms}\rangle}{E^0_{ms}-E^0_{ns'}} \qquad (3.6)$$

In the second-order PT one obtains for the eigenvalues

$$E^2_{ms} = \langle\Psi^0_{ms}|H_2|\Psi^0_{ms}\rangle + \sum_{\substack{ns'\\(n\neq m)}}\frac{|\langle\Psi^0_{ns'}|H_1|\Psi^0_{ms}\rangle|^2}{E^0_{ms}-E^0_{ns'}} \qquad (3.7)$$

$$\left(H_2=\frac{e^2}{\hbar^2c^2}\vec A^2\right).$$

It follows from (3.7) that E^2_{ms} does not vanish trivially. The substitution of the unperturbed eigenfunctions (3.1) and vector-potentials (3.5) into (3.7) leads to cumbersome relations between the radial integrals. Fortunately, they are simplified for $R \ll d$ (i.e. when the radius of the available cylindrical cavity is small). Then inside C

$$A_\rho \simeq \frac{\Phi d}{\pi}\cos\varphi,\quad A_\varphi = -\frac{\Phi d}{\pi}\sin\varphi,\quad \vec A^2=\frac{\Phi^2d^2}{\pi^2}.$$

Insert these expressions into (3.7)

$$\frac{2\mu}{\hbar^2}\,\frac{\pi^2d^2}{4\gamma^2}E^2_{ms} = \frac14 - \lambda^2_{ms}\sum_{s'}\frac{\lambda^2_{m+1,s'}}{(\lambda^2_{m+1,s'}-\lambda^2_{ms})^3} - \lambda^2_{ms}\sum_{s'}\frac{\lambda^2_{m-1,s'}}{(\lambda^2_{m-1,s'}-\lambda^2_{ms})^3}.$$

The requirement for E^2_{ms} to vanish suggests the following sum rule for the zeroes of the Bessel functions:

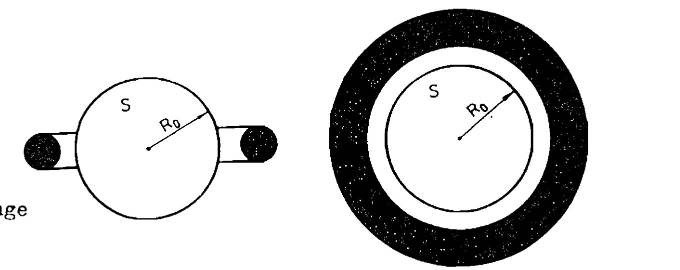
$$\frac{1}{4\lambda^2_{ms}} = \sum_{s'}\frac{\lambda^2_{m+1,s'}}{(\lambda^2_{m+1,s'}-\lambda^2_{ms})^3} + \sum_{s'}\frac{\lambda^2_{m-1,s'}}{(\lambda^2_{m-1,s'}-\lambda^2_{ms})^3} \qquad (3.8)$$

This expression is simplified for $m=0$

$$\frac{1}{8\lambda^2_{0s}} = \sum_{s'}\frac{\lambda^2_{1s'}}{(\lambda^2_{1s'}-\lambda^2_{0s})^3}. \qquad (3.9)$$

3.2. Let the space available for particles be a sphere of the radius R_0 (i.e. inside the sphere the eigen-function satisfies the free Schroedinger equation with the boundary condition $\Psi(r=R_0)=0$).

Fig. 2.
Outside the spherical cavity S there is a toroidal solenoid (darkened). The magnetic field of the solenoid does not change the eigenfrequencies of the spherical cavity.



Installing outside the sphere a toroidal solenoid (fig. 2) we create inside S a magnetic field with $\vec{H}=0$, but $\vec{A}\neq 0$. The space accessible for particles is simply connected. Thus, the existence of a nonzero A inside S should not change the energy levels. Using the vector-potentials of the toroidal solenoid given in ref./7/, we solve the Schroedinger equation with $\vec{A}\neq 0$ and require the energy shift to vanish in each order of PT in the parameter $\gamma=\frac{e\Phi}{\hbar c}$. The corrections of the first order vanish automatically. In the second order one arrives at the following two nontrivial sum rules

$$\frac{2l+3}{16\,\omega_{ls}^2} = -\sum_{s'} \frac{\omega_{l+1,s'}^2}{(\omega_{ls}^2-\omega_{l+1,s'}^2)^3}, \tag{3.10}$$

$$\frac{2l-1}{16\,\omega_{ls}^2} = \sum_{s'} \frac{\omega_{l-1,s'}^2}{(\omega_{ls}^2-\omega_{l-1,s'}^2)^3} \tag{3.11}$$

Here ω_{ls} is an s-th root of the equation

$$J_{l+\frac{1}{2}}(x)=0.$$

For $l=0,1$ one finds

$$\frac{3}{16}\,\frac{1}{\omega_{0s}^2} = -\sum_{s'} \frac{\omega_{1s'}^2}{(\omega_{0s}^2-\omega_{1s'}^2)^3}, \tag{3.12}$$

$$\frac{1}{16}\,\frac{1}{\omega_{1s}^2} = \sum_{s'} \frac{\omega_{0s'}^2}{(\omega_{1s}^2-\omega_{0s'}^2)^3} \tag{3.13}$$

4. Review of the Results Obtained

Here we shall collect the formulas obtained. As we mentioned earlier, they are absent in mathematical references:

$$\pi = \int_0^1 dx\left[\frac{1}{\sqrt{x}}\,Q_{\frac{1}{2}}\left(\frac{1+x^2}{2x}\right)+\sqrt{x}\;Q_{-\frac{1}{2}}\left(\frac{1+x^2}{2x}\right)\right] \tag{4.1}$$

$$\int_1^\infty Q_{n-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}} = \tag{4.2}$$

$$=\sqrt{2}\left[1-\pi n\cdot(-1)^n-4n\cdot(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right].$$

Particular cases of (4.2):

$$\int_1^\infty Q_{-\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}}=\sqrt{2},\quad \int_1^\infty Q_{\frac{1}{2}}(x)\,\frac{dx}{(1+x)^{3/2}}=\sqrt{2}\,(\pi-3),$$

$$\int_1^\infty Q_{\frac{3}{2}}(x)\,\frac{dx}{(1+x)^{3/2}}=\sqrt{2}\left(\frac{19}{3}-2\pi\right).$$

$$\int_0^\mu d\mu\cdot\exp\left(-\tfrac{3}{2}\mu\right)\cdot P^n_{-\frac{1}{2}}(\operatorname{ch}\mu) = \tag{4.3}$$

$$=\frac{2}{\sqrt{\pi}}\,\frac{1}{\Gamma(\frac{1}{2}-n)}\cdot\left[1-\pi n\cdot(-1)^n-4n\cdot(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right].$$

$$\frac{1}{4\lambda_{ms}^2}=\sum_{s'}\frac{\lambda_{m+1,s'}^2}{(\lambda_{m+1,s'}^2-\lambda_{ms}^2)^3}+\sum_{s'}\frac{\lambda_{m-1,s'}^2}{(\lambda_{m-1,s'}^2-\lambda_{ms}^2)^3} \tag{4.4}$$

Here λ_{ms} is the s-th root of the equation $J_m(x)=0$.

For $m=0$ (4.4) is simplified:

$$\frac{1}{8\lambda_{0s}^2}=\sum_{s'}\frac{\lambda_{1s'}^2}{(\lambda_{1s'}^2-\lambda_{0s}^2)^3}.$$

$$\frac{2l+3}{16\,\omega_{ls}^2} = -\sum_{s'} \frac{\omega_{l+1,s'}^2}{(\omega_{ls}^2-\omega_{l+1,s'}^2)^3} \tag{4.5}$$

$$\frac{2l-1}{16\,\omega_{ls}^2} = \sum_{s'} \frac{\omega_{l-1,s'}^2}{(\omega_{ls}^2-\omega_{l-1,s'}^2)^3}. \tag{4.6}$$

Here ω_{ls} is an s-th root of the equation

$$J_{l+\frac{1}{2}}(x)=0.$$

For $l=0,1$ Eqs. (4.5) and (4.6) reduce to

$$\frac{3}{16}\cdot\frac{1}{\omega_{0s}^2} = -\sum_{s'} \frac{\omega_{1s'}^2}{(\omega_{0s}^2-\omega_{1s'}^2)^3},$$

$$\frac{1}{16}\,\frac{1}{\omega_{1s}^2} = \sum_{s'} \frac{\omega_{0s'}^2}{(\omega_{1s}^2-\omega_{0s'}^2)^3}.$$

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Афанасьев Г.Н. E5-87-801

Замкнутые выражения для некоторых полезных интегралов, содержащих функции Лежандра и правила сумм для нулей функций Бесселя

Путем сравнения различных интегральных представлений одной и той же функции найдены явные выражения для интегралов, содержащих функции Лежандра. Из требования неизменяемости собственных частот цилиндрической или сферической полости при включении безвихревого векторного магнитного потенциала получены правила сумм для нулей функций Бесселя.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Afanasiev G.N. E5-87-801

Closed Expressions for Some Useful Integrals Involving Legendre Functions and Sum Rules for Zeroes of Bessel Functions

Comparing different integral representations of the same function we find closed expressions for a number of useful integrals involving Legendre functions. Switching on the curlless vector magnetic potential inside the cylindrical or spherical cavities and requiring the nonvariance of their eigenvalues we obtain the sum rules for zeroes of Bessel functions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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