ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА



E5-87-710

B.A.Dubrovin[1], I.M.Krichever[2], T.G.Malanyuk[1],
V.G.Makhankov

# EXACT SOLUTIONS TO A TIME DEPENDENT SCHRÖDINGER EQUATION WITH SELFCONSISTENT POTENTIAL

Submitted to "ЭЧАЯ"

[1] Moscow State University
[2] Energetical Institute, Moscow

1987

## Introduction

A unified approach for describing integrable models associated with the non-stationary Schrödinger equation along with constructing their n-soliton formulae is given. Among such models there are, e.g., a vector version of NLS with various internal symmetry groups, an analogous extension for the Yajima-Oikawa model and so on. "Integrability" of some of these systems follows as a rule from the existence for them of commutation representations(i.e.associated linear problems based on the L-A pair or L-A-B triad). Now it is however known that for the models with noncompact symmetries where condensate boundary conditions are of physical sense the conventional inverse spectral technique is nonconstructive.

Our approach does not use commutation relations and arises in fact in the depths of the algebrogeometrical theory of integrable systems. This theory is known to be used for constructing periodical and quasiperiodical solutions to such systems but what is less known the algebrogeometrical technique also allows us to obtain quite effectively all the known up to now their exact solutions (many-soliton and rational formulae and their combinations). We are going to show this in the form accessible for nonmathematicians taking as an example the models described by the Schrödinger equation with self-consistent potentials. The paper is organized as follows. In the first chapter we discuss the way how such models occur in physics, namely we consider a generalized version of the Heisenberg ferromagnet. The method for constructing and studying exact solutions for such models is given in the second chapter. Some specific examples are investigated in the third chapter and corresponding formulae are presented. In conclusion we discuss the results obtained.

## Chapter I

### Physical models related to NLS with self-consistent potentials

Systems of differential equations for a number of interacting waves and wave-packets occur frequently when one considers non-linear wave phenomena in various physical systems. The scalar NLS (SNLS)

$$i\Psi_t + \Psi_{xx} + \varepsilon|\Psi|^2\Psi = 0 \quad , \quad (\varepsilon = \pm 1) \qquad (1)$$

is the simplest mathematical model of this kind describing selfinteraction of the high-frequency wave packet (at $\varepsilon = -1$ it is the so-called Gross-Pitaevskii equation) in particular, the self-interaction of spin-waves (magnous) in ferromagnets, excitons in molecular crystals, Langmuir waves in plasmas and so on. Eq. (1) is now the most popular and studied (along with KdV) nonlinear model of mathematical physics integrable on both the classical and quantum levels. Moreover quantum (or quasiclassical) approach allows us to use the particle physics language in addition to the wave one. The simplest physical model described by (1) is a Bose-gas with point-like pair interaction of particles at zero temperature (see, e.g.[14]). This model gives us a visual picture of the results obtained which up to the definitions and redenotations may be used for other physical models.

A natural generalization of (1) is the system describing interaction of a h.f. wave packet $\Psi(x,t)$ with a l.f. wave $U(x,t)$. In this case the complex function $\Psi(x,t)$ obeys as above SNLS

$$i\Psi_t + \Psi_{xx} + U\Psi + \lambda|\Psi|^2\Psi = 0, \qquad (2)$$

which contains a potential $U$ - l.f. wave - described with one of the following equations (self-consistency)

$$\Box U = -|\Psi|^2_{xx} \qquad \text{(Zakharov[16])} \quad (3a)$$

$$(\partial_t + \partial_x)U = |\Psi|^2_x \qquad \text{(Yajima-Oikawa[3])} \quad (3b)$$

$$(\partial_t + \partial_x + \alpha\partial_x^3 + \beta U\partial_x)U = |\Psi|^2_x \qquad \text{(Nishikawa et al.[4])} \quad (3c)$$

$$(\Box + \alpha\partial_x^4)U + \beta\partial_x^2 U^2 = -|\Psi|^2_{xx} \qquad \text{(Makhankov[5])} \quad (3d)$$

Systems (2), (3) at $\lambda$ =0 occured in plasma physics where they described the interaction of Langmuir and ion-sound waves. Analogous equations were later shown to appear when one investigates spin waves interacting with phonons in ferromagnets[16] and excitons interacting with phonons in molecular crystals [7], here, however, $\lambda \neq 0$ in the general case.

Another natural generalization of (1) is a transition from the scalar version of NLS to a vector one: $\Psi \longrightarrow \Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)^{Tr}$ with simultaneous changing $|\Psi|^2$ by the inner product

$$(\Psi,\Psi) \overset{def}{=} \sum_{i,j=1}^{n} g_{ij}\bar{\Psi}_i\Psi_j \quad , \qquad (4)$$

where $g_{ij}$ is the metric of the isotopic space.



The Hamiltonian of the system is often invariant under transformations of an internal symmetry group compact or not depending on the signature of the matrix $g_{ij} = \lambda_i \delta_{ij}$. For Hermitian Hamiltonians this group is U(p,q). Such models describe a Bose-gas with internal quasispin ("coloured") degrees of freedom, they also appear in describing the propagation of plane circular polarized h.f. wave in plasmas[1]. Spin waves in ferromagnets with a multi-layered structure are related to a classical continuum analogue of the Hubbard model and so on. A part of these models are integrable[2] and may be studied completely. Finally, combining both the generalizations we come to vector versions of NLS with a self-consistent potential (l.f. mode) of one of the above forms (3)(although others are also possible) i.e.

$$i\Psi_t + \Psi_{xx} + U\Psi + \lambda(\Psi,\Psi)\Psi = 0 \qquad (5)$$

plus one of Eq.(3) in which the right-hand side contains the invariant form $(\Psi,\Psi)$.

For all of these models there is an interpretation in terms of multi-component Bose-gas (with internal degrees of freedom) language and the interaction between the gas particles may be various including the phonon mode. In other words (5) and (3) describe a mixture of gases with the attractive or repulsive interaction between particles if $\lambda \neq 0$ and

$$g_{ik} = \mathrm{diag}(1,1,\ldots,-1,-1,\ldots)^{i,k=\overline{1,n}} \qquad (6)$$

The particles may also emit or absorb phonon waves and thereby interact. That is why we shall name models (5) the Bose-gas models abstracting from the statement and results interpretation of concrete physical problem.

At the same time since nowadays namely in condensed matter physics there appear and are studied models of type (5) the behaviour of such systems is of a growing interest in this area. Most of the crystals, as experimental studies show, have layered or multi-chain structures[8]. More than that for the majority of them the interlayer or interchain interactions have a considerable effect on the general dynamics behaviour of crystals. Typical representatives of such systems are the salt crystals[8], however, analogous structures may also be seen in organic compounds in the form of molecular chains[9]. Theoretical description of multilayer structures is based on the many component generalization of the Heisenberg spin model[10]. The introduction of "colour" degrees of freedom for interacting spins in one-dimensional chains may also describe many layer quasi-two-dimensional magnetic systems with weak coupling. It is also well known[11] that the one-dimensional Hubbard model with a half-filled zone corresponds to the two-component Heisenberg spin chain with nontrivial intercomponent interactions. Many component spin chain which corresponds, consequently, to some generalized Hubbard model may be used for describing collective excitations (and also their statistical properties) in the system with different sorts of spins[12].

In all the above cases we come to Bose-gas models (5) which give a dynamical description of the corresponding system, strictly speaking, at zero temperature. Even in the very low temperature region some averaged characteristics are only measured experimentally such as static or dynamical structure factors. To calculate these theoretically the partition function given by the Feinmann integral ( $Z=\int D\varphi_t D\varphi \exp(-\beta H)$ ), $\beta = T^{-1}$ for real fields) is usually used. For the Bose-gas models (5) such an approach faced the difficulties[14] and the so-called phenomenological approach is widely (often) applied after the work by Krumhausl and Schrieffer[13]. They noticed that the partition functions obtained via the transfer matrix technique as well as in the ideal gas of kinks approximation are practically the same for the $\varphi^4$ model. Later on the phenomenological approach is used to calculate the structural factors for various models (see the review [14] and the references cited therein). Notice that the stability of solitons and the fact that they interact ellastically (or quasi ellastically) justifys mostly the way this approach may be used. Such a soliton behaviour used to be in the framework of integrable models with a sufficiently small number of interacting fields (waves). If it is not a case, the distribution function of solitons in velocities and amplitudes (frequencies) have to be found via some other kind of theory (for example in ref.[15] an approximate kinetic equation was written and solved for solitons in system (2)(3a) with $\lambda$ =0 on the grounds of computer experiments).

For the integrable systems of type (5) (with n>1) it is thus very important to know in an analytical form not only the whole spectrum of one-soliton solutions but also two-soliton and sometimes three-soliton formulae (especially their asymptotics) for the phenomenological approach may be applied.



## 1. Generalized Heisenberg chains and Bose-gas models

Consider a "colour" generalization of a spin chain with the Hamiltonian[17].

$$H = H_S + H_L \tag{7}$$

with

$$H_S = -\frac{1}{2}\Big\{\sum_{i,j,\alpha,\beta}\Big[\frac{1}{2}J_{ij}^{\alpha\beta}\big(S_i^{+\alpha}S_j^{-\beta}+S_i^{-\alpha}S_j^{+\beta}\big)+R_{ij}^{\alpha\beta}S_i^{z\alpha}S_j^{z\beta}\Big]\Big\} \tag{7a}$$

$$H_L = T+U_0,\quad T=\frac{m}{2}\sum_j \dot{x}_j^2,\quad U_0=\frac{mv_0^2}{2a_0^2}\sum_j (x_{j+1}-x_j-a_0)^2 \tag{7b}$$

describing the interaction of several "color" (types) spins ($\alpha=\overline{1,n}$). Neglecting the color-space interaction in the nearest neighbour approximation we have

$$J_{ij}^{\alpha\beta} = J_{jj+\sigma}\cdot K^{\alpha\beta}\ ,\quad R_{jj+\sigma}^{\alpha\beta}L_1^{\alpha}L_2^{\beta}\ , \tag{8}$$

where $J_{jj+\sigma}\equiv J(|x_j-x_{j+\sigma}|)$ is the exchange integral of the nearest neighbour spins, $S^{\pm}=S^x\pm iS^y$ and $S^z$ are the spin operators.

When $S^{\alpha}$ is sufficiently large, Hamiltonian (7) may be rewritten in terms of annihilation $a_j^{+\alpha}$ and creation $a_j^{\alpha}$ Bose operators via the generalized Holstein-Primakoff transformations

$$S_j^{+\alpha}=\sqrt{2s^{\alpha}}\Big(1-\frac{n_j^{\alpha}}{2s^{\alpha}}\Big)^{1/2}a_j^{\alpha}\ ;\quad S_j^{-\alpha}=\sqrt{2s^{\alpha}}\,a_j^{+\alpha}\Big(1-\frac{n_j^{\alpha}}{2s^{\alpha}}\Big)^{1/2}$$

$$n_j^{\alpha}=a_j^{+\alpha}a_j^{\alpha}:$$

$$H_S = \mathrm{const}-\frac{1}{2}\sum_{j,\sigma}\Big\{sJ_{jj+\sigma}\sum_{\alpha\beta}K^{\alpha\beta}\big(a_j^{+\alpha}a_{j+\sigma}^{\beta}+a_{j+\sigma}^{+\beta}a_j^{\alpha}\big)-$$
$$-\tilde{J}_{jj+\sigma}\Big[s\sum_{\alpha}\big(\ell_2L_1^{\alpha}n_j^{\alpha}+\ell_1L_2^{\alpha}n_{j+\sigma}^{\alpha}\big)+$$
$$+\sum_{\alpha\beta}L_1^{\alpha}L_2^{\beta}n_j^{\alpha}n_{j+\sigma}^{\beta}\Big]\Big\} \tag{9}$$

where $\ell_i=\mathrm{Tr}\,L_i\ ,\ s^{\alpha}\equiv s$ .

Evolution of the operator $a_j^{\alpha}(t)$ is given by the Heisenberg equation $i\hbar\,\dot{a}_j^{\alpha}(t)=[a_j^{\alpha},H_S]$ . To get a classical analog of the quantum Hamiltonian (9) we use the reduction procedure based on the coherent states of the Heisenberg-Weil group[17]

$$|\varphi^\alpha\rangle = \prod_j |\varphi_j^\alpha\rangle = \prod_j e^{-\frac{1}{2}|\varphi_j^\alpha|^2} e^{\varphi_j^\alpha a_j^{+\alpha}}|0\rangle .$$

These have the following important property: given an operator in the normal form

$$\hat A = \sum_{m,n} C_{mn}(a_j^{+\alpha})^m (a_j^\alpha)^n,$$

then

$$A \equiv \langle\varphi_j^\alpha|\hat A|\varphi_j^\alpha\rangle = \sum_{m,n} C_{mn}(\bar\varphi_j^\alpha)^m(\varphi_j^\alpha)^n. \qquad (10)$$

We employ this relation and go to the continuum limit by means of the well-known procedure: $\varphi^\alpha(\xi) = \varphi_j^\alpha$ and $\varphi^\alpha_{j+1} = \varphi^\alpha(\xi) + a_0\varphi^\alpha_\xi(\xi) + \frac{1}{2}a_0^2\varphi^\alpha_{\xi\xi}(\xi)+\dots$ . Representing the exchange integral as $J(|x_{j+1}-x_j|) = J_0 - J_1(x_{j+1}-x_j-a_0)$ (the same for $\tilde J$ ) we get the system

$$\ddot x = v_0^2 x_{\xi\xi} + \frac{s}{m}\sum_{\alpha\beta}\tilde T^{\alpha\beta}(\bar\varphi^\alpha\varphi^\beta)_\xi , \qquad (11)$$

$$i\varphi^\alpha_t = -b\sum_\beta \left(K_{(\alpha,\beta)}\varphi^\beta_{\xi\xi} - s\,T_{\alpha\beta}\varphi^\beta + s\,\tilde T_{\alpha\beta}\varphi^\beta x_\xi\right) -$$

$$- J_0\varphi^\alpha\sum_\beta\left(L_1^\beta L_2^\alpha + L_2^\alpha L_1^\beta\right)|\varphi^\beta|^2 , \qquad (12)$$

where

$$T_{\alpha\beta} = J_0 K_{(\alpha,\beta)} - \tilde J_0(\ell_1 L_2^\alpha + \ell_2 L_1^\alpha)\delta_{\alpha\beta}$$
$$\tilde T_{\alpha\beta} = J_1 K_{(\alpha,\beta)} - \tilde J_1(\ell_1 L_2^\alpha + \ell_2 L_1^\alpha)\delta_{\alpha\beta} \qquad b = J_0 s/2 ,$$

and ( $\alpha,\beta$ ) implies the symmetrization with respect to $\alpha$ and $\beta$ indices. Further investigation of system (11), (12) depends on the constraints we put on the coefficient matrices T and L.

## 2. Some particular reductions

Example I. Let the exchange integrals of colour degrees of freedom be diagonal and proportional to each other

$$K_{(\alpha,\beta)} = 2b_1 L_1^\beta\delta_{\alpha\beta} = 2b_2 L_2^\beta\delta_{\alpha\beta} \equiv \lambda_\alpha\delta_{\alpha\beta} ,$$

then the system (11), (12) is reduced to the system of the form (5), (3a). In a quasisteady (inertionless) limit when the second time derivative in (11) may be dropped one has

$$v(\xi,t) \equiv x_\xi = -\frac{s}{m v_0^2}\sum_{\alpha\beta}\tilde T_{\alpha\beta}(\bar\varphi^\alpha\varphi^\beta) + C \qquad (13)$$



and (12) assumes the VNLS form given by the Hamiltonian

$$H = \int d\xi\left[b(\varphi^+_\xi K\varphi_\xi) - d(\varphi^+K\varphi)^2 - \tilde\mu(\varphi^+K\varphi)\right], \qquad (14)$$

in which

$$(\varphi^+K\varphi) \equiv \sum_{\alpha\beta}\bar\varphi^\alpha K_{(\alpha,\beta)}\varphi^\beta = \sum_\alpha \lambda^\alpha|\varphi^\alpha|^2 ,$$

$$d = \frac{s^2\nu^2}{m v_0^2} + \frac{\tilde J_0}{2b_1b_2} , \quad \tilde\mu = s(\mu - c\nu),$$

$$T_{\alpha\beta} = \mu\lambda^\alpha\delta_{\alpha\beta} , \quad \tilde T_{\alpha\beta} = \nu\lambda^\alpha\delta_{\alpha\beta} ,$$

$$\mu = \left[J_0 - \frac{\tilde J_0}{2b_1b_2}(b_1\ell_1+b_2\ell_2)\right], \quad \nu = \left[J_1 - \frac{\tilde J_1}{2b_1b_2}(b_1\ell_1+b_2\ell_2)\right].$$

Functions $\varphi^\alpha(\xi,t)$ and $\bar\varphi^\alpha(\xi,t)$ are the conjugate variables

$$\{\varphi^\alpha(x),\bar\varphi^\beta(y)\} = i\delta^{\alpha\beta}\delta(x-y)$$

with the conventional Poisson brackets.

$$\{A,B\} = i\sum_{\alpha=1}^{n}\int d\xi\left(\frac{\delta A}{\delta\varphi^\alpha}\frac{\delta B}{\delta\bar\varphi^\alpha} - \frac{\delta B}{\delta\varphi^\alpha}\frac{\delta A}{\delta\bar\varphi^\alpha}\right).$$

Hamiltonian (14) is simplified when it is invariant under internal symmetry group transformations. Then we have $\lambda^\alpha = \varepsilon_\alpha$ :

$$\varepsilon_\alpha = \begin{cases}1 , & \alpha = 1,\dots,p\\ -1 , & \alpha = p+1,\dots,p+q\end{cases} \qquad (p+q=n) .$$

Denoting

$$\Psi_\alpha(x,t) = \begin{cases}\varphi^\alpha(\xi,t) , & \alpha = 1,\dots,p\\ \varepsilon_\alpha\varphi^\alpha(\xi,t) , & \alpha = p+1,\dots,n\end{cases}$$

$$(\Gamma_0)_{\alpha\beta} \equiv K_{(\alpha,\beta)} = \varepsilon_\alpha\delta_{\alpha\beta} , \quad \frac{d}{b} = \varkappa , \quad \frac{\mu'}{b} = \rho , \quad H \to H/b ,$$

we get

$$H = \int d\xi\left[(\Psi_\xi,\Psi) - \varkappa(\Psi,\Psi)^2 - \rho(\Psi,\Psi)\right]$$

$$\{\Psi^\alpha(\xi)\overset{*}{\Psi}{}^\beta(\eta)\} = i\delta^{\alpha\beta}\delta(\xi-\eta),$$

where

$$\overset{*}{\Psi} = \Psi^+\Gamma_0 , \quad \Gamma_0 = \begin{pmatrix}I_p & 0\\ 0 & -I_p\end{pmatrix}$$

and

$$(\Psi,\Psi) = \sum_{\alpha=1}^{p}|\Psi^\alpha|^2 - \sum_{\alpha=p+1}^{n}|\Psi^\alpha|^2 \equiv (\Psi^+\Gamma_0\Psi)$$

is a $U(p,q)$ internal product. Equations of motion

$$i\Psi_t + \Psi_{\xi\xi} + 2\varkappa(\Psi,\Psi)\Psi + \rho\Psi = 0 \qquad (15)$$

is now the $U(p,q)$ VNLS, the integrable system/2/.

The same reduction applied to system (11), (12) gives system (5)+(3a) that is in dimensionless variables

$$\partial_t^2 U - \partial_\xi^2 U - (\Psi,\Psi)_{xx} = 0 \;, \qquad (16a)$$

$$i\Psi_t + \Psi_{\xi\xi} - U\Psi + \lambda(\Psi,\Psi)\Psi = 0 \;. \qquad (16b)$$

It is worth to note that the last term in eq. (16b) is generated by the term $(S^z)^2$ of an initial spin Hamiltonian written in Holstein-Primakoff representation. It is proportional to an anisotrophy magnitude (relation $\tilde{J}_0/J_0$) and survives when the magnon-phonon interaction vanishes. Generalized Yajima-Oikawa system may be obtained from(16) via the conventional procedure of getting a unidirectional wave equation

$$\partial_t^2 - \partial_\xi^2 \simeq -2\partial_\xi(\partial_t + \partial_\xi) \qquad (17)$$

and integrating over $\xi$ .

Example 2. For taking into account a weak interaction between "colour" components in a chain dropped above suppose the nearest neibour interaction to be prominent also in the colour space. Then we have

$$R_{ij}^{\alpha\beta} = \rho J_{ij}^{\alpha\beta}, \quad J_{ij}^{\alpha\beta} = (J_{jj+\sigma}M^{\alpha\beta} + J^1 V_{ij}^{\alpha\beta}) \qquad (18)$$
$$M^{\alpha\beta} = \delta^{\alpha\beta} + \varepsilon\delta^{\beta,\alpha+\delta}, \quad V_{ij}^{\alpha\beta} = \delta_{ij}\delta^{\beta,\alpha+\delta} \;,$$

where $J^1/J \ll 1$, and J, $J^1$ are the intercell and interchain exchange integrals respectively. Making use of (23) one can get equations of type (5) and (8) with small terms allowing for the fact that the intercolour interaction matrix is nondiagonal. The effect of these on the system dynamics may be investigated by means of the "standard" soliton perturbation theory, otherwise by direct methods or by methods using the inverse transform/18/.

The above procedure when applied to the Hubbard model (more precise to its multichain spin analog) gives also under definite assumptions systems of type (5)+(3) now with the $U(\frac{n}{2},\frac{n}{2})$ internal symmetry group for an antiferromagnetic ground state and $U(n,0)$ for a ferromagnetic one (see /17/).



Example 3. Allowance for anharmonizm in the Hamiltonian $H_L$

$$U_{ung} = \frac{U_{III}}{3!}\sum_j (x_{j+1} - x_j - a_0)^3$$

and the phonon dispersion

$$x_{j\pm 1} = x \pm a_0 x_\xi + \frac{1}{2}a_0^2 x_{\xi\xi} \pm \frac{1}{6}a_0^3 x_{\xi\xi\xi} + \frac{1}{4!}a_0^4 x_{\xi\xi\xi\xi} + \ldots$$

changes wave equation (11) by an inhomogeneous Boussinesq equation

$$\partial_t^2 x - \partial_\xi^2(v_0^2 - \alpha\partial_\xi^2 - \beta x)x = g\partial_\xi^2(\Psi,\Psi) \qquad (19)$$

with $\alpha,\beta$ and $g$ being defined by the initial system parameters. Scale transformations of $\xi$,t,x and $\Psi$ give rise to system (5)+(3d). The unidirectional version of (19) is given by (5)+(3c).

We have considered a multicomponent spin system and found that under some assumptions (the long wave approximation and so on) it may be reduced to field models with internal ("colour") symmetries. A part of these turns out to be integrable, among them there are $U(p,q)$ VNLS (obtained in the quasistatic limit), the colour generalization of the Yajima-Oikawa system (obtained in the near-sound limit), finally system (5)+(3d) at $\lambda$ =0. Other nonintegrable versions may be often considered as nearly integrable systems. All the above equations have in addition to linear (phonon and magnon) solutions also essentially nonlinear (soliton) ones. The properties of these nonlinear solutions (solitons) we discuss in what follows. Just these solutions along with linear modes describe elementary excitations of the corresponding systems at low temperatures/20/.

In conclusion we notice that the models considered occur in many branches of physics, in particular, a part of them arises apparantly for the first time in plasma physics (see e.g. reviews/21/).

## Chapter II

### The general scheme of the method

**In this chapter we shall describe the method of the simultaneous construction of the integrable models which are associated with nonstationary Schrödinger equation and their exact soliton-like solutions.**

This method is a particular case of the general algebro-geometrical(or finite-gap) scheme, but its description can be presented in the closed form without using the results of the algebraic geometry. The authors assume that the "algebro-geometrical" approach to the construction of the multisoliton solutions is one of the most simple and elementary methods which can be used even in situations with no complete solutions of the direct and inverse spectral problems of the auxiliary linear problems.

It should be noted that our way of constructing the solutions of the non-stationary Schrödinger equation with the self-consistent conditions differs from the standard inverse transform method. All these equations have the Lax representations or the so-called L,A,B-representations. The corresponding auxiliary linear problems are notably different in each cases. In our construction the solutions of all these equations would be obtained in one general scheme using only one linear operator

$$L = i\partial_y - \partial_x^2 + u(x,y)$$

but not a few operators as in the inverse transform method. It is noteworthy that L is not only an auxiliary operator, but enters the initial systems of the equations.

The similar approach to the constructions of the finite-gap solutions of the non-linear Schrödinger equation and its vector generalizations has been first used in /22/. The periodic and quasi-periodic solutions of the equations with other self-consistent conditions have been constructed in paper /23/ which has stimulated our work.

## 1. The construction of the "integrable" potentials of the non-stationary Schrödinger equation associated with the rational algebraic curve.

The potential u(x,t) of the non-stationary Schrödinger equation would be called the "integrable" potential, associated with the rational algebraic curve if the equation

$$[i\partial_t - \partial_x^2 + u(x,t)]\Psi(x,t,\kappa) = 0 \tag{2.1}$$



has the solution of the form

$$\Psi(x,t,\kappa) = Q_N(x,t,\kappa)\, e^{i\kappa x + i\kappa^2 t}, \tag{2.2}$$

where

$$Q_N(x,t,\kappa) = \kappa^N + a_1(x,t)\kappa^{N-1} + \dots + a_N(x,t)$$

-is the polynomial of some degree N.

It is possible that the construction of such potentials which would be given below is not most general. But it contains as its particular cases, the multisoliton and rational solutions of the non-linear equations under consideration.

To begin with we shall construct the complex integrable potentials. Let's present the different complex of numbers $\varkappa_1,\dots,\varkappa_M$, $(\alpha_{ij}^s)$, where $i=1,\dots,N$; $j=1,\dots,M$; $s=0,\dots,m_j$ with $m_1+\dots+m_M+M \geq N$. They are free parameters of our construction. For any set of these parameters we shall uniquely determine the function $\Psi(x,t,\kappa)$ of the form (2.2) with the help of the following system of the linear conditions

$$\sum_{j=1}^{M}\sum_{s=0}^{m_j} \alpha_{ij}^s \partial_\kappa^s \Psi(x,t,\kappa)\Big|_{\kappa=\varkappa_j} = 0, \quad i=\overline{1,N}. \tag{2.3}$$

Conditions (2.3) are equivalent to the system of N linear equations on the coefficients $a_1,\dots,a_N$. Let's introduce the polynomials

$$P_{r,s}(x,t,\kappa) = e^{-i\kappa x - i\kappa^2 t}\partial_\kappa^s\left(\kappa^r e^{i\kappa x + i\kappa^2 t}\right) = e^{-i\kappa x - i\kappa^2 t}\left(\tfrac{1}{i}\partial_x\right)^r \partial_\kappa^s e^{i\kappa x + i\kappa^2 t} = (\partial_\kappa + ix + 2i\kappa t)^s \kappa^r, \tag{2.4}$$

and linear functions $\omega_j = \omega_j(x,t)$

$$\omega_j(x,t) = \varkappa_j x + \varkappa_j^2 t, \quad j=1,\dots,M. \tag{2.5}$$

Then the equation (2.3) can be written in the following form

$$\sum_{k=1}^{N} a_k(x,t)\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{ij}^s P_{N-k,s}(x,t,\varkappa_j)\, e^{i\omega_j} = -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{ij}^s P_{N,s}(x,t,\varkappa_j)\, e^{i\omega_j}, \quad i=\overline{1,N}. \tag{2.6}$$

Let's denote the (N×N) matrix of the coefficients at $a_k$ in the equations (2.6) by $A(x,t)=(A_{ik}(x,t))$ and denote the ((N+1)×(N+1)) matrix $\hat{A}(x,t,k)$

$$\hat{A}(\varkappa,t,k)=\begin{pmatrix} k^N & k^{N-1} \dots\dots 1 \\ -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{1j}^{s}P_{N,s}(x,t,\varkappa_j)e^{i\omega_j} & \\ \vdots & A(x,t) \\ -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{Nj}^{s}P_{N,s}(x,t,\varkappa_j)e^{i\omega_j} & \end{pmatrix} \qquad (27)$$

by $\hat{A}(x,t,k)$.

Theorem 1. Let the matrix $A(x,t)$ of the system be not identically (in $x,t$) singular. Then the function $\Psi(x,t,k)$ of the form (2.2) which is determined by conditions (2.3) satisfies equation (2.1) with its potential equal to

$$u(x,t)=2i\partial_x a_1(x,t)=2\partial_x^2 \ln\det A(x,t)\,. \qquad (2.8)$$

The proof of this theorem is standard in the theory of the finite-gap integration and can be obtained using only the form of $\Psi$ and conditions (2.3).

If we take $u(x,t)=2i\partial_x a_1(x,t)$ the substitution of (2.2) in (2.1) provides that the left side of this equality is the function $\tilde{\Psi}(x,t,k)$ of the form

$$\tilde{\Psi}(x,t,k)=(\tilde{a}_1(x,t)k^{N-1}+\dots+\tilde{a}_N(x,t))e^{ikx+ik^2t}, \qquad (2.9)$$

which is similar to (2.2), but has not the term $k^N$ in the pre-exponential factor. The conditions (2.3) are linear and do not depend on $x,t$, therefore for any linear operator $\Lambda=\Lambda(\partial_x,\partial_t)$ the function $\tilde{\Psi}(x,t,k)=\Lambda\Psi(x,t,k)$ satisfies conditions (2.3). Due to this $\tilde{a}_1,\dots,\tilde{a}_N$ satisfy the system of linear equations with the same coefficients as in the system for $a_1,\dots,a_N$. Unlike the latter system the system for $a_1,\dots,a_N$ is homogeneous. Consequently $\tilde{a}_1=\dots=0$, and the equality (2.1) is proved.

The proof of the second formula (2.8) follows from the Cramer's formula for the solution $a_1$ of the system (2.6) and from the evident relation

$$\frac{1}{i}\partial_x\left[P_{N-1,s}(x,t,k)e^{ikx+ik^2t}\right]=P_{N,s}(x,t,k)e^{ikx+ik^2t}$$

The theorem is proved now.

Remark. Below we shall assume that the rank of the matrix $(\alpha_{ij}^{s})$ is equal to N. This condition is necessary for non-singularity of the matrix $A(x,t)$. It must be mentioned also that the function $\Psi(x,t,k)$ would not change if we multiply the matrix $(\alpha_{ij}^{s})$ from the left by an arbitrary constant non-singular $(N\times N)$-matrix.



As it was mentioned before the potentials $u(x,t)$ corresponding to the arbitrary parameters $\varkappa_j,(\alpha_{ij}^{s})$ are the complex and meromorphic functions of $x,t$. Now we shall describe the restrictions which are sufficient for reality and regularity (for real $x,t$) of the corresponding potentials $u(x,t)$.

Here we shall consider the case $m_1=\dots=m_M=0$ only. (The example, in which the rational solutions of the scalar non-linear Schrödinger equation have been obtained with the help of our construction with $N=2$, $m_j\neq 0$, has been considered in /24/). In this case we shall assume that $M=2N$ and that the values $\varkappa_1,\dots,\varkappa_{2N}$ have non-zero imaginary parts and are subdivided into complex conjugated pairs

$$\varkappa_{N+i}=\bar{\varkappa}_i\ , \quad i=\overline{1,N}. \qquad (2.10)$$

We can assume without losing generality that the minor of the matrix $(\alpha_{ij})\equiv(\alpha_{ij}^{c})$ consisting of the columns with the numbers $j=N+1,\dots,2N$ is non-singular. As it follows from the previous remark the general case can be reduced to the case where this minor equals the unit matrix. In this case conditions (2.3) have the form

$$\Psi(\bar{\varkappa}_i)=-\sum_{j=1}^{N}\alpha_{ij}\Psi(\varkappa_j)\ , \quad i=\overline{1,N}\ , \qquad (2.11)$$

where $(\alpha_{ij})$ is the constant matrix of the dimension $N\times N$. It is convenient to renormalize the function $\Psi(x,t,k)$ in the following way

$$\Psi(x,t,k)=\frac{\Psi(x,t,k)}{(k-\varkappa_1)\cdots(k-\varkappa_N)}\equiv\left(1+\sum_{j=1}^{N}\frac{r_j(x,t)}{k-\varkappa_j}\right)e^{ikx+ik^2t} \qquad (2.12)$$

The conditions (2.11) for this renormalized function $\Psi$ will take the form

$$\Psi(x,t,\bar{\varkappa}_i)=-\sum_{j=1}^{N}c_{ij}\operatorname*{res}_{k=\varkappa_j}\Psi(x,t,k)\ , \qquad (2.13)$$

where the constant matrix$(c_{ij})$equals to

$$c_{ij}=\left[R(\bar{\varkappa}_i)\right]^{-1}\alpha_{ij}R'(\varkappa_j)\ , \quad i,j=\overline{1,N} \qquad (2.14)$$

$R(k)=(k-\varkappa_1)\cdots(k-\varkappa_N)$, and the prime denotes the derivative with respect to k.

Theorem 2. Let the parameters $\varkappa_1,\dots,\varkappa_N,(c_{ij})$ which determine the function $\Psi(x,t,k)$ of the form (2.12) with the help of the conditions (2.13) satisfy the following requirements:

a) the matrix $c_{ij}$ is skew-hermitian $c_{ij}=-\bar{c}_{ij}$ ;

b) if the points $\varkappa_1,\dots,\varkappa_N$ are numerated in such a way that

$\operatorname{Im} \varkappa_i > 0$, $i=1,\dots,p$ ; $\operatorname{Im} \varkappa_i < 0$, $i=p+1,\dots,N$,
then the hermitian matrix

$$\left( \frac{1}{i} c_{k\ell} \right), \quad 1 \le k,\ell \le p ,$$

must be positively defined and hermitian matrix

$$\left( \frac{1}{i} c_{k\ell} \right), \quad p+1 \le k,\ell \le N$$

must be negatively defined (these matrices can be non-negative as well ).

If those conditions on the parameters are fulfilled, the function $\Psi(x,t,k)$ is the smooth function of real $x,t$ for all $k \ne \varkappa_j$ and satisfies the equation (2.1) with a real smooth potential $u(x,t)$. For these functions we have

$$\Psi(x,t,k) = \frac{\det \hat{M}(x,t,k)}{\det M(x,t)} e^{ikx + ik^2 t}, \tag{2.15}$$

$$u(x,t) = 2\partial_x^2 \ln \det M(x,t) , \tag{2.16}$$

where

$$M_{ij}(x,t) = c_{ij} + \frac{e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j}, \quad \omega_i = \varkappa_i x + \varkappa_i^2 t , \quad i,j = \overline{1,N} , \tag{2.17}$$

$$\hat{M}_{ij} = M_{ij} : \ i,j = \overline{1,N}; \ \hat{M}_{00} = 1, \ \hat{M}_{i0} = e^{i\bar{\omega}_i}, \quad \hat{M}_{0i} = (k - \varkappa_i)^{-1} e^{-i\omega_i}, \quad i = \overline{1,N} . \tag{2.18}$$

The proof. Let's consider the function

$$\Omega(x,t,k) = \Psi(x,t,k)\overline{\Psi(x,t,\bar{k})} . \tag{2.19}$$

The residue of this function in $k=\infty$ equals ( $a_1(x,t) + \overline{a_1(x,t)}$). This function has simple poles in the points $k=\varkappa_i$, $k=\bar{\varkappa}_i$ with the residues

$$\operatorname*{res}_{\varkappa_i} \Omega(x,t,k) = \operatorname*{res}_{\varkappa_i} \Psi(x,t,k)\overline{\Psi(x,t,\bar{k})} = -\sum_{j=1}^{N} \bar{c}_{ij} \Psi_i \bar{\Psi}_j , \tag{2.20}$$

where

$$\Psi_i = \Psi_i(x,t) = \operatorname*{res}_{\varkappa_i} \Psi(x,t,k) = r_i(x,t) e^{i\omega_i}, \quad i = \overline{1,N} . \tag{2.21}$$



Similarly,

$$\operatorname*{res}_{\bar{\varkappa}_i} \Omega(x,t,k) = \Psi(x,t,\bar{\varkappa}_i) \operatorname*{res}_{\varkappa_i} \overline{\Psi(x,t,k)} = -\sum_{j=1}^{N} c_{ij} \Psi_j \bar{\Psi}_i . \tag{2.22}$$

The sum of the residues of $\Omega$ in all points $\varkappa_i$ is equal to zero because the matrix $C_{ij}$ is a skew-hermitian. Consequently, the residue $\Omega$ in the infinity is equal to zero, i.e. $\bar{a}_1 = -a_1$. The relation $u(x,t) = 2i\partial_x a_1$ provides that potential $u(x,t)$ is real.

The regularity of $u$ and $\Psi(x,t,k)$, $k \ne \varkappa_j$ is equivalent to the non-singularity of the matrix $M(x,t)$. This matrix is the matrix of the coefficients at $\Psi_j$ in the system (2.23) which is equivalent to the system (2.13). Let's prove that the system (2.13) has the unique solution for all the real $x,t$. This system can be re-written in the form

$$\sum_{j=1}^{N} \left( c_{ij} + \frac{e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j} \right) \Psi_j = - e^{i\bar{\omega}_i}, \quad i = \overline{1,N} , \tag{2.23}$$

where $\Psi_j$ is determined in (2.21).

The matrix of the coefficients of this system is degenerated if there exists the solution of the corresponding homogeneous system. The latter is equivalent to the existence of the non-zero function $\Psi'(x,t,k)$ of the form

$$\Psi'(x,t,k) = \sum_{j=1}^{N} \frac{\tilde{r}_j(x,t)}{k - \varkappa_j} e^{ikx + ik^2 t} , \tag{2.24}$$

satisfying the conditions (2.13). Let's show that this is impossible.

Consider the integral over the real axis

$$0 < \int_{-\infty}^{\infty} |\Psi'(x,t,k)|^2 dk = \int_{-\infty}^{\infty} \Omega'(x,t,k)\, dk = I , \tag{2.25}$$

where $\Omega'(x,t,k)$ is constructed from $\Psi'(x,t,k)$ with the help of the (2.19). This integral can be expressed in terms of the residues of $\Omega'$ in the upper half-plane. The residues $\Omega$ are given by formulae (2.20) and (2.22) where we must substitute $\Psi_i$ by $\Psi_i' = \operatorname*{res}_{\varkappa_i} \Psi'$. Because of this

$$\begin{aligned} I &= 2\pi i \left( \sum_{i=1}^{p} \sum_{j=1}^{N} c_{ji} \Psi_i' \bar{\Psi}_j' - \sum_{i=p+1}^{N} \sum_{j=1}^{N} c_{ij} \Psi_j' \bar{\Psi}_i' \right) = \\ &= 2\pi i \left( \sum_{i,j=1}^{p} c_{ji} \Psi_i' \bar{\Psi}_j' - \sum_{i,j=p+1}^{N} c_{ij} \Psi_j' \bar{\Psi}_i' + \right. \\ &\quad \left. + \sum_{i=1}^{p} \sum_{j=p+1}^{N} c_{ji} \Psi_i' \bar{\Psi}_j' - \sum_{i=p+1}^{N} \sum_{j=1}^{p} c_{ij} \Psi_j' \bar{\Psi}_i' \right) . \end{aligned} \tag{2.26}$$

From this we obtain

$$I = 2\pi i \left( \sum_{i,j=1}^{p} \overline{\Psi}_i' C_{ij} \Psi_j' - \sum_{i,j=p+1}^{N} \overline{\Psi}_i' C_{ij} \Psi_j' \right) \leq 0$$

using the condition b) of the theorem. This contradiction proves the regularity of $\Psi(x,t,k)$ and $u(x,t)$ for all real x,t. The formulae (2.15) and (2.16) can be obtained similarly to the formulae (2.7) and (2.8). The theorem is proved.

<u>Definition 1.</u> We shall call the integrable potential $u(x,t)$ which is given by our construction with N parameters $\varkappa_1,\dots,\varkappa_N$ and $N\times N$ matrix $C_{ij}$ the N-soliton potential.

This definition coincides with the ordinary one for the scalar non-linear Schrödinger equation. In the vector case our definition of the number of solitons does not always agree with the intuitive definition /25/.

Let's find out in which case the two sets of the "spectral data" $\varkappa_i$, $C_{ij}$ and $\varkappa_i'$, $C_{ij}'$ determine the same Schrödinger operator and the same function $\Psi(x,t,k)$. Consider the relation (2.11). Let's represent the matrix $(\alpha_{ij})$, which is related to $C_{ij}$ with the help of (2.14) in the block form

$$(\alpha_{ij}) = \left( \begin{array}{c|c} \alpha_+ & \beta \\ \hline \gamma & \alpha_- \end{array} \right),$$

where matrices $\alpha_+$ and $\alpha_-$ have the dimensions $p\times p$ and $(N-p)\times(N-p)$, respectively. Assume that the matrix $\alpha_-$ is invertible. Then the transformation $(\varkappa_i,(\alpha_{ij}))\Rightarrow(\varkappa_i',(\alpha_{ij}'))$, where

$$\varkappa_i' = \begin{cases} \varkappa_i, & i = \overline{1,p} \\ \overline{\varkappa}_i, & i = \overline{p+1,N} \end{cases}$$

$$(\alpha_{ij}') = \left( \begin{array}{c|c} \alpha_+ - \beta\alpha_-^{-1}\gamma & -\beta\alpha_-^{-1} \\ \hline \alpha_-^{-1}\gamma & \alpha_-^{-1} \end{array} \right) \qquad (2.27)$$

does not change the relations (2.11), which determine the function $\Psi(x,t,k)$. Hence, for the invertible minor $\alpha_-$ the points $\varkappa_{p+1},\dots,\varkappa_N$ may be transformed from the lower to the upper half-plane without changing the Schrödinger operator and his eigenfunctions.

It must be mentioned, that if for some $i_0$

$$C_{i_0 j} = C_{j i_0} = 0\ ,\ j = 1,\dots,N \qquad (2.28)$$

then the corresponding function $\Psi(x,t,k)$ has the form



$$\Psi(x,t,k) = \frac{k-\overline{\varkappa}_{i_0}}{k-\varkappa_{i_0}} \widetilde{\Psi}(x,t,k) \qquad (2.29)$$

where $\widetilde{\Psi}$ does not depend on $\varkappa_{i_0}$ and is determined by the system (2.13) with $i\neq i_0$ . The potential $u(x,t)$ does not depend on $\varkappa_{i_0}$ either.

We shall present now the transformations of the spectral data which are corresponding to the Galilei , scale and other simplest transformations of the Schrödinger operator:

a) The Galilei transformation

$$x' = x + vt\ ,\ t' = t\ . \qquad (2.30)$$

In this case

$$\varkappa_i' = \varkappa_i - \frac{v}{2}\ ,\quad i = \overline{1,N}$$
$$(C_{ij}') = (C_{ij})\ . \qquad (2.31)$$

The corresponding potential and eigenfunctions are equal to

$$u(x,t) = u'(x',t')$$
$$\Psi'(x',t',k') = \Psi(x,t,k)e^{-i\frac{v}{2}(x+\frac{v}{2}t)}\ ,\ k' = k - \frac{v}{2}\ . \qquad (2.32)$$

b) Translations

$$x' = x + x_0\ ,\quad t' = t + t_0\ . \qquad (2.33)$$

In this case

$$\varkappa_i' = \varkappa_i\ ,\ i = \overline{1,N}$$
$$C_{ij}' = C_{ij}\exp\{i[(\overline{\varkappa}_i - \varkappa_j)x_0 + (\overline{\varkappa}_i^2 - \varkappa_j^2)t_0]\}\ ,\ i,j = \overline{1,N} \qquad (2.34)$$

and

$$u'(x',t') = u(x,t)$$
$$\Psi'(x',t',k') = \Psi(x,t,k)\, e^{ik(x_0+kt_0)} \qquad (2.35)$$

c) The scaling transformations

$$x' = \lambda x\ ,\quad t' = \lambda^2 t\ . \qquad (2.36)$$

The corresponding transformation of the spectral data has the form

$$\varkappa_i' = \lambda^{-1}\varkappa_i\ ,\ i = \overline{1,N}$$
$$(C_{ij}') = (C_{ij})\ . \qquad (2.37)$$

For the potential and eigenfunction we have

$$u'(x',t') = \lambda^{-2}u(x,t)$$
$$\Psi'(x',t',k') = \Psi(x,t,k)\ ,\ k' = \lambda^{-1}k\ . \qquad (2.38)$$

d) The space and time reflection

$$x' = -x\ ,\ t' = -t\ . \qquad (2.39)$$

Then

$$\varkappa_i' = \bar{\varkappa}_i \ , \quad i=\overline{1,N}$$
$$C_{ij}' = \bar{C}_{ij} \ , \quad i,j=\overline{1,N} \ . \tag{2.40a}$$

As this takes place

$$u'(x',t') = u(x,t) \ , \quad \Psi'(x',t',k') = \overline{\Psi(x,t,k)} \ , \quad k' = \bar{k} \ . \tag{2.40b}$$

## 2. The asymptotic properties of the constructed potentials and eigenfunctions.

At the begining we shall consider the case N=1. The system (2.23) is reduced to the equation

$$\left( c + \frac{e^{i(\bar{\omega}-\omega)}}{\bar{\varkappa} - \varkappa} \right) \Psi_1(x,t) = -e^{i\bar{\omega}} \tag{2.41}$$

Here $\varkappa \equiv \varkappa_1$ (let $\mathrm{Im}\,\varkappa > 0$), $C = C_{11}$, $\mathrm{Re}\, C_{11} = 0$, $\mathrm{Im}\, C > 0$, (the case $C=0$ is trivial), $\omega = \varkappa x + \varkappa^2 t$ .

$$\Psi_1(x,t) = -e^{i\bar{\omega}} \left( c + \frac{e^{i(\bar{\omega}-\omega)}}{\bar{\varkappa}-\varkappa} \right)^{-1} . \tag{2.42}$$

Let's denote $\varkappa = \alpha + i\beta$ . Then

$$\Psi_1(x,t) = \frac{i\beta}{\sqrt{(\bar{\varkappa}-\varkappa)c}} \cdot \frac{e^{i\alpha x + i(\alpha^2-\beta^2)t}}{ch[\beta(x-x_0)+2\alpha\beta t]} \ , \tag{2.43}$$

where

$$x_0 = \frac{1}{\beta} \ln \sqrt{(\bar{\varkappa}-\varkappa)c} \ . \tag{2.44}$$

For $r = r_1(x,t)$ we have the formula

$$r(x,t) = i\beta \left\{ 1 + th\left[\beta(x-x_0) + 2\alpha\beta t\right] \right\} . \tag{2.45}$$

Therefore, for the case N=1 it corresponds to the well-known one-soliton potential of the Schrödinger equation which is decreasing in all directions except $x = -2\alpha t$ + const :

$$u(x,t) = 2i\partial_x r(x,t) = -2\beta^2 ch^{-2}\left[\beta(x-x_0)+2\alpha\beta t\right] \ . \tag{2.46}$$

The eigenfunction of the corresponding Schrödinger operator has the form

$$\Psi(x,t,k) = \left[ 1 + i\beta \frac{1 + th[\beta(x-x_0)+2\alpha\beta t]}{k-\varkappa} \right] e^{ik(x+kt)} \ . \tag{2.47}$$

Consider now the case $N>1$. The asymptotic behavior of $\Psi(x,t,k)$ for the general $(\varkappa_i)$, $(c_{ij})$ is too complicated to be analysed. Here we shall consider only the simplest case, when $\mathrm{Im}\,\varkappa_i > 0$, $i=1,\dots,N$ and

$$\det\,(C_{ij}) \neq 0 \ . \tag{2.48}$$

Some other examples would be considered in sec.4.

For the fixed t and $x \to -\infty$ we have



$$e^{i\bar{\omega}_j} = e^{i\bar{\varkappa}_j x + i\bar{\varkappa}_j^2 t} \to 0$$
$$e^{-i\omega_j} \to 0 \ , \quad j=\overline{1,N} \ . \tag{2.49}$$

Because of this the system (2.23) would be reduced to the form

$$\sum_{j=1}^{N} C_{ij}\Psi_j = 0 \ , \quad i=\overline{1,N} \ . \tag{2.50}$$

Consequently, $\Psi_j \to 0$ for all j . It is easy to see that this decrease has the exponential form, i.e. for $x \to -\infty$

$$\Psi_j(x,t) = \Psi_j^0(t)\, e^{\beta x} \ , \quad j=\overline{1,N} \ , \tag{2.51}$$

where $\Psi_j^0(t)$ are some functions depending on t and

$$\beta = \min_j |\mathrm{Im}\,\varkappa_j| \tag{2.52}$$

Since $r_j = \Psi_j e^{-i\omega_j}$ we have

$$r_j(x,t) \to 0 \ , \quad j=\overline{1,N} \tag{2.53}$$

and

$$u(x,t) = O\left(e^{2\beta x}\right) \ , \quad x \to -\infty \tag{2.54a}$$

As follows from (2.53)

$$\Psi(x,t,k) \to e^{ikx+ik^2t} \ , \quad x \to -\infty \ . \tag{2.54b}$$

The case $x \to +\infty$ is a bit more difficult. The system (2.23) can be re-written as the system for $r_j = \Psi_j \exp(-i\omega_j)$ in the form

$$\sum_{j=1}^{N} \left( c_{ij} e^{-i\bar{\omega}_i} + \frac{1}{\bar{\varkappa}_i - \varkappa_j} \right) r_j = -1 \ , \quad i=\overline{1,N} \ .$$

For $x \to +\infty$ this system turns into

$$1 + \sum_{j=1}^{N} \frac{r_j^0}{\bar{\varkappa}_i - \varkappa_j} = 0 \ , \quad i=\overline{1,N} \tag{2.55}$$

where $r_i^0$ is the limit of $r_i$ .

The rational function

$$f = 1 + \sum \frac{r_i^0}{k - \varkappa_i}$$

can be represented in the form

$$f = \frac{P(k)}{\prod (k-\varkappa_i)} \ ,$$

where P(k) is the polynomial of the degree N the highest coefficient of which equals 1. From (2.55) it follows that $f(\bar{\varkappa}_i) = 0$. Hence, $P(k) = \prod (k - \bar{\varkappa}_i)$ and

$$\Psi(x,t,k) \longrightarrow \prod_{j=1}^{N} \frac{k-\bar{\varkappa}_j}{k-\varkappa_j} \, e^{ikx+ik^2t} \ , \quad x \to +\infty \ . \tag{2.56}$$

The functions $\Psi_j$ are exponentially decreasing

$$\Psi_j(x,t) \to r_j^{\circ} e^{i\omega_j}, \quad x \to +\infty, \quad j=\overline{1,N}. \tag{2.57}$$

It may be shown that

$$u(x,t) = O(e^{-2\beta x}), \quad x \to +\infty, \tag{2.58}$$

where $\beta$ is given with the help of (2.52).

For the singular matrix $C_{ij}$ the asymptotics are more complicated. It must be mentioned that if Im $\varkappa_i > 0$, Im $\varkappa_i \neq$ Im $\varkappa_j$ and det $(C_{ij}) = 0$, at least one of the functions $\Psi_1(x,t),\dots,\Psi_N(x,t)$ tends to infinity for $x \to -\infty$. Actually, let $\lambda_i$ be the non-zero solution of the equations

$$\sum_{i=1}^{N} \lambda_i C_{ij} = 0, \quad j = \overline{1,N}.$$

If we multiply the i-th equation of (2.23) by $\lambda_i$, and take their sum we shall obtain

$$\sum_{i,j=1}^{N} \lambda_i \frac{e^{i(\bar\omega_i - \omega_i)}}{\bar\varkappa_i - \varkappa_j} \Psi_j = -\sum_{i=1}^{N} \lambda_i e^{i\bar\omega_i}.$$

If the functions $\Psi_j$ are bounded, it follows for $x \to -\infty$ that all $\lambda_i$ must equal to zero. This contradiction proves that functions $\Psi_j$ are unbounded. It can be noted also that for special selection of the spectral data $\varkappa_i$, $(C_{ij})$ the corresponding potentials are periodic or quasi-periodic functions of $\varkappa$. The periodic potentials with their period equal to T correspond to such data that

$$C_{ij} \exp i[(\bar\varkappa_i - \varkappa_j)T] = C_{ij}, \quad i,j = \overline{1,N}. \tag{2.59}$$

If $\varkappa_j = \alpha_j + i\beta_j$, then from (2.59) it follows for $C_{ij} \neq 0$

$$\begin{cases} \alpha_i - \alpha_j = \dfrac{2\pi n_{ij}}{T} \\ \beta_i + \beta_j = 0 \end{cases} \tag{2.60}$$

where $n_{ij}$ - integers.

The conditions

$$\beta_i + \beta_j = 0, \quad \text{if} \quad C_{ij} \neq 0, \tag{2.61}$$

lead us to the quasi-periodic $u(x,t)$ (with respect to $x$).

These conditions can be fulfilled if the matrix $C_{ij}$ satisfies the following relations

$$C_{ii} = 0, \; C_{ij} \cdot C_{jk} \cdot C_{ki} = 0, \quad i,j,k = 1,\dots,N. \tag{2.62}$$

The conditions of the quasi-periodicity of $u(x,t)$ with respect to t have the similar form.



Let's consider now the asymptotics for the large t and fixed x. We shall assume again that Im $\varkappa_i > 0$, $i=1,\dots,N$, det $C_{ij} \neq 0$. There exist two different cases. The first one is the case when Im $\varkappa_j^2 \neq 0$ for all $j=1,\dots,N$. Then it can be shown that all functions $\Psi_1(x,t),\dots,\Psi_N(x,t)$ are exponentially decreasing for $|t| \to \infty$ and the rate of their decrease is determined by the number $\min_j [\mathrm{Im}\,\varkappa_j^2]$.

In the second case we shall consider only the simpliest situation when

$$\mathrm{Im}\,\varkappa_1^2 > \dots > \mathrm{Im}\,\varkappa_{N-1}^2 > \mathrm{Im}\,\varkappa_N^2 = 0. \tag{2.63}$$

For $t \to -\infty$, the system (2.23) has the asymptotic form:

$$\begin{gathered} \sum_{j=1}^{N} C_{ij} \Psi_j = 0, \quad i = \overline{1,N-1} \\ \sum_{j=1}^{N-1} C_{Nj} \Psi_j + \left( C_{NN} + \frac{e^{i(\bar\omega_N - \omega_N)}}{\bar\varkappa_N - \varkappa_N} \right) \Psi_N = -e^{i\bar\omega_N}. \end{gathered} \tag{2.64}$$

Let's denote by $(C^{ij})$ the matrix which is inverse to $(C_{ij})$. Changing variables in (2.64) which are determined in (2.65)

$$\Psi_j = \sum_{\ell=1}^{N} C^{j\ell} \Phi_\ell, \quad j = \overline{1,N}, \tag{2.65}$$

give us

$$\begin{gathered} \Phi_1 = \dots = \Phi_{N-1} = 0, \\ \Phi_N = -e^{i\bar\omega_N} \left[ 1 + \frac{e^{i(\bar\omega_N - \omega_N)}}{\bar\varkappa_N - \varkappa_N} C_{NN} \right]^{-1}. \end{gathered} \tag{2.66}$$

Consequently for $t \to -\infty$

$$\Psi_j \to -\frac{C^{jN}}{C^{NN}} \cdot \frac{e^{i\bar\omega_N}}{\dfrac{1}{C^{NN}} + \dfrac{e^{i(\bar\omega_N - \omega_N)}}{\bar\varkappa_N - \varkappa_N}} = \frac{C^{jN}}{C^{NN}} \sqrt{\frac{C^{NN}}{\bar\varkappa_N - \varkappa_N}}\, i\beta_N \frac{e^{-i\beta_N^2 t}}{ch[\beta_N(x - x_0^-)]}, \tag{2.67}$$

$j = \overline{1,N}$,

where $\varkappa_N = i\beta_N$,

$$x_0^- = \frac{1}{\beta_N} \ln \sqrt{\frac{\bar\varkappa_N - \varkappa_N}{C^{NN}}}. \tag{2.68}$$

The corresponding potential $u(x,t)$ has the soliton asymptotic (stationary soliton) of the form

$$u(x,t) \to -2\beta_N^2\, ch^{-2}[\beta_N(x - x_0^-)], \quad t \to -\infty. \tag{2.69}$$

For the calculation of the asymptotics for $t \to +\infty$ it is convenient to consider the system for $r_1(x,t),\dots,r_N(x,t)$. The system (2.23) has the asymptotic form:

$$1+\sum_{j=1}^{N}\frac{r_j}{\bar{\varkappa}_i-\varkappa_j}=0,\quad i=\overline{1,N-1}$$
$$\sum_{j=1}^{N-1}\left(C_{Nj}e^{-i\bar{\omega}_N}+\frac{1}{\bar{\varkappa}_N-\varkappa_j}\right)r_j+\left(e^{-i(\bar{\omega}_N-\omega_N)}+\frac{1}{\bar{\varkappa}_N-\varkappa_N}\right)r_N=-1. \tag{2.70}$$

The first equations show that (N-1) zeros of the rational function

$$R(k)=1+\sum_{j=1}^{N}\frac{r_j}{k-\varkappa_j} \tag{2.71}$$

are the points $\bar{\varkappa}_1,\dots \bar{\varkappa}_{N-1}$. Therefore, this function can be written in the form:

$$R(k)=(k-a)\prod_{i=1}^{N-1}(k-\bar{\varkappa}_i)\left(\prod_{i=1}^{N}(k-\varkappa_i)\right)^{-1}. \tag{2.72}$$

In the terms of the unknown function a(x,t) we have from (2.72) that

$$r_j=\frac{(\varkappa_j-a)\prod_{i=1}^{N-1}(\varkappa_j-\bar{\varkappa}_i)}{\prod_{i\neq j}(\varkappa_j-\varkappa_i)},\quad j=\overline{1,N}. \tag{2.73}$$

The function a(x,t) can be found now from the last equation (2.70). Finally, we obtain that for $t\to\infty$

$$\Psi_N(x,t)\to\frac{i\beta_N}{\sqrt{C_{NN}(\bar{\varkappa}_N-\varkappa_N)}}\,\frac{e^{-i\beta_N^2t+i\varphi_0^+}}{\operatorname{ch}[\beta_N(x-x_0^+)]}, \tag{2.74}$$

where

$$\varphi_0^+=\arg Z_N,\quad x_0^+=\frac{1}{\beta_N}\ln\left[\sqrt{C_{NN}(\bar{\varkappa}_N-\varkappa_N)}\,|Z_N|\right],$$
$$Z_N=\prod_{i\neq N}\frac{\varkappa_N-\bar{\varkappa}_i}{\varkappa_N-\varkappa_i}. \tag{2.75}$$

The asymptotics of $\Psi(x,t,k)$ has the form

$$\Psi(x,t,k)\to\prod_{j=1}^{N-1}\frac{k-\bar{\varkappa}_j}{k-\varkappa_j}\left\{1+\frac{1}{2}\frac{\varkappa_N-\bar{\varkappa}_N}{k-\varkappa_N}\left[1+\operatorname{th}\beta_N(x-x_0^+)\right]\right\}e^{ik(x+kt)}. \tag{2.76}$$

The potential has the soliton asymptotic

$$u(x,t)\to-2\beta_N^2\operatorname{ch}^{-2}[\beta_N(x-x_0^+)],\quad t\to+\infty \tag{2.77}$$

again.

The corresponding shift of the phase of the stationary soliton is equal to

$$x_0^+-x_0^-=\frac{1}{\beta_N}\ln\left(\sqrt{C_{NN}C^{NN}}\left|\prod_{i\neq N}\frac{\varkappa_N-\bar{\varkappa}_i}{\varkappa_N-\varkappa_i}\right.\right). \tag{2.78}$$



Similarly the asymptotics for $t\to\pm\infty$ along the lines $x=vt+x_0$ can be determined (using the formulae (2.31),(2.32) for the Galileo's transformation). In doing so we obtain that the potential would decay into solitons of the form (2.46) moving with the speeds $v_j=-\operatorname{Im}\varkappa_j^2/\operatorname{Im}\varkappa_j$, j=1,...,N. The shifts of their phases are given by formulae of the form (2.78). It must be mentioned that the interactions are not reduced to the pairwise interaction because of the term $\frac{1}{\beta_N}\ln\sqrt{C_{NN}C^{NN}}$ in (2.78).

It should be particularly emphasized that the asymptotic falling of the potential into solitons which were described above holds only for the solutions corresponding to the parameters with the different values $\operatorname{Im}\varkappa_1^2,\dots,\operatorname{Im}\varkappa_N^2$. If some of these values coincide the bound states of the solitons would occur. Consider as an example the case when

$$\operatorname{Im}\varkappa_1^2\geqslant\operatorname{Im}\varkappa_2^2\geqslant\dots>\operatorname{Im}\varkappa_{N-m+1}^2=\dots=\operatorname{Im}\varkappa_N^2=0,$$

and $\operatorname{Im}\varkappa_j>0$, $\det(C_{ij})\neq0$, $j=\overline{1,N}$.

a) Let $t\to-\infty$. The matrix which is inverse to $(C_{ij})$ would be denoted as $C^{ij}$. Through $(C_{ij}^-)_{N-m+1\leq i,j\leq N}$ we denote the matrix which is inverse to

$$\sum_{s=N-m+1}^{N}C_{is}^-C^{sj}=\delta_i^j,\quad i,j=\overline{N-m+1,N}. \tag{2.79}$$

For $t\to-\infty$ the function $\Psi(x,t,k)$ has the oscillating asymptotic (i.e. the quasiperiodic for real k) of the form

$$\Psi(x,t,k)\to\Psi^-(x,t,k), \tag{2.80}$$

where the function

$$\Psi^-(x,t,k)=\left(1+\sum_{j=N-m+1}^{N}\frac{r_j^-}{k-\varkappa_j}\right)e^{ik(x+kt)}$$

is determined according to our main construction with the help of the matrix $C_{ij}^-$. Let

$$\Psi_j^-(x,t)=r_j^-e^{i\omega_j},\quad j=\overline{N-m+1,N} \tag{2.81}$$

be the residues of this function. For $t\to-\infty$ the residues $\Psi_1(x,t),\dots\Psi_N(x,t)$ of the function (x,t,k) have the oscillating asymptotic of the form

$$\Psi_j(x,t)\to\Psi_j^-(x,t),\quad j=\overline{N-m+1,N},$$
$$\Psi_j(x,t)\to\sum_{l,s=N-m+1}^{N}C^{jl}C_{ls}^-\Psi_s^-(x,t),\quad j=\overline{1,N-m}. \tag{2.82}$$

The corresponding asymptotics of the potential $u(x,t) \to u^-(x,t)$ is the m-soliton one corresponding in our construction to the set of data $\varkappa_{N-m+1}, \dots, \varkappa_N$, $C^-_{ij}$. The function $u^-(x,t)$ is the quasi-periodic function of the variable t.

b) $t \to +\infty$. Let R(k) denote the rational function

$$R(k) = \prod_{i=1}^{N-m} \frac{k-\bar{\varkappa}_i}{k-\varkappa_i} , \tag{2.83}$$

and $C^+_{ij}$ denote $m \times m$-matrix

$$C^+_{ij} = R^{-1}(\bar{\varkappa}_i)\, C_{ij}\, R(\varkappa_j) , \quad i,j=\overline{N-m+1,N} . \tag{2.84}$$

Then for $t \to +\infty$ the asymptotic of the function $\Psi^-(x,t,k)$ has the form

$$\Psi^-(x,t,k) \to R(k) \overset{+}{\Psi}(x,t,k) , \tag{2.85}$$

where the function $\overset{+}{\Psi}(x,t,k)$ is given with the help of our construction and corresponds to the set of data $\varkappa_{N-m+1}, \dots, \varkappa_N$, $(C^+_{ij})$.

The asymptotics of the functions $\Psi^-_j(x,t)$ can be easely obtained from the formula (2.85). The asymptotic of the potential $u(x,t) \to u^+(x,t)$ is m-soliton one corresponding to $\varkappa_{N-m+1}, \dots, \varkappa_N, (C^+_{ij})$ and quasi-periodic function of t. The transformation of the matrix $C^-_{ij}$ to $C^+_{ij}$ determines the interaction between the bound states of m-solitons and the other components of the N-soliton solution.

### 3. The self-consistent conditions

The function $\Psi^-(x,t,k)$ which has been defined in the first section can be represented in the neighbourhood of $k=\infty$ in the form

$$\Psi^-(x,t,k) = \left(1 + \sum_{s=1}^{\infty} \xi_s(x,t) k^{-s}\right) e^{ik(x+kt)} . \tag{2.86}$$

(The first factor in (2.86) is the expansion in $k^{-1}$ of the pre-exponential factor in (2.12)). From (2.12) it follows that:

$$\xi_1 = a_1 = \sum_{j=1}^{N} r_j , \tag{2.87}$$

$$\xi_1 + \bar{\xi}_1 = 0 \tag{2.88}$$

The substitution of (2.86) into (2.1) gives us the equalities

$$i\dot{\xi}_s - 2i\xi'_{s+1} - \xi''_s + u\xi_s = 0 , \quad s = 0,1,\dots ; \ \xi_0 = 1 . \tag{2.89}$$

(The dot denotes the time derivative and the prime denotes the x derivative).

Consider once again the meromorphic function

$$\Omega(x,t,k) = \Psi^-(x,t,k)\, \overline{\Psi^-(x,t,\bar{k})} . \tag{2.90}$$



It's expansion in the infinity has the form

$$\Omega(x,t,k) = 1 + \sum_{s=2}^{\infty} J_s(x,t) k^{-s} . \tag{2.91}$$

A few first coefficients have the form

$$J_2 = \xi_2 + \bar{\xi}_2 - \xi_1^2 , \quad J_3 = \xi_3 + \bar{\xi}_3 + \xi_1(\bar{\xi}_2 - \xi_2)$$
$$J_4 = \xi_4 + \bar{\xi}_4 + \xi_1(\bar{\xi}_3 - \xi_3) + |\xi_2|^2 . \tag{2.92}$$

Using the equations (2.89) we can find the representation of $J_s$ in terms of the potential u(x,t).

Lemma 1. The following relations hold for any formal solution $\Psi^-(x,t,k)$ of the equation (2.1) which has the form (2.86)

$$J_2(x,t) = \tfrac{1}{2} u(x,t) + C_2 , \quad C_2 = \text{const} , \tag{2.93}$$

$$\partial_x J_3(x,t) = \tfrac{1}{2} \dot{u}(x,t) , \tag{2.94}$$

$$\partial^2_x J_4(x,t) = \tfrac{3}{8} \ddot{u} - \tfrac{1}{8}(u_{xxx} - 6uu_x)_x . \tag{2.95}$$

The relation (2.93) was found in /22/ and the relations (2.94) and (2.95) were found in /23/. The constant $C_2$ in (2.93) can be determined from the asymptotic of $\Omega(x,t,k)$ for $|x| \to \infty$. For example, in the case considered in the previous section where $\operatorname{Im} \varkappa_i > 0$ and the matrix $C_{ij}$ is invertible we have $\Omega(x,t,k) \to 1$, $u(x,t) \to 0$, $x \to \pm\infty$. Therefore, $C_2=0$.

The relations (2.93 - 2.95) are the basis of the constructions of the solutions with all self-consistent conditions (see below (2.101),(2.103),(2.104)). Let E(k) be the rational function of the forms

$$E(k) = k + \sum_{i=1}^{n} \varepsilon_i \frac{b_i^2}{k-k_i} , \tag{2.96}$$

$$E(k) = k^2 + \alpha k + \sum_{i=1}^{n} \varepsilon_i \frac{b_i^2}{k-k_i} , \tag{2.97}$$

$$E(k) = k^3 + \beta k^2 + \gamma k + \sum_{i=1}^{n} \varepsilon_i \frac{b_i^2}{k-k_i} . \tag{2.98}$$

Here $\alpha, \beta, \gamma, k_i, b_i$ are arbitrary real constants. The constants $\varepsilon_i = \pm 1$. We shall denote

$$\Phi_i(x,t) = b_i \Psi^-(x,t,k_i) , \quad i = \overline{1,n} . \tag{2.99}$$

The functions $\Phi_i$ satisfy the equation

$$i\dot{\Phi}_j - \Phi''_j + u(x,t)\Phi_j = 0 , \quad j = \overline{1,n}$$

by the definition. The functions $\Psi_j(x,t)$ which are determined with the help of (2.21) satisfy the same equation

$$i\dot{\Psi}_j - \Psi_j'' + u(x,t)\Psi_j = 0\,, \quad j = \overline{1,N} \tag{2.100}$$

too.

Theorem 3. Let the functions $\Phi_i(x,t)$, $\Psi_j(x,t)$ correspond to the set of data $\varkappa_1,..,\varkappa_N$, $C_{ij}$ and to the rational function which has one of the form (2.96 - 2.98). Then they satisfy one of the self-consistent conditions:

1°. If E(k) has the form (2.96), then

$$\frac{u}{2} + \sum_{i=1}^{n} \varepsilon_i b_i^2 + C_2 = \sum_{i=1}^{n} \varepsilon_i |\Phi_i(x,t)|^2 - \sum_{i,j=1}^{N} \overline{\Psi_i(x,t)} E_{ij} \Psi_j(x,t), \tag{2.101}$$

where

$$E_{ij} = C_{ij}\left(\overline{E(\varkappa_i)} - E(\varkappa_j)\right), \quad i,j = \overline{1,N}\,. \tag{2.102}$$

2°. If E(k) has the form (2.97) then

$$\frac{\dot{u}}{2} + \alpha\frac{u'}{2} = \sum_{i=1}^{n} \varepsilon_i |\Phi_i(x,t)|_x^2 - \left(\sum_{ij=1}^{N} \overline{\Psi_i(x,t)} E_{ij} \Psi_j(x,t)\right)_x . \tag{2.103}$$

Here the matrix $E_{ij}$ is the same as in (2.102).

3°. If E(k) has the form (2.98), then

$$\frac{3}{8}\ddot{u} - \frac{1}{8}\left(u_{xxx} - 6uu_x\right)_x + \beta\frac{\dot{u}_x}{2} + \gamma\frac{u_{xx}}{2} = \tag{2.104}$$

$$= \sum_{i=1}^{n} \varepsilon_i |\Phi_i(x,t)|_{xx}^2 - \left(\sum_{ij=1}^{N} \overline{\Psi_i(x,t)} E_{ij} \Psi_j(x,t)\right)_{xx}$$

(the matrix $E_{ij}$ is given with the help of (2.102)).

The proof. Consider the rational function

$$\hat{\Omega}(x,t,k) = E(k)\Psi(x,t,k)\overline{\Psi(x,t,k)}. \tag{2.105}$$

Using the residue theorem for this function and the relations (2.20), (2.22) we obtain the self-consistent conditions. The theorem is proved.

It must be mentioned that the matrix $E_{ij}$ is Hermitian. Therefore, this matrix can be reduced to the diagonal form with the help of the linear transformation of $\Psi_1, \ldots, \Psi_N$. By doing so we obtain the diagonal element equal to $\pm 1$, or 0.

For the general data $\varkappa_i$, $C_{ij}$ and rational functions of the form (2.96),(2.98) the Hermitian forms in the right sides of self-consistent conditions have big ranks which are equal to N+n . In the case of the self-consistent condition (2.101) it means that the functions $\Phi_1,\ldots,\Phi_n$, $\Psi_1,\ldots,\Psi_N$ are the solution of the (N+n)-component vector non-linear Schrödinger equation the symmetry of which is determined by the signature of the Hermitian matrix

$$\begin{pmatrix} -\varepsilon_1 & & & \\ & \ddots & & 0 \\ & & -\varepsilon_n & \\ & 0 & & \boxed{E_{ij}} \end{pmatrix} . \tag{2.106}$$

The rank of this matrix decreases under some special conditions for the parameters of the construction. Consequently, the number of the component of the vector non-linear Schrödinger equations (and for other self-consistent conditions) would decrease.

The difference between the solutions ( $\Psi$ , $\Phi$ ) of the self-consistent equation corresponding to the matrix (2.106) with the same rank and signature but with different number n of the poles of the functions E(k) is as follows. As was shown in the previous section the functions $\Psi_1,\ldots,\Psi_N$, $\Phi_1,\ldots,\Phi_n$ have the different asymptotics for large x . The functions $\Phi_i$ have as a rule the oscillating asymptotics but the functions $\Psi_j(x,t)$ are exponentially decreasing. This difference must be taken into account in the construction of the multisoliton solutions, and the choice of the function E(k) must depend on the required boundary conditions.

## Chapter III

## Some examples

### 1. Scalar models.

Example 1. Let us obtain using the formalism of Ch.II the well-known /30/ multisolitons solutions of the scalar NLS equation (with attraction). Let us take E(k) = k . For the scalar case matrix $E_{ij}$ of the form (2.95) must be of rank 1. Hence the matrix $C_{ij}$ must have the form:

$$C_{ij} = \lambda\,\frac{\bar{\gamma}_i \gamma_j}{\bar{\varkappa}_i - \varkappa_j}\,, \quad i,j = \overline{1,N}\,, \tag{3.1}$$

where $\gamma_1,\ldots,\gamma_N$ are arbitrary complex constants and $\lambda$ is a real number. The constants $\gamma_1,\ldots,\gamma_N$ can be supposed non-zero. They are to satisfy a normalizing condition

$$|\gamma_1|^2 + \cdots + |\gamma_N|^2 = 1\,. \tag{3.2}$$

(If $\gamma_i = 0$ for some i then there are zero i-th line and i-th column in the matrix $C_{ij}$; hence the system (2.13) is trivialized (see S.2.1)). Because of that we can consider only the case where all the constants $\varkappa_1,\ldots,\varkappa_N$ are in upper halfplane. It can be deduced from the re-

sults of S.2.1 because of non-degenerateness of matrix $((\bar{\varkappa}_i - \varkappa_j)^{-1})$.*)
The condition for the matrix $C_{ij}$ (3.1) to be positive is equivalent to the inequality $\lambda > 0$ (we suppose now that $\mathrm{Im}\,\varkappa_i > 0$, $i=1,..,N$). The Hermitian form (2.95) in this case reduces to

$$-\lambda \sum_{i,j=1}^{N} \bar{\gamma}_i \gamma_j \overline{\Psi_i(x,t)}\, \Psi_j(x,t) = -\lambda \left| \sum_{i=1}^{N} \gamma_i \Psi_i(x,t) \right|^2 .$$

The constant $C_2$ in (2.94) in this case equals zero. Finally we obtain the function:

$$\varphi(x,t) = \sqrt{|\lambda|} \sum_{i=1}^{N} \gamma_i \Psi_i(x,t) = \sqrt{|\lambda|}\, \frac{\det \hat{M}(x,t)}{\det M(x,t)} , \qquad (3.3)$$

where the $N \times N$-matrix $M(x,t)$

$$M_{ij}(x,t) = \frac{\lambda \bar{\gamma}_i \gamma_j + e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j} , \qquad (3.4)$$

$$\hat{M}_{ij}(x,t) = M_{ij}(x,t) ,\ i,j = \overline{1,N} , \qquad (3.5)$$

$$\hat{M}_{00} = 0 ,\ \hat{M}_{i0} = e^{i\bar{\omega}_i} ,\ \hat{M}_{0i} = \gamma_i ,\ i = \overline{1,N}$$

is a decreasing with $|x| \to \infty$ solution of the NLS-equation

$$i\varphi_t = \varphi_{xx} + 2|\varphi|^2 \varphi . \qquad (3.6)$$

Example 2. In analogous way decreasing solutions of the Schrö-dinger equation

$$i\varphi_t = \varphi_{xx} - u\varphi \qquad (3.7)$$

with self-consistent conditions of the forms

$$\frac{u_t}{2} = -|\varphi|^2_x \qquad (3.8)$$

or

$$3\partial_t^2 u - (u_{xxx} - 6uu_x)_x = -8|\varphi|^2_{xx} \qquad (3.9)$$

can be constructed (we consider the case $\alpha = \beta = \gamma = 0$ in the formulae (2.96),(2.97)). For these conditions a solution has the form (3.3), where the matrix $M(x,t)$ is

---

*) Such a simple assertion will be usefull below: if all the numbers $\varkappa_1,..,\varkappa_N$, $\bar{\varkappa}_1,...,\bar{\varkappa}_N$ are distinct, and $\mathrm{Im}\,\varkappa_i > 0$, $i=\overline{1,p}$; $\mathrm{Im}\,\varkappa_j < 0$, $j=p+1,..,N$; then the Hermitian matrix $[i(\bar{\varkappa}_i - \varkappa_j)]^{-1}$ has the signature$(p,N-p)$.



$$M_{ij} = \frac{\lambda \bar{\gamma}_i \gamma_j}{\bar{\varkappa}_i^q - \varkappa_j^q} + \frac{e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j} , \quad q = 2,3,\dots \qquad (3.10)$$

the matrix $\hat{M}$ is determined by formula (3.5), $\lambda > 0$. Here $q=2$ for equation (3.8). In this case numbers $\varkappa_1,\dots,\varkappa_N$ should be taken from the 1st quadrant of the complex plane, i.e.

$$\mathrm{Im}\,\varkappa_i > 0 ,\ \mathrm{Re}\,\varkappa_i > 0 ,\ i = \overline{1,N} . \qquad (3.11)$$

For the equation (3.9) we have $q=3$ ; the numbers $\varkappa_1,\dots,\varkappa_N$ are in sectors

$$0 < \arg \varkappa_i < \frac{\pi}{3} ,\ \frac{2\pi}{3} < \arg \varkappa_i < \pi ,\ i = \overline{1,N} , \qquad (3.12)$$

and $\varkappa_i^3 \neq \varkappa_j^3$ for $i \neq j$.

For other self-consistent conditions of the type

$$\frac{1}{2}\dot{u} = |\varphi|^2_x \qquad (3.13)$$

or

$$3\ddot{u} - (u_{xxx} - 6uu_x)_x = 8|\varphi|^2_{xx} \qquad (3.14)$$

solution has the same form, but $\lambda < 0$ and the numbers $\varkappa_1,\dots,\varkappa_N$ satisfied other restrictions. For (3.13) (where $q=2$) it is required the following inequalities being fulfilled:

$$\mathrm{Im}\,\varkappa_i > 0 ,\ \mathrm{Re}\,\varkappa_i < 0 ,\ i = \overline{1,N} . \qquad (3.15)$$

For equation (3.14) (where $q=3$) the restrictions are

$$\frac{\pi}{3} < \arg \varkappa_i < \frac{2\pi}{3} ,\ i = \overline{1,N} . \qquad (3.16)$$

Example 3. The technique of construction of non-decreasing conditions for various self-consistent conditions we demonstrate at first by a simple example of scalar NLS-equations. First of all let us take the NLS with attraction. To construct non-decreasing (oscillating) with $|x| \to \infty$ solutions of this equation we need the function $E(k)$ having the form

$$E(k) = k - \frac{b_1^2}{k - k_1} . \qquad (3.17)$$

The Hermitian form $E_{ij}$ (2.95) must be zero (otherwise the scalar NLS will not be obtained). In other words the following stick conditions must be fulfilled:

$$E(\bar{\varkappa}_i) = E(\varkappa_j) \quad \text{for } C_{ij} \neq 0 \qquad (3.18)$$

(we have used that the coefficients $b_1, k_1$ are real). For every value of i equality $E(\bar{\varkappa}_i)=E(\varkappa_j)$ can be fulfilled only for the single value of j because the degree of the function E(k) equals two. Hence for each i there are only one value $j=\nu(i)$ such that $C_{ij}\neq 0$ (we recall that matrices $C_{ij}$ with zero lines are not considered). The requirement of matrix $C_{ij}$ being Hermitian implies that $\nu$ is an involution of the set of indices (1,2,..,N). This involution has no fixed points because $E(\bar{\varkappa}_i)\neq E(\varkappa_i)$ for non-real $\varkappa_i$. Hence N is even and the points $\varkappa_i$ can be numerated in such a way that

$$E(\varkappa_{N-i+1}) = E(\bar{\varkappa}_i) \ , \quad i=\overline{1, N/2} \ . \tag{3.19}$$

The matrix $C_{ij}$ is antidiagonal. It is easy to see that the points $\varkappa_1,\dots,\varkappa_{N/2}$ can be taken only from upper half-plane. Then the points

$$\varkappa_{N-i+1} = k_1 - \frac{b_1^2}{\bar{\varkappa}_i - k_1} \ , \qquad i=\overline{1, N/2} \tag{3.20}$$

are in lower half-plane. Finally, we obtain the following formulae for non-decreasing solutions of NLS-equation

$$i\varphi_t = \varphi_{xx} + 2(|\varphi|^2 - b_1^2)\varphi \ , \tag{3.21}$$

$$\varphi(x,t) = b_1 e^{i\eta} \frac{\det \hat{M}(x,t)}{\det M(x,t)} e^{ik_1(x+k_1 t)} \ , \tag{3.22}$$

where $\eta$ is an arbitrary real constant and the N×N matrix M(x,t) has the form

$$M_{ij}(x,t) = C_i \delta_{i,N-j+1} + \frac{e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j} \ , \tag{3.23}$$

here $C_1,\dots,C_N$ are non-zero complex numbers satisfying the skew Hermitian conditions

$$C_{N-i+1} = -\bar{C}_i \ , \quad i=\overline{1, N/2} \ , \tag{3.24}$$

numbers $\varkappa_1,\dots,\varkappa_N$ satisfy (3.20); the matrix $\hat{M}(x,t)$ has the form

$$\hat{M}_{ij} = M_{ij} \ , \quad 1\le i,j\le N \tag{3.25}$$
$$\hat{M}_{00} = 1 \ , \ \hat{M}_{i0} = e^{i\bar{\omega}_i} \ , \ \hat{M}_{0i} = e^{-i\omega_i}(k_1-\varkappa_i)^{-1} \ , \ i=\overline{1,N} \ .$$

It should be noted that the solution $\varphi(x,t)$ will bethe quasi-periodic function of x (see S.2.2) if one requires the numbers $\varkappa_1,\dots,\varkappa_N$ satisfying conditions

$$\mathrm{Im}\,\varkappa_i + \mathrm{Im}\,\varkappa_{N-i+1} = 0 \ , \quad i=\overline{1, N/2} \tag{3.26}$$

(the restrictions (3.20) in this case imply that the points $\varkappa_i$ and $\varkappa_{N-i+1}$ are situated in the circle of radius $b_1$ with centre in $k_1$).



If the equalities (3.26) are not fulfilled for any value of i then the solution $\varphi(x,t)$ (3.22) has the following asymptotics:

$$\varphi(x,t) \to b_1 e^{i\eta} \prod{}' \left(\frac{k_1-\bar{\varkappa}_j}{k_1-\varkappa_j}\right) e^{ik_1(x+k_1 t)} \ , \quad x\to -\infty \ , \tag{3.27}$$

where one takes the product over all values of j such that

$$\mathrm{Im}(\varkappa_j + \varkappa_{N-j+1}) < 0 \ ;$$

$$\varphi(x,t) \to b_1 e^{i\eta} \prod{}'' \left(\frac{k_1-\bar{\varkappa}_j}{k_1-\varkappa_j}\right) e^{ik_1(x+k_1 t)} \ , \quad x\to +\infty \ , \tag{3.28}$$

where one takes the product over the rest values of index j. We omit the proof of these formulae. Asymptotics with $|t|\to\infty$ can be calculated in analogous way but depend on relations between the values $\mathrm{Im}\,\varkappa_j^2$.

Let us investigate now the NLS with repulsion:

$$i\varphi_t = \varphi_{xx} - 2(|\varphi|^2 - b_1^2)\varphi \ . \tag{3.29}$$

The function E(k) here must have the form

$$E(k) = k + \frac{b_1^2}{k-k_1} \tag{3.30}$$

and the stick conditions must be fulfilled.

As well as above we obtain an involution $\nu$ on the set of indices (1,2,..,N) such that $C_{ij}\neq 0$ only for $j=\nu(i)$. But this involution now can have fixed points. Let us numerate the points $\varkappa_1,\dots,\varkappa_N$ in such a way that the points $\varkappa_1,\dots,\varkappa_l$ lie in the circle of the radius $b_1$ and with the centre $k_1$ and the rest points are pairwise symmetric in this circle, i.e.

$$|\varkappa_i - k_1| = b_1 \ , \quad i=\overline{1,l} \ ;$$
$$\varkappa_{N-i+l+1} = k_1 + \frac{b_1^2}{\bar{\varkappa}_i - k_1} \ , \quad i=\overline{l+1, l+m} \ . \tag{3.31}$$

The matrix $C_{ij}$ must have the following form:

$$(C_{ij}) = \begin{pmatrix} \begin{matrix} C_1 & & \\ & \ddots & \\ & & C_l \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & & & d_1 \\ & \ddots & d_2 & \\ & -d_2 & \ddots & \\ -d_1 & & & 0 \end{matrix} \end{pmatrix} \ , \tag{3.32}$$

where numbers $c_1,\dots,c_l$ are purely imaginary and $d_1,\dots,d_m$ are arbitrary; all these numbers are non-zero. The points $\varkappa_1,\dots,\varkappa_l$ can be supposed being situated in the upper half-plane. But for each $i > l$ the points $\varkappa_i$ and $\varkappa_{N-i+l+1}$ are situated in one half-plane. From the positive definiteness of the matrix $(i^{-1}c_{ij})$ we obtain that $d_1=\dots=d_m=0$. So we can consider only the case m=0 and the matrix $C_{ij}$ being diagonal. Finally we have that the solutions of NLS with repulsion take the form (3.22) where the matrix $(M_{ij}(x,t))$ is

$$M_{ij}(x,t) = i\tilde{c}_i\delta_{ij} + \frac{e^{i(\bar{\omega}_i-\omega_j)}}{\bar{\varkappa}_i-\varkappa_j}, \tag{3.33}$$

the numbers $\varkappa_1,\dots,\varkappa_N$ are situated in the upper half-plane satisfying restrictions $|\varkappa_i-k_1|=b_1$, $i=1,\dots,N$, and the numbers $\tilde{c}_1,\dots,\tilde{c}_N$ are real and positive; the matrix M is defined by the matrix M via (3.25). The simplest form of such solutions (N=1) has the form of a kink

$$\varphi(x,t) = b_1\left[1+i\beta\frac{1+\operatorname{th}\tau}{k_1-\varkappa}\right]e^{ik_1(x+k_1t)+i\eta}, \tag{3.34}$$

where $\varkappa=\alpha+i\beta=k_1+b_1(\cos\xi+i\sin\xi)$, $\xi\neq 0,\pi$ is an arbitrary parameter,

$$\tau=b_1\sin\xi\left[(x-x_0)+2(k_1+b_1\cos\xi)t\right],\ x_0=\frac{1}{\beta}\ln\sqrt{\tilde{c}(\bar{\varkappa}-\varkappa)i} \tag{3.35}$$

(see the formulae (2.44)-(2.47) above). For $N>1$ our solution is a non-linear superposition of steps.

2. Vector models

Example 1. Let us construct vanishing with $|x|\to\infty$ solutions of the vector NLS with U(n,o) symmetry. Firstly let us supposed that $n\leq N$. To obtain decreasing solutions we need the function E(k) having the form E(k)=k. The Hermitian matrix $(E_{ij})$ takes on the form

$$E_{ij}=(\bar{\varkappa}_i-\varkappa_j)C_{ij},\quad i,j=\overline{1,N}. \tag{3.36}$$

This matrix must be non-negatively definite of rank n. Let us represent it in a form $E=\Gamma^+\Gamma$, where $\Gamma$ is a matrix of rank n, i.e.

$$E_{ij}=\sum_{s=1}^{n}\bar{\gamma}_{si}\gamma_{sj}. \tag{3.37}$$

We obtain the following form of matrix $C_{ij}$ :

$$C_{ij}=\frac{\sum_{s=1}^{n}\bar{\gamma}_{si}\gamma_{sj}}{\bar{\varkappa}_i-\varkappa_j},\quad i,j=\overline{1,N}. \tag{3.38}$$



It is easy to see that for any matrix $\Gamma=(\gamma_{ij})$ with no zero columns the matrix $(C_{ij})$ (3.38) will be positively definite if all the numbers $\varkappa_1,\dots,\varkappa_N$ are situated in the upper half-plane. And if there are zero columns in the matrix $\Gamma$ then the matrix $(C_{ij})$ also has zero columns and hence a reduction of number of the parameters $\varkappa_1,\dots\varkappa_N$ takes place.

Finally we have : if the numbers $\varkappa_1,\dots,\varkappa_N$ situate in the upper half-plane for any $n\times N$-matrix $\Gamma=(\gamma_{ij})$ the functions $\Phi_1(x,t),\dots,\Phi_n(x,t)$ of the form

$$\Phi_k(x,t)=\frac{\det M^{(k)}(x,t)}{\det M(x,t)},\quad k=\overline{1,n}, \tag{3.39}$$

where the $N\times N$-matrix $M(x,t)=(M_{ij})$ has the form

$$M_{ij}(x,t)=\frac{\sum_{1}^{n}\bar{\gamma}_{si}\gamma_{sj}+\exp(i(\bar{\omega}_i-\omega_j))}{\bar{\varkappa}_i-\varkappa_j},\quad i,j=\overline{1,N}, \tag{3.40}$$

$(N+1)\times(N+1)$-matrices $M^{(k)}(x,t)=(M^{(k)}_{ij}(x,t))$ have the form

$$\begin{aligned}&M^{(k)}_{ij}=M_{ij},\ i,j=\overline{1,N};\quad M^{(k)}_{00}=0;\\ &M^{(k)}_{0j}=\gamma_{kj};\quad M^{(k)}_{j0}=e^{i\bar{\omega}_j},\quad j=\overline{1,N},\end{aligned} \tag{3.41}$$

are solutions of system of equations

$$i\dot{\Phi}_k=\Phi''_k+2\left(\sum_{s=1}^{n}|\Phi_s|^2\right)\Phi_k,\quad k=\overline{1,n}. \tag{3.42}$$

These solutions exponentially vanish with $|x|\to\infty$ and fixed t because of results of s.2.2 (the matrix $(C_{ij})$ here is non-degenerate). Asymptotics with $|t|\to\infty$ we shall describe below. From this description it will be clear that these solutions are a non-linear superposition of N one-soliton solutions of the form

$$\Phi_{k,s}(x,t)=\Phi^{\pm}_{k,s}(\varkappa_s-\bar{\varkappa}_s)\frac{\exp\left[i(\alpha_sx+(\alpha_s^2-\beta_s^2)t)\right]}{2\operatorname{ch}\left[\beta_s(x-x_0^{\pm})+2\alpha_s\beta_st\right]} \tag{3.43}$$

$$\varkappa_s=\alpha_s+i\beta_s,\quad s=\overline{1,N}.$$

Here $\Phi^{\pm}_{k,s}$ are some constant vectors of unity length, they are different for $t\to+\infty$ and $t\to-\infty$. Recall that the asymptotical decay of an initial packet into the solitons and hence formulae (3.43) take place in the case of the generic position only, when magnitudes of $\operatorname{Im}\varkappa_j^2$ are different in pairs. A general N-soliton solution with some of those magnitudes being equal is a conglomerate of the solitons and their bound states.

We have constructed yet only N-soliton solutions of equation (3.42) with $N \geqslant n$. For $N < n$ all N-soliton equations of n component NLS can be obtained of N-soliton solution of NLS (with U(N,o)-symmetry) by means of action of the group U(n).

Remark. In view of definition of N-soliton potential given above in Ch.II the N-soliton solution of vector NLS is given by N poles $\varkappa_1, \dots, \varkappa_N$. In particular, independently of the vector dimension we call the solution one-soliton if it is defined by one pole $\varkappa = \varkappa_1$. It always can be obtained from a solution of the scalar NLS by isorotation.

It also should be noted that the two solutions (3.39)-(3.41) which are given by fixed poles $\varkappa_1, \dots, \varkappa_N$ and by different matrices $\Gamma = (\gamma_{ij})$ but with the same Hermitian form $(E_{ij})$ of the form (3.37) can be obtained one from another by means of action of the unitary group U(n).

Asymptotics for $|t| \to \infty$ (for x is fixed) can be found using the formulae of s.2.2 and the following relations between the components $\Phi_1, \dots, \Phi_n$ of solution of vector NLS and the residues $\Psi_1, \dots, \Psi_N$ of the function $\Psi(x,t,k)$:

$$\Phi_k(x,t) = \sum_{j=1}^{N} \gamma_{kj} \Psi_j(x,t), \quad k = \overline{1,n} \quad . \tag{3.44}$$

As well as in s.2.2 let us suppose the conditions (2.63) being valid (i.e. the N-th soliton is stationary and the rest solitons move from right to left). Then we obtain from (2.67) that with $t \to -\infty$ the functions $\Phi_k(x,t)$ have the following asymptotics

$$\Phi_k(x,t) \to \sum_{j=1}^{N} \frac{\gamma_{kj} c^{jN} (\bar{\varkappa}_N - \varkappa_N) \exp(-i\beta_N^2 t)}{\sqrt{c^{NN}(\bar{\varkappa}_N - \varkappa_N)} \cdot 2\,\mathrm{ch}[\beta_N (x - x_o^-)]}, \quad k = \overline{1,n}, \tag{3.45}$$

$$\varkappa_N = i\beta_N .$$

The phase $x_o^-$ has the form (2.68).

For $t \to +\infty$ we have the following asymptotics

$$\Phi_k(x,t) \to \gamma_{kN} \left( \frac{|Z_N|}{C_{NN}(\bar{\varkappa}_N - \varkappa_N)} \right)^{1/2} i\beta_N \frac{e^{-i\beta_N^2 t + i\varphi_o^+}}{\mathrm{ch}(\beta_N (x - x_o^+))}, \tag{3.46}$$

$$k = \overline{1,n},$$



where the numbers $Z_N$, $x_o^+$ and $\varphi_o^+$ are given by formulae (2.75). Hence the phase shift $x_o^+ - x_o^-$ can be calculated by formula (2.68). The unit vectors $\Phi^{\pm}_{k,N}$ from (3.43) have the form

$$\Phi^-_{k,N} = \sum_{j=1}^{N} \gamma_{kj} c^{jN} \left( c^{NN} (\bar{\varkappa}_N - \varkappa_N) \right)^{-1/2}$$
$$\Phi^+_{k,N} = \gamma_{kN} e^{i\varphi_o^+} \left( \frac{|Z_N|}{C_{NN}(\bar{\varkappa}_N - \varkappa_N)} \right)^{1/2} \tag{3.47}$$

<u>Example 2</u>. Let us construct solutions of the two-component NLS with oscillating asymptotics. We shall consider in detail only two-soliton solutions.

Case I. Both components oscillate when $|x| \to \infty$. The function E(k) should be taken in the form

$$E(k) = k + \varepsilon_1 \frac{b_1^2}{k - k_1} + \varepsilon_2 \frac{b_2^2}{k - k_2} \quad . \tag{3.48}$$

Here $\varepsilon_1, \varepsilon_2 = \pm 1$. These signs response for the type of symmetry of vector NLS. The Hermitian form $(E_{ij})$ (2.95) must vanish, i.e. the stick conditions

$$E(\bar{\varkappa}_i) = E(\varkappa_i) \quad \text{for } c_{ij} \neq 0, i,j=1,\dots,N \tag{3.49}$$

are to be valid. If the stick conditions for the parameters $(\varkappa_i)$, $(C_{ij})$ are fulfilled for the given function E(k) then the function $\Psi(x,t,k)$ which is given by these parameters by means of formulae (2.15),(2.17),(2.18) gives a solution $\Phi = (\Phi_1, \Phi_2)$ of the vector NLS

$$i\dot{\Phi}_j = \Phi_j'' - 2\left[\varepsilon_1 |\Phi_1|^2 + \varepsilon_2 |\Phi_2|^2 - \varepsilon_1 b_1^2 - \varepsilon_2 b_2^2\right] \Phi_j = 0 \tag{3.50}$$

by formula

$$\Phi_j(x,t) = b_j \Psi(x,t,k), \quad j = 1,2 . \tag{3.51}$$

For N = 1 we obtain one-soliton solution

$$\Phi_j(x,t) = b_j \left\{ 1 + \frac{1}{2} \frac{\varkappa - \bar{\varkappa}}{k_j - \varkappa} \left[ 1 + \mathrm{th}(\beta(x - x_o) + 2\alpha\beta t) \right] e^{ik_j(x + k_j t)} \right., \quad j=1,2, \tag{3.52}$$

where relation between $\varkappa \equiv \varkappa_1 = \alpha + i\beta$ and the parameters $k_1, k_2, b_1, b_2$

is given by the stick condition

$$E(\bar{\varkappa}) = E(\varkappa) . \tag{3.53}$$

The signs $\varepsilon_1, \varepsilon_2$ in (3.50) can be arbitrary except $\varepsilon_1 = \varepsilon_2 = -1$ (in this case the equation (3.53) has no solutions).

In two-soliton case (N=2) there are two types of matrices $(C_{ij})$ which are consistent with the stick conditions (3.49). The first type consists of diagonal matrices $(C_{ij})$, i.e. $C_{12}=0$; the second type consists of antidiagonal matrices, i.e. $C_{11}=C_{22}=0$. Really if $C_{11}\neq 0$ and $C_{12}\neq 0$ then the following stick conditions are to be fulfilled:

$$E(\bar{\varkappa}_1) = E(\varkappa_1) , \quad E(\bar{\varkappa}_1) = E(\varkappa_2) .$$

The first of them implies the number $r = E(\varkappa_1)$ being real. Hence we have that the numbers $\varkappa_1, \bar{\varkappa}_1, \varkappa_2$ are the three roots of the cubic equation $E(k) = r$ with real coefficients. But this is impossible because all three numbers $\varkappa_1, \bar{\varkappa}_1, \varkappa_2$ being non-real and distinct.

Let us consider in detail both the types of two-soliton solutions.

Type 1. $C_{12}=0$, $C_{11}\neq 0$, $C_{22}\neq 0$. One may assume that $\operatorname{Im}\varkappa_1 > 0$, $\operatorname{Im}\varkappa_2 > 0$. The stick conditions have the form

$$E(\bar{\varkappa}_1) = E(\varkappa_1) , \quad E(\bar{\varkappa}_2) = E(\varkappa_2) . \tag{3.54}$$

For $\varepsilon_1 = \varepsilon_2 = -1$ this equations have no solutions. For other sings $(\varepsilon_1, \varepsilon_2)$ restrictions have the form of inequalities. It turns out that these restrictions can be formulated in terms of disposition of the point $[\varkappa_1 ; \varkappa_2]$

$$a = \frac{|\varkappa_2|^2 - |\varkappa_1|^2}{2(\varkappa_2 + \bar{\varkappa}_2 - \varkappa_1 - \bar{\varkappa}_1)} \tag{3.55}$$

of intersection of the middle perpendicular to the segment $[\varkappa_1 ; \varkappa_2]$ and the real axis within the interval $[k_1, k_2]$. For the U(0,2)-symmetry (i.e. $\varepsilon_1 = \varepsilon_2 = 1$) the point a (3.55) must lie within the interval $[k_1, k_2]$. For the U(1,1)-symmetry (i.e. $\varepsilon_1 \cdot \varepsilon_2 < 0$) the point a must be situated out of the interval $[k_1, k_2]$ (including the limit case when the segment $[\varkappa_1, \varkappa_2]$ being vertical).

Asymptotics of these solutions for $|x| \to \infty$ and fixed t can be calculated as in 2. We have

$$\Phi(x,t) \to \begin{pmatrix} b_1 \exp(ik_1(x+k_1 t)) \\ b_2 \exp(ik_2(x+k_2 t)) \end{pmatrix} , \quad x \to -\infty , \tag{3.56}$$



$$\Phi(x,t) \underset{x\to+\infty}{\longrightarrow} \begin{pmatrix} b_1 \dfrac{(k_1-\bar{\varkappa}_1)(k_1-\bar{\varkappa}_2)}{(k_1-\varkappa_1)(k_1-\varkappa_2)} e^{ik_1(x+k_1t)} \\ b_2 \dfrac{(k_2-\bar{\varkappa}_1)(k_2-\bar{\varkappa}_2)}{(k_2-\varkappa_1)(k_2-\varkappa_2)} e^{ik_2(x+k_2t)} \end{pmatrix} . \tag{3.57}$$

Asymptotics at $|t| \to \infty$ and fixed x also can be calculated via methods of s.2.2. We give here such asymptotics for the case $\operatorname{Im}\varkappa_1^2 > 0$, $\operatorname{Im}\varkappa_2^2 = 0$ (calculations are omitted). At $t \to -\infty$ we have

$$\Phi_j(x,t) \to b_j \left\{ 1 + \frac{1}{2} \frac{\varkappa_2 - \bar{\varkappa}_2}{k_j - \varkappa_2} \left[ 1 + \operatorname{th} \beta_2 (x - x_0^-) \right] \right\} e^{ik_j(x+k_jt)} , \tag{3.58}$$

$j = 1,2$

where

$$\varkappa_2 = i\beta_2 , \quad \beta_2 > 0$$

$$x_0^- = \frac{1}{\beta_2} \ln \sqrt{C_{22}(\bar{\varkappa}_2 - \varkappa_2)} . \tag{3.59}$$

At $t \to +\infty$ asymptotics has the following form:

$$\Phi_j(x,t) \to b_j \left\{ 1 + \frac{1}{2} \frac{\varkappa_2 - \bar{\varkappa}_2}{k_j - \varkappa_2} \left[ 1 + \operatorname{th} \beta_2 (x - x_0^+) \right] \right\} e^{ik_j(x+k_jt) + i\eta_j} , \tag{3.60}$$

$j = 1,2$

where

$$x_0^+ - x_0^- = \frac{1}{\beta_2} \ln \left| \frac{\varkappa_2 - \bar{\varkappa}_1}{\varkappa_2 - \varkappa_1} \right| , \tag{3.61}$$

$$\eta_j = \arg \frac{k_j - \bar{\varkappa}_1}{k_j - \varkappa_1} , \quad j = 1,2 . \tag{3.62}$$

Asymptotics on lines $x = -2\alpha_1 t + x_0$ at $|t| \to \infty$ have similar form. So we have obtained a non-linear superposition of the one-soliton solutions (3.52).

Type 2. $C_{11}=C_{22}=0$, $C_{12}\neq 0$. One may assume that $\operatorname{Im}\varkappa_1 > 0$, $\operatorname{Im}\varkappa_2 < 0$. The stick conditions have the form

$$E(\bar{\varkappa}_1) = E(\varkappa_2) . \tag{3.63}$$

Let us consider the problem of solvability of this equation (here the sings $\varepsilon_1, \varepsilon_2$ can be arbitrary). Let

$$b = \frac{\bar{\varkappa}_2 \varkappa_1 - \varkappa_2 \bar{\varkappa}_1}{(\varkappa_1 - \bar{\varkappa}_1) - (\varkappa_2 - \bar{\varkappa}_2)} \tag{3.64}$$

be a point of intersection of segment $[\varkappa_1, \varkappa_2]$ with the real axis. Let us introduce the notation

$$d(k) = b - \frac{(\varkappa_1 - \bar{\varkappa}_1)(\bar{\varkappa}_2 - \varkappa_2)(\varkappa_1 - \varkappa_2)(\bar{\varkappa}_2 - \bar{\varkappa}_1)}{(b-k)(\varkappa_1 + \bar{\varkappa}_2 - \bar{\varkappa}_1 - \varkappa_2)^2} . \tag{3.65}$$

For solvability of the equation (3.63) it is necessary that the points $k_1, k_2, \varkappa_1, \varkappa_2$ are not situated on a circle, i.e.

$$k_2 \neq d(k_1) . \tag{3.66}$$

The sings $\varepsilon_1, \varepsilon_2$ depend on $k_1, k_2, \varkappa_1, \varkappa_2$ as follows:

$$\begin{aligned}
&k_1 < k_2 < b \Rightarrow \varepsilon_1 = -\varepsilon_2 = 1 \\
&b < k_1 < k_2 \Rightarrow \varepsilon_1 = -\varepsilon_2 = -1 \\
&k_2 = b \Rightarrow b_1 = 0 \\
&k_1 = b \Rightarrow b_2 = 0 \\
&k_1 < b < k_2 < d(k_1) \Rightarrow \varepsilon_1 = \varepsilon_2 = 1 \\
&k_1 < b < d(k_1) < k_2 \Rightarrow \varepsilon_1 = \varepsilon_2 = -1 .
\end{aligned} \tag{3.67}$$

Asymptotics of the solutions $\Phi_j(x,t)$ at $|x| \to \infty$ and fixed t depend on relations between $\operatorname{Im} \varkappa_1$ and $\operatorname{Im} \varkappa_2$. Namely, for $\operatorname{Im}(\varkappa_1 + \varkappa_2) > 0$ asymptotics have the form (3.56), (3.57). For $\operatorname{Im}(\varkappa_1 + \varkappa_2) = 0$ the solution $\Phi(x,t)$ is quasi-periodic function of x. And for $\operatorname{Im}(\varkappa_1 + \varkappa_2) < 0$ asymptotics in (3.56), (3.57) at $x \to \pm\infty$ change over.

Asymptotics at $|t| \to \infty$ and fixed x can be calculated very easily. For the case $\operatorname{Im} \varkappa_1^2 > 0$, $\operatorname{Im} \varkappa_2^2 = 0$ we shall have for $t \to -\infty$

$$\Phi_j(x,t) \to b_j e^{ik_j(x+k_jt)} , \quad j = 1,2 . \tag{3.68}$$

For $t \to +\infty$

$$\Phi_j(x,t) \to b_j \frac{(k_j - \bar{\varkappa}_1)(k_j - \bar{\varkappa}_2)}{(k_j - \varkappa_1)(k_j - \varkappa_2)} e^{ik_j(x+k_jt)} , \quad j = 1,2 . \tag{3.69}$$



Hence the asymptotics are purely exponential for solutions of such a type. These solutions cannot be reduced to a superposition of one-soliton solutions. Because of that it is naturally to call them <u>double solitons</u> (they are analogous to the well-known bions of the scalar NLS /30/ ).

It should be noted that for arbitrary N the solutions (3.51) of the equation (3.50) can be reduced to a non-linear superposition of solitons and double solitons. For the U(2,0)-case there are supplementary triple soliton solutions. The triple soliton is a solution with N=3 and the matrix $(C_{ij})$ as follows:

$$(C_{ij}) = \begin{pmatrix} 0 & C_{12} & C_{13} \\ C_{21} & 0 & 0 \\ C_{31} & 0 & 0 \end{pmatrix} , \tag{3.70}$$

the stick conditions have the following form:

$$E(\bar{\varkappa}_1) = E(\varkappa_2) = E(\varkappa_3) , \tag{3.71}$$

($\varepsilon_1 = \varepsilon_2 = -1$). Here the points $\varkappa_2, \varkappa_3$ lie in the upper half-plane, and the point $\varkappa_1$ lies in the lower half-plane. Proof of this assertion we omit.

Case 2. The component $\Phi_1(x,t)$ oscillates at $|x| \to \infty$ and $\Phi_2(x,t)$ vanishes when $|x| \to \infty$. The function E(k) one should take as follows:

$$E(k) = k + \varepsilon_1 \frac{b_1^2}{k - k_1} . \tag{3.72}$$

The simplest one-soliton solution of such a type can be constructed via our formalism for N=1. It is given by parameters $\varkappa \equiv \varkappa_1 = \alpha + i\beta$ (let us suppose that $\beta > 0$) and $C_{11} = i\tilde{C}_{11}$, $\tilde{C}_{11} > 0$. This solution was obtained in /31/, /2/. It has the following form:

$$\begin{aligned}
\Phi_1(x,t) &= b_1 \left\{ 1 + \frac{1}{2} \frac{\varkappa - \bar{\varkappa}}{k_1 - \varkappa} \left[ 1 + \operatorname{th} \beta (x - x_0 + 2\alpha t) \right] \right\} e^{ik_1(x+k_1t)} \\
\Phi_2(x,t) &= \frac{(\varkappa - \bar{\varkappa})(|\varkappa - k_1|^2 - \varepsilon_1 b_1^2)^{1/2}}{|\varkappa - k_1|} \cdot \frac{\exp i(\alpha x + (\alpha^2 - \beta^2)t)}{2 \operatorname{ch} \beta (x - x_0 + 2\alpha t)} .
\end{aligned} \tag{3.73}$$

Here

$$x_0 = \frac{1}{\beta} \ln \sqrt{C_{11}(\bar{\varkappa} - \varkappa)} . \tag{3.74}$$

The vector function $\Phi = (\Phi_1, \Phi_2)$ (3.73) is a solution of equations

$$i\dot{\Phi}_j = \Phi_j'' - 2\left[\varepsilon_1|\Phi_1|^2 + \varepsilon_2|\Phi_2|^2 - \varepsilon_1 b_1^2\right]\Phi_j \ , \ j = 1,2 . \qquad (3.75)$$

Here the sign $\varepsilon_2$ is defined if $|\varkappa - k_1|^2 \neq \varepsilon_1 b_1^2$ via the following formula:

$$\varepsilon_2 = \operatorname{sgn}\left[\varepsilon_1 b_1^2 - |\varkappa - k_1|^2\right] . \qquad (3.76)$$

For $\varepsilon_1 = -1$ we also have $\varepsilon_2 = -1$. Hence in this case (3.73) gives one-soliton solution of the U(2,0)-NLS.

For

$$\varepsilon_1 = 1 \ , \ |\varkappa - k_1| > b_1 \qquad (3.77)$$

we have $\varepsilon_2 = -1$. Hence in this case we have solution of the U(1,1) NLS. For

$$\varepsilon_1 = 1 \ , \ |\varkappa - k_1| < b_1 \qquad (3.78)$$

we obtain solution of the U(0,2)-NLS.

If the equation

$$|\varkappa - k_1|^2 = \varepsilon_1 b_1^2 \iff E(\bar{\varkappa}) = E(\varkappa) \qquad (3.79)$$

is satisfied (it is possible only for $\varepsilon_1 = 1$) the component $\Phi_2$ is identically zero and the solution (3.73) reduces to one-soliton solution (3.74) of NLS with repulsion.

Let us prove that for $\varepsilon_1 = 1$ multi-soliton solutions are non-linear superposition of solitons. The Hermitean form $(E_{ij})$ (2.95) is to be of rank 1. Let us suppose first of all that the points $\varkappa_1, \dots, \varkappa_N$ satisfy no stick conditions, i.e.

$$E(\bar{\varkappa}_i) \neq E(\varkappa_j) \ , \quad i,j = \overline{1,N} . \qquad (3.80)$$

Then the corresponding matrix $(C_{ij})$ has the following form:

$$C_{ij} = \lambda \frac{\bar{\gamma}_i \gamma_j}{E(\bar{\varkappa}_i) - E(\varkappa_j)} \ , \quad i,j = \overline{1,N} . \qquad (3.81)$$

Here $\gamma_1, \dots, \gamma_N$ are arbitrary complex constants such that

$$|\gamma_1|^2 + |\gamma_2|^2 + \dots + |\gamma_N|^2 = 1 \ , \qquad (3.82)$$

$\lambda$ is a real number. Assuming that $\gamma_1, \dots, \gamma_{N-1}, \gamma_N \neq 0$ (cf. the example 1 above) we prove non-degeneracy of the matrix $(C_{ij})$. Hence one may suppose the points $\varkappa_1, \dots, \varkappa_N$ being situated in the upper half-plane. In view of non-negative definiteness of the matrix $(\frac{1}{i} C_{ij})$ we have one of the following conditions for the points $\varkappa_1, \dots, \varkappa_N$ and $\lambda$ being valid:

$$1)\ \lambda > 0 \ , \ \operatorname{Im} E(\varkappa_i) > 0 \iff |\varkappa_i - k_1| > b_1 \ ; \quad i = \overline{1,N} \qquad (3.83)$$

$$2)\ \lambda < 0 \ , \ \operatorname{Im} E(\varkappa_i) < 0 \iff |\varkappa_i - k_1| < b_1 \ . \quad i = \overline{1,N} \qquad (3.84)$$

We have the U(1,1)-NLS for the first possibility and the U(0,2)-NLS for the second. Hence if points $\varkappa_1, \dots, \varkappa_N$ are situated in the upper half-plane and satisfy (3.83) or (3.84) and the matrix $(C_{ij})$ has the form (3.81) (in these formulae we put $E(k) = k + b_1^2(k - k_1)^{-1}$), then the corresponding function $\Psi(x,t,k)$ (see (2.15),(2.17),(2.18)) gives solution of U(1,1)-NLS or U(0,2)-NLS via the formulae

$$\begin{aligned} \Phi_1(x,t) &= b_1 \Psi(x,t,k_1) \\ \Phi_2(x,t) &= \sqrt{|\lambda|} \sum_{i=1}^{N} \gamma_i \operatorname*{res}_{k=\varkappa_i} \Psi(x,t,k) \ . \end{aligned} \qquad (3.85)$$

We shall see in what follows that these solutions actually describe a non-linear superposition of the one-soliton solutions (3.73). But firstly let us analyse the stick conditions. Let us suppose that for some i,j the stick conditions $E(\bar{\varkappa}_i) = E(\varkappa_j)$ are fulfilled. Then the i-th and j-th lines (and columns) of the matrix $(E_{ij})$ are zero. Hence in the i-th line of the matrix $(C_{ij})$ only the element $c_{ij}$ might be nonzero. But the numbers $\varkappa_i$, $\varkappa_j$ are situated in the same half-plane because of the stick condition. Hence the definiteness of the corresponding block of the matrix $(C_{ij})$ can be valid only for j=i. That means that the stick condition has the following form:

$$E(\bar{\varkappa}_i) = E(\varkappa_i) \iff |\varkappa_i - k_1| = b \ . \qquad (3.86)$$

As a consequence we have the general form of the matrix $(C_{ij})$ which one needs to construct the solutions of considered type to the U(1,1)-NLS or the U(0,2)-NLS:

$$C_{ij} = \begin{cases} \lambda \dfrac{\bar{\gamma}_i \gamma_j}{E(\bar{\varkappa}_i) - E(\varkappa_j)} & \text{when } \lambda\{|\varkappa_i - k_1| - b_1\} > 0 \\ C_{ii}\delta_{ij} & , \ |\varkappa_i - k_1| = b_1 \ . \end{cases} \qquad (3.87)$$

One has the U(1,1)-symmetry for $\lambda > 0$ and the U(0,2)-symmetry for $\lambda < 0$. The solution is defined via formulae (3.85). The constants $\gamma_1, \dots, \gamma_N$ are satisfied (3.82).

Let us pay our attention to the calculation of asymptotics.

In the considered case all the points $\varkappa_1,\dots,\varkappa_N$ are situated in the upper half-plane and the matrix $(C_{ij})$ is non-degenerate. Hence we can use the asymptotic formula from s.2.2 . Let $t$ be fixed. Then for $x\to-\infty$ we have

$$\Phi(x,t)\to\begin{pmatrix} b_1 e^{ik_1(x+k_1t)} \\ 0\end{pmatrix}. \tag{3.88}$$

At $x\to+\infty$ we have

$$\Phi(x,t)\to\begin{pmatrix} b_1 \prod_j \dfrac{k_1-\bar\varkappa_i}{k_1-\varkappa_i}\, e^{ik_1(x+k_1t)} \\ 0\end{pmatrix}. \tag{3.89}$$

Let us calculate the asymptotics at $|t|\to\infty$ and fixed $x$ under assumption $\operatorname{Im}\varkappa_i^2>0$ for $i=1,\dots,N-1$, $\operatorname{Im}\varkappa_N^2=0$. Using the formulae from s.2.2 we have at $t\to-\infty$:

$$\Phi_1(x,t)\to b_1\left\{1+\frac{1}{2}\,\frac{\bar\varkappa_N-\varkappa_N}{k_1-\varkappa_N}\left(1+\operatorname{th}\beta_N(x-x_0^-)\right)\right\}e^{ik_1(x+k_1t)}, \tag{3.90}$$

$$\Phi_2(x,t)\to\frac{\varkappa_N-\bar\varkappa_N}{|\varkappa_N-k_1|}\left(\left||\varkappa_N-k_1|^2-b_1^2\right|\right)^{1/2}\cdot\frac{\exp i(-\beta_N^2t+\varphi_2^-)}{2\operatorname{ch}\beta_N(x-x_0^-)}.$$

Here $x_0^-$ is defined via (2.68).

$$\varphi_2^-=\arg\gamma_N-\arg\prod_{s\neq N}\frac{E_N-\bar E_s}{E_N-E_s}, \tag{3.91}$$

where we defined $E_s$ as follows:

$$E_s=E(\varkappa_s)=\varkappa_s+\frac{b_1^2}{\varkappa_s-k_1}. \tag{3.92}$$

For $t\to+\infty$ the asymptotics has the following form:

$$\Phi_1(x,t)\to b_1\left\{1+\frac{1}{2}\,\frac{\bar\varkappa_N-\varkappa_N}{k_1-\varkappa_N}\left(1+\operatorname{th}\beta_N(x-x_0^+)\right)\right\}e^{ik_1(x+k_1t)+i\varphi_1^+}$$

$$\Phi_2(x,t)\to\frac{(\varkappa_N-\bar\varkappa_N)}{|\varkappa_N-k_1|}\left\{\left||\varkappa_N-k_1|^2-b_1^2\right|\right\}^{1/2}\cdot\frac{\exp i(-\beta_N^2t+\varphi_2^+)}{2\operatorname{ch}\beta_N(x-x_0^+)}. \tag{3.93}$$

The phase $x_0^+$ has the form (2.75) and



$$\varphi_1^+=\arg\prod_{j\neq N}\frac{k_1-\bar\varkappa_j}{k_1-\varkappa_j}, \tag{3.94}$$

$$\varphi_2^+=\arg\gamma_N+\arg\prod_{s\neq N}\frac{\varkappa_N-\bar\varkappa_s}{\varkappa_N-\varkappa_s}. \tag{3.95}$$

We have obtained the one-soliton asymptotics. The interaction between solitons is pairwise as can be shown by simple calculations. This follows from the formulae for the phase shifts of the N-th soliton:

$$\begin{aligned}
\Delta x_0 &= x_0^+-x_0^- = \sum_{j\neq N}\frac{1}{\beta_N}\ln\left\{\left|\frac{E_N-\bar E_j}{E_N-E_j}\right|\left|\frac{\varkappa_N-\bar\varkappa_j}{\varkappa_N-\varkappa_j}\right|\right\},\\
\Delta\varphi_1 &= \varphi_1^+-\varphi_1^- = \sum_{j\neq N}\arg\frac{k_1-\bar\varkappa_j}{k_1-\varkappa_j},\\
\Delta\varphi_2 &= \varphi_2^+-\varphi_2^- = \sum_{j\neq N}\arg\frac{(\varkappa_N-\bar\varkappa_j)(E_N-\bar E_j)}{(\varkappa_N-\varkappa_j)(E_N-E_j)}.
\end{aligned} \tag{3.96}$$

In conclusion let us note that in the case of U(2,0)-symmetry (where $\varepsilon_1=-1$) the multi-soliton solutions are the non-linear superposition of one-soliton and also of double solitons. The simplest double soliton corresponds to the case N=2 with the points $\varkappa_1$, $\varkappa_2$ being satisfied the stick conditions

$$\bar\varkappa_1-\frac{b_1^2}{\bar\varkappa_1-k_1}=\varkappa_2-\frac{b_1^2}{\varkappa_2-k_1}, \tag{3.97}$$

and the matrix $(C_{ij})$ having the form

$$(C_{ij})=\begin{pmatrix}0 & C_{12}\\ C_{21} & 0\end{pmatrix}. \tag{3.98}$$

We shall not discuss properties of such solutions.

## Conclusion

We give above a modern state of problems related to a class of models we name Bose-gas models. From the point of view of condensed matter theory there arises an important question whether localized excitations of the soliton (or soliton-like) type can exist in a given ordered system (crystals, ferromagnets and so on). To understand statistical properties of such excitations (if any) the stability problem of a separate soliton-like object and that of their interaction should be solved. A part of these have been solved by the constructive way for the above models associated with a nonstationary Schrödinger equation. Namely: the general method developed in chapter II was applied to get and study asymptotic behaviour of multi-soliton solutions to some integrable versions of NLS with selfconsistent potentials. Such solutions describe well a dilute soliton gas and one can tell about an ideal gas, weakly non-ideal gas and so on depending on the result of soliton interactions.

First we discuss formulae given in chapter III having in mind their stability. It is well known that plane wave solutions (condensate) and those obtained from them via local modifications are unstable in the framework of compact versions of the VNLS with attraction (U(p,o) versions). The instability is of a gravitational type. Unlikely, condensate solutions are stable for compact versions of the VNLS with repulsion (U(o,q) versions) /2,26/. The stability of localized solutions under vanishing boundary conditions in the case of U(p,o) VNLS and the condensate boundary conditions in the case of U(o,q) VNLS is stated rigorously for only some simplest (one-soliton) solutions /26,27/. Question is still open of stability of arbitrary non-soliton solutions to U(p,o) NLS and the answer aparently depends on the type of equation as well as the solution under consideration. At any rate one-soliton solutions to U(p,o) VNLS are stable that follows from qualitative ideas based on the inverse transform (see also a generalization of the Q-theorem given in /27/).

Stability of the condensate for non-compact U(p,q) models is given by the condition

$$(\Psi_c, \Psi_c) = \sum_1^p |\Psi_c^{(i)}|^2 - \sum_1^q |\Psi_c^{(i)}|^2 > 0 .$$

The above multi-soliton formulae make sense under condensate boundary conditions only when this condition holds.

Stability of the two-soliton solutions (according to our definition and one-soliton solutions in a naive one) has been investigated by means of computer in Dubna (JINR) for the simplest non-compact U(1,1) VNLS. Results tell in favour of stability of such solitons. Multi-soliton solutions asymptotical behaviour obtained above makes us to be sure that the intéraction between solitons is reduced to the pair elastic interaction in the framework of compact models (with an arbitrary signature viz.,U(p,o) see also /28/, or U(o,q)). This interaction results only in changing soliton phases in the usual space and in the colour one. The change of colour as a result of the soliton interaction is possible as well which was established first in /1/.

All this means that even in the framework of one model gas of soliton-like excitations may be regarded as an ideal one (the soliton density is less than unity) and at the same time as a non- ideal gas if one considers e.g. the colour exchange.

There are physical situations when the soliton gas can be with a sufficient accuracy regarded as an ideal gas, then one can imploy a phenomenological approach /13/ for calculating e.g. dynamical structure factors /14/ in vector models and the signature of the colour space metric is arbitrary at $N \geqslant 2$.

In this sense the method to study the vector NLS equation proposed above can be thought of as a tool for the furhter research of corresponding models, in particular, those of condense matter physics (see chapter I).



## References


1. S.V.Manakov. ZETF, 1973, 65, p.505-516 (in Russian).
2. V.G.Makhankov, O.K.Pashaev. TMF, 1982, 53 p.55-67 (in Russian).
3. N.Yajima, M.Oikawa. Progr.Theor.Phys., 1976, 56, p.1719-1739.
4. K.Nishikawa et al. Phys.Rev.Lett., 1974, 33, p.148-151.
5. V.G.Makhankov. Phys.Lett., 1974, 50A, p.42-44.
6. A.Kundu, V.Makhankov, O.Pashaev. Physica, 1984, 11D, p.375-380.
7. A.S.Davydov. Solitons in molecular systems. Naukova Dumka. Kiev, 1984, p.287 (in Russian).
8. L.J.Jongh, A.R.Miedema. Adv. in Physics, 1974, 23, p.1-260.
9. A.A.Ovchinnikov, I.I.Ukrainskii, G.F.Kvencel'. UFN, 1972, 108, p.81-112 (in Russian).
10. M.Ito. Progr.Theor.Phys., 1981, 65, p.1773-1786.
A.B.Zolotovitzkii, V.P.Kalashnikov. TMF, 1981, 49, p.273-282 (in Russian).
11. H.Shiba. Progr.Theor.Phys., 1972, 48, p.2171-2186.
R.A.Bari. Phys.Rev., 1973, B7, p.4318-4320.
I.Egri. Solid State Comm., 1975, 17, p.441-444.
12. V.P.Kalashnikov, N.V.Kozhevnikov. TMF, 1978, 37, p.402-415.
A.B.Zolotovitzkii, V.P.Kalashnikov. Phys.Lett., 1982, 88A, p.315-317.
13. J.Krumhansl, J.Schrieffer. Phys.Rev., 1975, B11, p.3535-3545.
14. V.G.Makhankov, V.K.Fedyanin. Phys.Rep., 1984, 104, p.1-86.
15. L.M.Degtyarev, et al. ZETF, 1974, 67, p.533-542 (in Russian).
16. V.E.Zakharov. ZETF, 1972, 62, p.1745-1759 (in Russian).
17. V.Makhankov, O.Pashaev, A.Kundu. Phys.Scripta, 1983, 28, p.229-234.
18. K.A.Gorshkov, L.A.Ostrovskii. Physica, 1983, 3D, p.428-438.
19. V.I.Karpman. Phys.Scripta, 1979, 20, p.462-478.

20. A.M.Kosevich, B.A.Ivanov, A.S.Kovalev. Non-linear waves of magnetization. Magnetic solitons. Kiev, Naukova Dumka, 1983, p.192 (in Russian).
21. V.G.Makhankov. Phys.Rep.,1978, 35C, p.1-128;
CPC, 1980, 21, p.1-49; Particles and Nuclei, 1983, 14, p.123-180.
22. I.V.Cherednik. Functional analysis and application. 1978, 12, p.45-52 (in Russian).
23. I.M.Krichever. Functional analysis and application. 1986, 20, Num.3, p.42-54 (in Russian).
24. V.M.Eleonskii, I.M.Krichever, N.E.Kulagin. DAN, 1986, 287, p.606-610 (in Russian).
25. A.Scott, F.Chu, D.Malaughlin. Proc. IEEE, 1973, 61, p.1443-1483.
26. V.G.Makhankov, O.K.Pashaev, S.Sergeenkov. Phys.Lett.,1983, 98A, p.227-232; Physica Scripta, 1984, 29, p.521-525.
27. I.V.Barashenkov. Acta.Phys.Austr.,1983, 55, p.155-165.
28. P.P.Kulish. Works of FVE Conference, Protvino, 1980, p.463-470.
29. Physica Scripta, 1979, 20. Special Soliton Issue.
30. Soliton Theory. Method of Inverse Problem. Editor S.P.Novikov. M.Nauka, 1979, p.318 (in Russian).
31. V.G.Makhankov. Phys.Lett.,1981, 81A, p.156-163.

Received by Publishing Department
on September 24, 1987.



Дубровин Б.А. и др. E5-87-710
Точные решения нестационарного уравнения Шредингера с самосогласованными потенциалами

В рамках единого подхода дается описание интегрируемых моделей,связанных с нестационарным уравнением Шредингера,вместе с построением их многосолитонных формул. К ним относятся векторные НУШ, модель Яджимы-Ойкавы и др. При построении явных решений не используются коммутационные представления. Рассмотрены конденсатные граничные условия для некомпактных моделей, где стандартная техника обратной задачи неконструктивна. Предлагаемый подход основывается на алгебро-геометрической теории интегрируемых систем и позволяет эффективно строить все известные на сегодняшний день их явные решения. Обзор содержит ряд оригинальных результатов и написан в доступной для не математиков форме.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Dubrovin B.A. et al. E5-87-710
Exact Solutions to a Time Dependent Schrödinger Equation with Selfconsistent Potential

Description of integrable models associated with a time-dependent Schrödinger equation is given along with constructing their multisoliton formulae. Among such models there are the vector versions of NLS, Yajima-Oikawa model and others. In constructing exact solutions the communication relations are not used. The condensate boundary conditions are considered for noncompact models where the conventional technique of the inverse transform is not effective. The proposed approach is based on the algebro-geometrical theory of integrable systems and allows to construct all known by now exact solutions of such systems.The review contains a number of original results and is addressed to nonmathematicians.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987