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THRESHOLD
OF KdV SOLITON PRODUCTION

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1. Introduction

The investigation of nonlinear excitations attracts attention of specialists of the various fields, from pure mathematics up to practical applications. The problem of great importance in this concern is the following: which fundamental modes are responsible for the behaviour of a physical system in various parameter regions and how dynamical regime can be transformed into a new coherent structures.[1]. As is known for nondissipative systems, which are described by nonlinear differential equations, such stable coherent structures are the solitary wave soluttions or solitons [2]. Separation of fundamental modes, viz., the solitons and linear waves, is more clearly observed in the complete integrable systems to which the Inverse Scattering Transform (IST) method is applicable. However, the problem of the nonlinear spectral transform of an initial pulse is still open. Which modes can be generated in a pulse decay? how they depend on the initial pulse parameters? It is clear that these questions are important in the investigations of soliton generation in plasma physics, in magnetic systems, in nonlinear optics and so on [2,3].

The problem may, in principle, be studied in the framework of the IST equations, but here the well-known difficulties arise and it may be solved for only a limited class of initial pulses. The difficulties, as is known, are connected with a numerical solution of the IST integral equations, when the reflection coefficient $R \neq 0$. On the other hand, the information about an integrable system (i.e. the higher integrals of motion, linear eigenvalue problem and so on) turns out to be useful in approximate describing soliton formation by taking into account their amplitude, profile, formation time and the number of solitons [4,5].

An approximate method for analitycal investigation of soliton formation as a result of the decay of initial pulses has been proposed by Karpman and Sokolov for the Korteweg-de Vries (KdV) equation [4]. However, their method is relevant only when the nonsoliton sector contains a small part of excitation. It is known that in the decay of "zero area" pulses $u(x,0)$,

$$\int_{-\infty}^{+\infty} u(x,0)\,dx = 0\,, \tag{1.1}$$

the nonsoliton tail takes away the considerable part of excitation energy and under some conditions solitons are not formed. This is true, for example, in the case of the Gaussian pulses [5]:

$$u(x,0) = u_c\, x\, e^{-k^2x^2}. \tag{1.2}$$

The condition (1.1) is satisfied by the harmonical pulses too, which importance does not give rise to any doubt since any smooth pulses may be expanded in the Fourier series.

One of examples of the harmonical pulse decay is the model of the anharmonical crystal with a cubic nonlinearity or the Toda lattice model [6], which longwave excitations are described by the KdV equation. Depending on the parameters of an initial pulse a phase transition can arise as a result of its decay leading to a soliton lattice that describes the stable configuration of an anharmonic chain.

The practical absence of results on the dynamics of the zero area pulses as well as the question about the soliton generation in the magnetized plasma waveguide [7] provokes us to study this problem.

In Section 2 of this paper a simple quantum mechanical problem of the existence of discrete energy levels in the exactly solvable potential (2.2) is considered. A transcendent equation for eigenvalues is numerically solved and a threshold for the discrete energy level appearence is shown to exist.

In Section 3 we simulate numerically the decay of some zero area pulses for the KdV equation. A procedure to approximate the pulses is proposed and the threshold existence region predicted theoretically is in a good agreement with the numerical simulation results.

In conclusion we discuss the results obtained.

2. The existence of the discrete energy levels in the zero "area" potential

As is known the reflectionless potentials of the stationary Schrödinger equation

$$-\Psi_{xx} + u(x)\Psi = E\,\Psi \tag{2.1}$$

correspond to KdV solitons in the framework of the IST method. In connection with the problem under consideration it is important to find the condition for a discrete energy level in potentials with the zero area to exist.

The exact solution of this problem is possible only for a narrow class of potentials. We restrict ourselves to the simple potential which consists of the rectangular well and barrier

$$u(x) = \begin{cases} 0\,, & x < -a\,, \\ -u_1\,, & -a \leqslant x \leqslant 0\,, \\ 0\,, & 0 < x < b-a\,, \\ u_2\,, & b-a \leqslant x \leqslant b\,, \\ 0\,, & x > b\,. \end{cases} \tag{2.2}$$

In the next section having used the results obtained here we shall approximate the zero area pulses of a more complicated form by this potential.

The solutions to Schrödinger equation (2.1) corresponding to discrete level $E=-|E|$, in the case of potential (2.2) look like

$$\Psi(x)=\begin{cases}\Psi_{01}\, e^{k_1 x}, & x<-a\\ \Psi_{02}\sin(k_2x+\delta_2), & -a\leqslant x\leqslant 0,\\ \Psi_{03}\, e^{k_1x}+\Psi'_{03}\, e^{-k_1x}, & 0<x<b-a,\\ \Psi_{04}\, e^{k_4x}+\Psi'_{04}\, e^{-k_4x}, & b-a\leqslant x\leqslant b,\\ \Psi_{05}\, e^{-k_1x} & x>b\end{cases}\qquad(2.3)$$

where $k_1=|E|^{1/2}$, $k_2=(U_0-|E|)^{1/2}$, $k_4=(U_0+|E|)^{1/2}$ and $U_0\equiv|U_1|=|U_2|$.
Using continuity of the logarithmic derivative of the wave functions (2.3) we get the following set of equations

$$k_1=k_2\,\mathrm{ctg}\,(-k_2a+\delta_2), \qquad(2.4)$$

$$k_2\,\mathrm{ctg}\,\delta_2=k_1\frac{1-\delta_3}{1+\delta_3}, \quad \delta_3=\Psi'_{03}/\Psi_{03}, \qquad(2.5)$$

$$k_1\frac{e^{k_1(b-a)}-\delta_3 e^{-k_1(b-a)}}{e^{k_1(b-a)}+\delta_3 e^{-k_1(b-a)}}=k_4\frac{e^{k_4(b-a)}-\delta_4 e^{-k_4(b-a)}}{e^{k_4(b-a)}+\delta_4 e^{-k_4(b-a)}}, \qquad(2.6)$$

$$-k_1=k_4\frac{e^{k_4b}-\delta_4e^{-k_4b}}{e^{k_4b}+\delta_4e^{-k_4b}}, \quad \delta_4=\Psi'_{04}/\Psi_{04}. \qquad(2.7)$$

From Eqs. (2.6) and (2.7) we have for the phase δ_3 the expression

$$\delta_3=\frac{e^{2k_1(b-a)}\left[(k_4+k_1)^2e^{2k_4a}-(k_4-k_1)^2\right]}{(k_4^2-k_1^2)\quad(1-e^{2k_4a})}.$$

Substituting this expression in Eq. (2.5) we have

$$\mathrm{ctg}\,\delta_2=f(E), \qquad(2.8a)$$

where $f(E)$ is given by

$$f(E)=\frac{k_1(k_4^2-k_1^2)(e^{2k_4a}-1)+\left[(k_4+k_1)^2e^{2k_4a}-(k_4-k_1)^2\right]e^{2k_1(b-a)}}{k_2(k_4^2-k_1^2)(e^{2k_4a}-1)-\left[(k_4+k_1)^2e^{2k_4a}-(k_4-k_1)^2\right]e^{2k_1(b-a)}}. \qquad(2.8b)$$

The equations (2.4) and (2.8) imply the following transcendent equation for the discrete energy levels

$$|E|=(U_0-|E|)\left\{\frac{f(E)\,\mathrm{ctg}\left[(U_0-|E|)^{1/2}a\right]+1}{\mathrm{ctg}\left[(U_0-|E|)^{1/2}a\right]-f(E)}\right\}^2. \qquad(2.9)$$

We have investigated equation (2.9) numerically. The solutions at different values of the well width a and depth U_0, different distances between the well and the barrier $\Delta\equiv b-a$ are presented in Tables I and II.

It is easily to see the number N of the discrete energy levels, i.e. the number of crossings of functions

$$\varphi_1\equiv|E|, \quad \varphi_2\equiv(U_0-|E|)\left\{\frac{f(E)\,\mathrm{ctg}\left[(U_0-|E|)^{1/2}a\right]+1}{\mathrm{ctg}\left[(U_0-|E|)^{1/2}a\right]-f(E)}\right\}^2$$

depends essentially on the well width a and increases with its growth.

For the values $a<1$ we find the threshold for a discrete energy level to exist. This means that the discrete energy levels do not exist always in the well, they depend on the well-barrier distance. At the same time it is well known that adding a second, symmetrically located barrier lowers the energy of levels. This in assense unexpected behaviour of quantum mechanical particles leads to the threshold for KdV solitons to be formed of the zero area pulses decay.

Table I

U	a	1	6	11	16	21
1	N	1	3	7	9	9
5	N	1	9	13	19	21
10	N	2	10	17	25	28

Table II

U	a	0.1	0.2	0.5	0.6	0.8
1	N	0	0($\Delta<10$) 1($\Delta\geq10$)	0($\Delta=0$) 1($\Delta>0$)	0($\Delta=0$) 1($\Delta>0$)	1
	a	0.1	0.2	0.5	0.6	1.0
2	N	0	0($\Delta=0$) 1($\Delta>0$)	0($\Delta=0$) 1($\Delta>0$)	1	1
	a	0.1	0.2	0.3	0.5	1.0
5	N	0($\Delta=0$) 1($\Delta>0$)	0($\Delta=0$) 1($\Delta>0$)	1	1	1

3. Numerical simulations of the decay of zero area pulses

From the condition for the discrete energy levels to exist, obtained in the previous section, it follows that the evolution of the initial zero area pulses in the framework of the KdV equation has the threshold behaviour. This means that there is a width a and amplitude u_0 **area of the initial pulses where** decay results in: 1) the formations of N soliton lattice, with N being the number of the descrete energy levels in the potential (2.2); 2) a nonsoliton tail production only (i.e. the soliton formation is forbidden).

In this section we present the numerical simulation results on evolution of the following zero area pulses:

1) the rectangular pulse

$$u(x,0)=\begin{cases} 0\,, & x<-a\,,\\ -u_0\,, & -a\leqslant x\leqslant 0\,,\\ +u_0\,, & 0<x\leqslant a\,,\\ 0\,, & x>a\,,\end{cases} \qquad (3.1)$$

2) the sinusoidal pulse

$$u(x,0)=\tilde{u}_0 \sin kx\,, \qquad (3.2)$$

3) the Gaussian pulse (1.2)

The results for pulses (3.1) are presented in Table III and Figure 1.

The analysis shows that for the fixed amplitude $u_0=1$ there is a critical magnitude a_0 under which the soliton formation is impossible. For example, at $a<a_c\sim 0.9$ one observes producing only an oscillating tail. On the contrary, the number of produced solitons increases with a and we have a soliton lattice. Thus in the evolution of the pulse (3.1) with $u_0=1$ and $a=6$ its positive part produces three solitons and the negative one is transformed into an oscillating tail. For $u_0=1$ and $a=11$ we observe the production of seven solitons. The Table III shows the comparison between the amplitudes A_n $(n=1,2,\ldots)$ of the produced solitons and the energies of discrete levels obtained from the solution of transcendent

Table III

a=6	An	1.69	0.81	0.15				
	$2E_n$	1.66	1.38	0.72				
a=11	An	1.88	1.57	1.0	0.31	0.25	0.10	0.04
	2En	1.88	1.82	1.5	1.3	0.92	0.4	0.2

equation (2.9) (remind that the energy E_n of the discrete level defines unambiguously the corresponding soliton amplitude A_n : $A_n = 2E_n$).

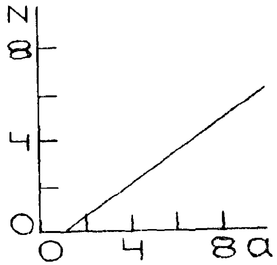


Fig. 1. The number of produced solitons as a function of the half-width of initial pulses (3.1).

In Figure 1 the number of solitons produced in the decay of pulses (3.1) is presented.

The number of produced solitons coincides with that obtained in Section 2. The discrepance in the amplitude values (obtained numerically and analytically) is due to a big nonsoliton tail created in the zero area pulse decay.

To study the decay of the harmonical (3.2) and Gaussian pulses (1.2) we get the condition for the threshold of soliton formation to exist by passing to the limit $E \to 0$.

From Eqs. (2.4) and (2.8) we get the following transcendent equation for discrete energy levels

$$K_2 a = \pi n - \sin^{-1}\left(\frac{K_2}{U_0^{1/2}}\right) - \sin^{-1}\left[\frac{1}{(1+f_0^2)^{1/2}}\right], \qquad (3.3)$$

where $n = 1, 2, \ldots$

It is easy to show that $f(E)$ in Eq. (2.8b) is a monotonic negative function so that $\mathrm{ctg}\,\delta_2 < 0$ and $f_0 = \lim_{E \to 0} f(E) = -1/U_0^{1/2}(b-a)$.

Since the values of $\sin^{-1}$ argument are not greater than 1, values of K_2 can lie only in the interval $0 \leq K_2 \leq U_0^{1/2}$. Hence the left hand side of Eq. (3.3) monotonically increases and the right hand side monotonically decreases. Therefore for

the existence of solutions of Eq. (3.3) it is necessary that at $K_2 = U_0^{1/2}$ the right-hand side magnitude should be less than of the left-hand side. So for $E \approx 0$, (i.e. in the transition region from a discrete spectrum to a continium one) we have the following inequality

$$a U_0^{1/2} \geq \frac{\pi}{2} - \sin^{-1}\frac{U_0^{1/2}(b-a)}{\left[1+U_0(b-a)^2\right]^{1/2}} . \qquad (3.4)$$

The last condition corresponds to the existence of, at least, one descrete level.

By introducing the variable $z = a U_0^{1/2}$ the inequality (3.4) is rewritten as

$$\frac{z(b-a)}{\left[a^2 + z^2(b-a)^2\right]^{1/2}} \geq \cos z . \qquad (3.5)$$

This condition will be used below as a restriction criterion for the soliton formation in the decay of pulses which are approximated by Eq. (3.1).

First let us consider the sinusoidal pulse (3.2).

We assume the well area and its approximation area to be equal:

$$S_1 = a U_0 , \quad S_2 = 2\tilde{U}_0 \int_0^{\pi/2k} \sin kx \, dx = \frac{2\tilde{U}_0}{k} ; \quad S_1 = S_2 .$$

Hence taking into account that $b = \pi/k$ we get $a = z^2 k/\tilde{U}_0$. Substituting a and b into Eq. (3.5) we have

$$\frac{\alpha^2 - z^2}{\left[z^2 + (\alpha^2 - z^2)^2\right]^{1/2}} \geq \cos z , \qquad (3.6)$$

where $\alpha^2 = 2\pi \tilde{U}_0/k^2$. It is easy to see that when $\alpha \geq \frac{\pi}{2}$ condition (3.6) is satisfied and for $\alpha < 1$ it is not satisfied for any z . Hence we have that when $\tilde{U}_0 \geq 0.392\ k^2$ at least one soliton is always produced and at $\tilde{U}_0 < 0.159\ k^2$ the soliton formation is impossible. In Figure 2 these two regions (I and III) are presented respectively. For region II our consideration cannot give an answer whether solitons are formed or not. In Figure 2 the computer simulation results are marked by crosses and dots and the first are related to the soliton formation processes, the second to their absence.

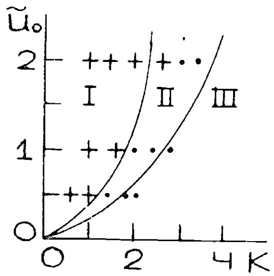


Fig. 2. Allowed (I) and forbidden (III) regions of the soliton generation from pulses of type (3.2).

Now we apply the condition (3.5) to the Gaussian pulses (1.2). The equality of areas $S_1 = a U_0 = S_2 = \tilde{\tilde{U}}_0 \int_0^\infty x e^{-K^2 x^2} dx = \frac{2\tilde{\tilde{U}}_0}{K^2}$ gives $a = 2 z^2 K^2 / \tilde{\tilde{U}}_0$. By dividing of the area $S = S_c + S_d$ into two parts with $c = a\left(1 - \frac{1}{e^{1/2}}\right)$, $d = \frac{a}{e^{1/2}}$, $a = c + d$, we have

$$b - a = \frac{\sqrt{2}}{K} - \frac{4 z^2 K^2}{\tilde{\tilde{U}}_0}\left(1 - \frac{1}{e^{1/2}}\right). \qquad (3.7)$$

Substituting Eq. (3.7) into the inequality (3.4) we get

$$\frac{\alpha^2 - z^2}{\left[\gamma^2 z^2 + (\alpha^2 - z^2)^2\right]^{1/2}} \geqslant \cos z, \qquad (3.8)$$

where

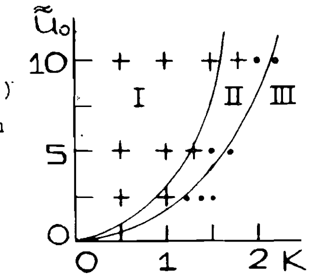
$$\gamma = 2\left(1 - \frac{1}{e^{1/2}}\right), \quad \alpha^2 = \frac{\tilde{\tilde{U}}_0}{2\sqrt{2}\left(1 - \frac{1}{e^{1/2}}\right)K^3}.$$

Introducing the variables $y \equiv z/\gamma$, $\tilde{\alpha} \equiv \alpha/\gamma$ the inequality (3.8) can be written as

$$\frac{\tilde{\alpha}^2 - y^2}{\left[y^2 + (\tilde{\alpha}^2 - y^2)^2\right]^{1/2}} \geqslant \cos \gamma y. \qquad (3.9)$$

It follows from Eq. (3.9) that when $\tilde{\tilde{U}}_0 \geqslant 2.8\ K^3$ the soliton formation is possible and impossible for $\tilde{\tilde{U}}_0 < 0.9\ K^3$. In Figure 3 these two regions (I and III) are presented correspondingly. Crosses and dots mean the numerical simulation results.

Fig. 3. Allowed (I) and forbidden (III) regions of the soliton generation from pulses of type (1.2).



Conclusion

For the KdV equation the problem of the soliton formation in the decay of zero area pulses is investigated both analytically and numerically. Using the linear representation for the KdV equation the problem is reduced to finding the discrete energy levels of the Schrödinger operator with the potential (2.2). The numerical analysis of the transcendent eigenvalue equation is carried out and the region for discrete energy levels to exist is found.

The analytical and numerical results are in a good agreement on the threshold generation of solitons in the decay zero area pulses processes.

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Пороговый характер образования солитонов уравнения Кортевега-де Вриза

Для уравнения Кортевега-де Вриза исследован распад импульсов нулевых площадей на солитоны. При использовании линейного представления уравнения Кортевега-де Вриза задача сведена к нахождению дискретных уровней энергии оператора Шредингера. Получено трансцендентное уравнение на собственные значения и проведен его численный анализ. Найдены области существования дискретных уровней энергий, соответствующих в рамках уравнения Кортевега-де Вриза решетке солитонов. Установлен порог для существования одного дискретного уровня. Проведено сравнение результатов численного моделирования уравнения Кортевега-де Вриза по распаду импульсов нулевой площади с дискретными уровнями энергий оператора Шредингера.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Threshold of KdV Soliton Production

Decay of the zero area pulses is investigated for the Korteweg-de Vries equation. By using a linear representation of the Korteweg-de Vries equation the problem is reduced to solution of a simple quantum-mechanical problem. The transcendent equation for eigenvalues is obtained and its numerical analysis is carried out. The existence region of the discrete energy levels, corresponding to a soliton lattice of the Korteweg-de Vries equation is found. The threshold existence of discrete energy level in the zero area potential is established. The numerical simulation results on the decay of zero area pulses are compared with the existence region which is predicted for the discrete energy levels.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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