ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E5-87-489

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A TAUBERIAN THEOREM FOR THE GENERALIZED S_2-TRANSFORM OF DISTRIBUTIONS

Submitted to "Mathematische Nachrichten"

1987

1. Introduction

Asymptotic behaviour of distributions plays a fundamental role in the analysis of singularities of integral transforms. Here we use the technique of quasiasymptotics of distributions (Zavialov /6/) in order to obtain some Tauberian theorems for the generalized S_2-transform.

The notation and terminology of this work will follow that of /5/ and Vladimirov et al. /6/ . Throughout the paper $\alpha, \beta, \gamma, \rho, \nu, a, b, k$ are real numbers, p,n are non-negative integers; S and S' denote respectively the space of test (C^∞-rapid decreasing) functions and tempered distributions on the real line. The elements f with the property $\text{supp}\, f \in [0,\infty)$ form a subspace in S' which we denote by S'_+ . S_+ denotes the space of C^∞ on $[0,\infty)$ functions equipped with a topology induced by S .

2. Quasiasymptotic behaviour of distributions

The quasiasymptotic behaviour of distributions $f \in S'$, respectively $f \in S'_+$, was introduced by B.I.Zavialov and later analysed in /6/ . A natural scale for the definition of quasiasymptotic behaviour is the class of regular varying functions /4/ . Therefore, we shall start with the definition.

A function $\varkappa(k)$ which is positive and continuous on $R_+ = (0,\infty)$ is called regular varying if for any $a > 0$ there exists the limit (depending on a)

$$\lim_{k \to \infty} \frac{\varkappa(ak)}{\varkappa(k)} = C(a) \neq 0$$

and the convergence is uniform with respect to any compact set of numbers a in R_+ /4/ .

It is not difficult to see that C(a) satisfies the functional equation

$$C(a) \cdot C(b) = C(a \cdot b)$$

from which it follows that $C(a) = a^{\gamma}$ for some real γ. In this case we call the function $\varkappa(k)$ regular varying of order γ.
The functions t^{γ} and $t^{\gamma}\ln^{\beta}(1+t)$ are regular varying of order γ. On the other hand, for all real β they are not asymptotically equivalent, i.e. the limit of their ratios as $t \longrightarrow +\infty$ does not exist or is equal to zero.
Let $f \in S'_+$ and $\varkappa(k)$ be a regular varying function of order γ. We say that the distribution f has in S'_+ a quasiasymptotic behaviour at infinity with respect to $\varkappa(k)$ if there exists the limit

$$\lim_{k\to\infty} \frac{f(kt)}{\varkappa(k)} = F(t) \quad \text{(in the sense of } S'_+ \text{)}$$

provided that $F \neq 0$ /6/.
One can prove that F is a homogeneous distribution of order γ (hence $F \in S'_+$) and $\operatorname{supp} F \subset \bar{R}_+$. Thus there exists a constant C such that $F(t) = C \cdot \theta_{\gamma+1}(t)$. Here

$$\theta_{\gamma+1}(t) = \theta(t)\,\frac{t^{\gamma}}{\Gamma(\gamma+1)} \qquad \text{for } \gamma > -1$$

and

$$\theta_{\gamma+1}(t) = \frac{d^n}{dt^n}\,\theta_{\gamma+n+1}(t) \qquad \text{for } \gamma \leqslant -1 \text{ and } \gamma+n > -1 .$$

As usual, $\theta(t)$ is the characteristic function of the interval R_+. The distributions $\theta(t)\,t$, $\theta(t)\ln t$, $\theta(t)\sin t$, $\delta(t)$, $\delta'(t)$ have in S'_+ a quasiasymptotic behaviour with respect to t, $\ln(1+t)$, t^{-1}, t^{-1}, t^{-2}.

3. The distributional generalized S_2-transform

In this section we reproduce that part of the theory of distributional generalized S_2-transform present in /5/ that will be used in subsequent sections.
We define for infinitely differentiable complex-valued functions $\phi(t)$ on R_+ and $a,b \in R$ the set of seminorms

$$(3.1) \quad \mu_{a,b,p}(\phi) = \sup_{t\in R_+} t^{1-a+p}(1+t)^{a-b}\,|\phi^{(p)}(t)|$$

The test function Space $M_{a,b}$ is given by

$$M_{a,b} = \{\phi \in C^{\infty}(R_+) : \mu_{a,b,p}(\phi) < \infty \text{ for all } p \in Z_+\},$$

where Z_+ is the set of all non-negative integers. $M_{a,b}$ is equipped with the topology generated by the seminorms (3.1). A sequence $\{\phi_n(t)\}$, $\phi_n(t) \in M_{a,b}$, converges in $M_{a,b}$ to $\phi(t)$ if $\mu_{a,b,p}(\phi_n - \phi)$ tends to zero as n goes to infinity for each $p \in Z_+$. We also use the directed set of seminorms

$$\tilde{\mu}_{a,b,p}(\phi) = \sup_{0\leq p'\leq p} \mu_{a,b,p'}(\phi) .$$

It can be proved that the two families of seminorms $\mu_{a,b,p}$ and $\tilde{\mu}_{a,b,p}$ define the same topology on $M_{a,b}$.
Note that $M_{a,b}$ is a Fréchet-space. The space $M_{a,b}$ is not nuclear. The dual space $M'_{a,b}$ consists of all continuous linear functionals on $M_{a,b}$ and is equipped with the usual weak topology. Obviously we have $M'_{a,b} \subset S'_+$ for $a \leqslant 1$ and $M'_{a,b} \subset M'_{c,d}$ for $a \leqslant c$ and $d \leqslant b$.

The generalized S_2-transform of ordinary functions $f(t)$ is defined by

$$\mathcal{S}[f;\rho,\nu](z) = \int_0^{\infty} \mathcal{K}(z,t;\rho,\nu)\,f(t)\,dt , \quad z \in C ,$$
$$\frac{-\pi}{\max(1,\rho)} \leqslant \arg z \leqslant \frac{\pi}{\max(1,\rho)},$$

where

$$\mathcal{K}(z,t;\rho,\nu) = \int_0^{\infty} (z+y)^{-\rho}(y+t)^{-\nu}\,dy , \qquad \rho > 0,\ \nu > 0,\quad \rho+\nu > 1 .$$

Boas and Widder /1/ studied the S_2-transform in the case $\rho = \nu = 1$. Some properties of the generalized S_2-transform and a distributional extention have recently been given in /5/.

Theorem 3.1 /5, theorem 2/ :
Let
(3.2) $\alpha > \max(0, \nu-1)$, $\beta < \nu + \min(0, \rho-1)$.
Then the generalized S_2-transform maps $M_{\alpha,\beta}$ continuously into $M_{a,b}$ if

$$(3.3)\quad \begin{array}{llll} a \leqslant 1 + \min(0,1-\rho) & \text{and} & a < 1 & \text{if } \rho = 1 \\ a \leqslant 2-\rho-\nu+\alpha & \text{and} & a < 2-\rho & \text{if } \nu = \alpha \\ b \geqslant 1-\rho+\max(0,1-\nu) & \text{and} & b > 1-\rho & \text{if } \nu = 1 \\ b \geqslant 2-\rho-\nu+\beta & \text{and} & b > 2-\rho-\nu & \text{if } \beta = 0 . \end{array}$$

Now suppose that α, β, a, b satisfy conditions (3.2) and (3.3). Let $f \in M'_{a,b}$. For each $\phi \in M_{\alpha,\beta}$ we have by theorem 3.1 $\mathcal{S}[\phi; \rho, \nu] \in M_{a,b}$. Thus the adjoint mapping

$$\langle \mathcal{S}[f; \nu, \rho], \phi \rangle = \langle f, \mathcal{S}[\phi; \rho, \nu] \rangle \tag{3.4}$$

defines the generalized S_2-transform $\mathcal{S}[f; \nu, \rho] \in M'_{\alpha,\beta}$ of $f \in M'_{a,b}$.

Remark that for ordinary functions with suitable integrability properties we may consider the integral

$$\int_0^\infty \int_0^\infty \mathcal{K}(x,t;\rho,\nu)\, f(x)\, \phi(t)\, dt\, dx .$$

If it is evaluated in two different ways, (3.4) follows.

The generalized S_2-transform can be inverted by using a differential operator of infinite order.

Let $L_{n,\rho,\nu}$ be an operator which acts on functions $\phi(t) \in C^\infty(R_+)$ as follows

$$L_{n,\rho,\nu}\, \phi(t) = L_{n,\rho,\nu,t}\, \phi(t) =$$

$$= \frac{\Gamma(\rho)\,\Gamma(\nu)}{n!\, n!\, \Gamma(n+\rho-1)\,\Gamma(n+\nu-1)} \left(\frac{d}{dt}\right)^n t^{2n+\nu-1} \left(\frac{d}{dt}\right)^{2n} t^{2n+\rho-1} \left(\frac{d}{dt}\right)^n \phi(t).$$

Theorem 3.2 /5, theorem 3/ :

Suppose

$$\alpha > \max(0, \nu-1), \quad \beta < \nu + \min(0, \rho-1)$$

and let $\phi \in M_{\alpha,\beta}$.

Then the sequence $\{ L_{n,\rho,\nu}\, \mathcal{S}[\phi; \rho, \nu] \}$ converges in $M_{\alpha,\beta}$ to ϕ.

The proof follows from the estimate

$$\begin{aligned} &\mu_{\alpha,\beta,p}(L_{n,\rho,\nu}\, \mathcal{S}[\phi; \rho, \nu] - \phi) \le \\ &\le \varepsilon_n \left(\mu_{\alpha,\beta,p+1}(\phi) + p\, \mu_{\alpha,\beta,p}(\phi) \right) \end{aligned} \tag{3.5}$$

where $\varepsilon_n \longrightarrow 0$ if $n \longrightarrow \infty$.

Let

$$\gamma > 1-\rho + \max(0, 1-\nu), \quad \delta < 1 + \min(0, 1-\rho)$$

and consider $\phi \in M_{\gamma,\delta}$. Then the following commutation relation holds for the operator $L_{n,\rho,\nu}$ /5, lemma 6/ :

$$\begin{aligned} &L_{n,\rho,\nu,x}\, \mathcal{S}[t^{\rho+\nu-2} \phi(t); \rho, \nu]\,(x) = \\ &= x^{\rho+\nu-2}\, \mathcal{S}[L_{n,\rho,\nu,t}\, \phi(t); \rho, \nu]\,(x). \end{aligned} \tag{3.6}$$

Once for all in this paper suppose

$$\max(0, \nu-1) < \alpha, \beta < \nu + \min(0, \rho-1) \tag{3.7}$$

and put

$$\begin{aligned} a &= 2-\rho-\nu+\alpha \\ b &= 2-\rho-\nu+\beta . \end{aligned} \tag{3.8}$$

In this case the results on the inversion of the generalized S_2-transform can be summarized as follows:

Let $f \in M'_{\alpha,\beta}$, $\phi \in M_{\alpha,\beta}$.

Then

$$\begin{aligned} &\langle \mathcal{S}[L_{n,\nu,\rho} f; \nu, \rho], \phi \rangle = \\ &= \langle f, L_{n,\rho,\nu}\, \mathcal{S}[\phi; \rho, \nu] \rangle \longrightarrow \langle f, \phi \rangle \end{aligned} \tag{3.9}$$

as $n \longrightarrow \infty$.

Let $f \in M'_{a,b}$, $\phi \in M_{a,b}$.

Then

$$\begin{aligned} &\langle L_{n,\nu,\rho}\, \mathcal{S}[f; \nu, \rho], \phi \rangle = \\ &= \langle f, \mathcal{S}[L_{n,\rho,\nu} \phi; \rho, \nu] \rangle \longrightarrow \langle f, \phi \rangle \end{aligned} \tag{3.10}$$

as $n \longrightarrow \infty$.

To get the Tauberian theorem, we need the additional

Lemma 3.3 :

Suppose α, β, a, b as in (3.7) and (3.8).

Then the set

$$\mathfrak{A} = \{ \mathcal{S}[\psi; \rho, \nu] \; : \; \psi \in M_{\alpha,\beta} \}$$

is dense in the space $M_{a,b}$.

Proof:

Consider $f \in M'_{a,b}$ and suppose $\langle f, \mathcal{S}[\psi; \rho, \nu] \rangle = 0$ for every $\psi \in M_{\alpha,\beta}$. Let $\phi \in M_{a,b}$. Then for all $n \in Z_+$ $L_{n,\rho,\nu} \phi \in M_{\alpha,\beta}$ so that $\langle f, \mathcal{S}[L_{n,\rho,\nu} \phi; \rho, \nu] \rangle = 0$. By (3.10) $\langle f, \mathcal{S}[L_{n,\rho,\nu} \phi; \rho, \nu] \rangle$ converges to $\langle f, \phi \rangle$ as $n \longrightarrow \infty$, so that $\langle f, \phi \rangle = 0$ for every $\phi \in M_{a,b}$ and consequently $f = 0$ in $M'_{a,b}$. This means that the set $\mathfrak{A}$ is dense in the space $M_{a,b}$.

4. The main theorem

Theorem 4 :
Let $f \in M'_{a,b}$ and $\varkappa(k)$ be a regular varying function of order γ, $b < -\gamma < a$. Let $\mathcal{S}[f;\nu,\varrho] \in M'_{\alpha,\beta}$ and suppose α,β,a,b as in (3.7) and (3.8).

Then the following statements are equivalent:

i) f has in $M'_{a,b}$ a quasiasymptotic behaviour at infinity with respect to $\varkappa(k)$.

ii) $\mathcal{S}[f;\nu,\varrho]$ has in $M'_{\alpha,\beta}$ a quasiasymptotic behaviour at infinity with respect to $\varkappa(k)\cdot k^{2-\varrho-\nu}$ and the set $\mathfrak{M}$

$$\mathfrak{M} = \bigcup_{k \geq k_0} \left\{ \frac{1}{\varkappa(k)} (L_{n,\nu,\varrho}\, \mathcal{S}[f;\nu,\varrho])(kx) \; : n \in Z_+ \right\} \tag{4.1}$$

is bounded in $M'_{a,b}$.

Proof:

i) $\longrightarrow$ ii).

Let $\phi \in M_{a,b}$ and $\psi \in M_{\alpha,\beta}$. We have

$$\lim_{k\to\infty} \frac{1}{\varkappa(k)} \langle f(kt), \phi(t) \rangle = \langle g(t), \phi(t) \rangle$$

For each $\psi \in M_{\alpha,\beta}$ by theorem 3.1 $\mathcal{S}[\psi;\varrho,\nu] \in M_{a,b}$. Hence

$\langle g(t), \mathcal{S}[\psi;\varrho,\nu] \rangle =$

$$\begin{aligned} &= \lim_{k\to\infty} \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[\psi;\varrho,\nu](t) \rangle \\ (4.2)\quad &= \lim_{k\to\infty} \frac{1}{k^{2-\varrho-\nu}\varkappa(k)} \langle \mathcal{S}[f;\nu,\varrho](kx), \psi(x) \rangle \\ &= \langle \mathcal{S}[g;\nu,\varrho](x), \psi(x) \rangle . \end{aligned}$$

The second equality follows from the homogeneity of the kernel of the generalized S_2-transform

$$\mathcal{K}(\alpha x, \alpha t; \varrho,\nu) = \alpha^{1-\varrho-\nu} \mathcal{K}(x,t;\varrho,\nu)$$

so that

$$\mathcal{S}[\phi(t);\varrho,\nu](kx) = k^{2-\varrho-\nu} \int_0^\infty \mathcal{K}(x,t;\varrho,\nu)\phi(kt)\, dt$$

and the last equality is a consequence of (3.4).

(4.2) means that $\mathcal{S}[f;\nu,\varrho]$ has in $M'_{\alpha,\beta}$ the quasiasymptotic behaviour at infinity with respect to $k^{2-\varrho-\nu}\varkappa(k)$.

Property (4.1) follows from the existence of the quasiasymptotic behaviour at infinity of f in $M'_{a,b}$ and (3.10). Really, because f has a quasiasymptotic behaviour with respect to $\varkappa(k)$ the set $\mathfrak{M}_1$

$$\mathfrak{M}_1 = \bigcup_{k \geq k_0} \frac{1}{\varkappa(k)} f(kt)$$

is weakly bounded in $M'_{a,b}$. Since the space $M_{a,b}$ is a Fréchet-space, $\mathfrak{M}_1$ is uniform bounded on bounded subsets of $M_{a,b}$, i.e. if $\mathcal{N} \subset M_{a,b}$ is a bounded subset, then there exists a constant $C_{\mathcal{N}}$ such that

$$\sup_{k \geq k_0} \left| \left\langle \frac{1}{\varkappa(k)} f(kt), \phi(t) \right\rangle \right| \leq C_{\mathcal{N}} \quad \forall\, \phi \in \mathcal{N} .$$

Consider now $\phi \in M_{a,b}$ arbitrary fixed and let

$$\mathcal{N}_1 = \{ \mathcal{S}[L_{n,\varrho,\nu}\phi \; ; \varrho,\nu] \; : n \in Z_+ \} .$$

Because the sequence $\{ \mathcal{S}[L_{n,\varrho,\nu}\phi ; \varrho,\nu] \}$ converges in $M_{a,b}$ to ϕ, the set $\mathcal{N}_1$ is a bounded subset of $M_{a,b}$ so that

$$\sup_{\substack{k \geq k_0 \\ n \in Z_+}} \left| \left\langle \frac{f(kt)}{\varkappa(k)}, \mathcal{S}[L_{n,\varrho,\nu}\phi \; ; \varrho,\nu](t) \right\rangle \right|$$

$$= \sup_{\substack{k \geq k_0 \\ n \in Z_+}} \left| \frac{1}{\varkappa(k)} \langle (L_{n,\nu,\varrho}\, \mathcal{S}[f;\nu,\varrho])(kx), \phi(x) \rangle \right| \leq C_{\phi} .$$

Consequently $\mathfrak{M}$ is bounded in $M'_{a,b}$.

ii) $\longrightarrow$ i).

We have

$$\lim_{k\to\infty} \frac{1}{k^{2-\varrho-\nu}\varkappa(k)} \langle \mathcal{S}[f;\nu,\varrho](kx), \psi(x) \rangle =$$

$$= \langle \mathcal{S}[g;\nu,\varrho](x), \psi(x) \rangle$$

so that

$$\lim_{k\to\infty} \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[\psi;\varrho,\nu](t)\rangle =$$

$$= \langle g(t), \mathcal{S}[\psi;\varrho,\nu](t)\rangle .$$

By lemma 3.3 this means that the limit

$$\lim_{k\to\infty} \frac{1}{\varkappa(k)} f(kt)$$

exists on a dense set of elements of the space $M_{a,b}$. If we show that the set $\mathfrak{M}_1$

$$\mathfrak{M}_1 = \bigcup_{k\geq k_0} \frac{1}{\varkappa(k)} f(kt)$$

is bounded in $M'_{a,b}$, then by the theorem of uniform convergence f has a quasiasymptotic behaviour with respect to $\varkappa(k)$.

Let $\phi \in M_{a,b}$. From (4.1) we have

$$\sup_{\substack{k\geq k_0\\ n\in Z_+}} \left|\frac{1}{\varkappa(k)}\langle (L_{n,\nu,\varrho}\, \mathcal{S}[f;\nu,\varrho])(kx), \phi(x)\rangle\right| \leq C_\phi$$

so that

$$\sup_{\substack{k\geq k_0\\ n\in Z_+}} \left|\frac{1}{\varkappa(k)}\langle f(kt), \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)\rangle\right| \leq C_\phi .$$

Fix at the moment arbitrary k , $k\geq k_0$, by (3.10) it follows that

$$\left|\frac{1}{\varkappa(k)}\langle f(kt), \phi(t)\rangle\right| \leq C_\phi .$$

Hence

$$\sup_{k\geq k_0} \left|\frac{1}{\varkappa(k)}\langle f(kt), \phi(t)\rangle\right| \leq C_\phi .$$

This means $\mathfrak{M}_1$ is bounded in $M'_{a,b}$.

The theorem is proved.

Remarks:

1. Since $-1 \leq -a < \gamma$ we have $g(t) = C\ \theta_{\gamma+1}(t)$. Because $b < -\gamma < a$, it follows that $g(t) \in M'_{a,b}$.

2. By formula 2.2.4.24 from /3/ we have

$$\mathcal{S}[\theta_\gamma;\nu,\varrho] = B(\gamma,\varrho-\gamma)\cdot B(\gamma+1-\varrho,\nu+\varrho-1-\gamma)\,\theta_{\gamma+2-\varrho-\nu}$$

$\max(0,\varrho-1) < \gamma < \varrho + \min(0,\nu-1)$,

where $B(i,j)$ is the usual Beta function. Thus

$$\mathcal{S}[g;\nu,\varrho](x) = C'\ \theta_{\gamma+3-\varrho-\nu}(x) .$$

5. Non-negative measures

In this section we show that our condition (4.1) is more general than the usual Tauberian condition by which $f(t)$ is a non-negative measure.

First we give a description of non-negative elements of $M'_{a,b}$ with the help of the generalized S_2-transform. This description is a straightforward verification of the classical one given by Boas and Widder in the case $\varrho = \nu = 1$. Remember, f is a non-negative element of a space of distributions if for every non-negative test function $\phi(t) \geq 0$ the inequality $\langle f, \phi\rangle \geq 0$ is valid.

Lemma 5.1 :

Let $f \in M'_{a,b}$,

then the following statements are equivalent:

i) f is non-negative.

ii) For every $n \in Z_+$ $L_{n,\nu,\varrho}\,\mathcal{S}[f;\nu,\varrho]$ is non-negative.

Proof:

i) $\longrightarrow$ ii) .

Let $\phi \in M_{a,b}$ and suppose $\phi(t) \geq 0$. Because of equality (3.6) ,

$$\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](x) =$$

$$= x^{2-\varrho-\nu}\, L_{n,\varrho,\nu,x}\, \mathcal{S}[t^{\varrho+\nu-2}\phi(t);\varrho,\nu](x)$$

$$= \frac{\Gamma(2n+\varrho)\,\Gamma(2n+\nu)}{n!\; n!\; \Gamma(n+\varrho-1)\,\Gamma(n+\nu-1)} \int_0^\infty\int_0^\infty \frac{x^n\, y^{2n}\, t^{n+\varrho+\nu-2}}{(x+y)^{2n+\varrho}\,(y+t)^{2n+\nu}}\,dy\,\phi(t)\,dt$$

so that $\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](x) \geq 0$. Thus

$$\langle L_{n,\nu,\varrho}\,\mathcal{S}[f;\nu,\varrho], \phi\rangle =$$

$$= \langle f, \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu]\rangle \geq 0 .$$

ii) $\longrightarrow$ i) .
Let $\phi \in M_{a,b}$, $\phi(t) \geq 0$ and suppose $\langle L_{n,\nu,\varrho}\, \mathcal{S}[f;\nu,\varrho], \phi \rangle \geq 0$, $n \in Z_+$. Using (3.4) we have

$$0 \leq \langle L_{n,\nu,\varrho}\, \mathcal{S}[f;\nu,\varrho], \phi \rangle = \langle f, \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu] \rangle .$$

By (3.10) $\langle f, \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu] \rangle$ converges to $\langle f, \phi \rangle$.
Hence $\langle f, \phi \rangle \geq 0$.
The lemma is proved.

Theorem 5.2 :
Let $f \in M'_{a,b}$ and let $\varkappa(k)$ be a regular varying function of order γ , $b < -\gamma < a$. Let $\mathcal{S}[f;\nu,\varrho] \in M'_{\alpha,\beta}$ and suppose α, β, a, b as in (3.7) and (3.8) . Suppose that $\mathcal{S}[f;\nu,\varrho]$ has in $M'_{\alpha,\beta}$ a quasiasymptotic behaviour at infinity with respect to $k^{2-\varrho-\nu}\varkappa(k)$ and suppose further that f is a non-negative element. Then condition (4.1) is valid.

Proof:
Let $f \in M'_{a,b}$. Thus $\mathcal{S}[f;\nu,\varrho] \in M'_{\alpha,\beta}$. Consider $\phi \in M_{a,b}$. Hence $L_{n,\varrho,\nu}\phi \in M_{\alpha,\beta}$ and $\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu] \in M_{a,b}$.
We have

$$\frac{1}{k^{2-\varrho-\nu}\,\varkappa(k)} \langle \mathcal{S}[f;\nu,\varrho](kx), L_{n,\varrho,\nu}\phi(x) \rangle =$$

$$= \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t) \rangle$$

and the limit for $k \longrightarrow \infty$ exists for every $n \in Z_+$.
Consider $\phi_0 = x^{\alpha-1}(1+x)^{\beta-\alpha} \in M_{\alpha,\beta}$. Because for $t > 0$

$$\mathcal{S}[\phi_0;\varrho,\nu](t) = \int_0^\infty \mathcal{K}(t,x;\varrho,\nu)\, x^{\alpha-1}(1+x)^{\beta-\alpha}\, dx > 0 ,$$

and f is non-negative, we have

$$\left| \frac{1}{k^{2-\varrho-\nu}\,\varkappa(k)} \langle \mathcal{S}[f;\nu,\varrho](kx), L_{n,\varrho,\nu}\phi(x) \rangle \right|$$

$$= \left| \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t) \rangle \right|$$

$$= \left| \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[\phi_0;\varrho,\nu](t) \frac{\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)}{\mathcal{S}[\phi_0;\varrho,\nu](t)} \rangle \right|$$

(5.1)

$$\leq \sup_{t \in R_+} \frac{|\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)|}{\mathcal{S}[\phi_0;\varrho,\nu](t)} \cdot \frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[\phi_0;\varrho,\nu](t) \rangle .$$

Since $\mathcal{S}[\phi_0;\varrho,\nu](t)$ is continuous, monotonically decreasing for $t > 0$ and

$$\mathcal{S}[\phi_0;\varrho,\nu](t) = O(t^{\alpha-\varrho-\nu+1}) = O(t^{a-1}) \quad t \longrightarrow +0$$

$$\mathcal{S}[\phi_0;\varrho,\nu](t) = O(t^{\beta-\varrho-\nu+1}) = O(t^{b-1}) \quad t \longrightarrow \infty$$

we can estimate the first term of (5.1) by

$$\sup_{t \in R_+} \frac{|\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)|}{\mathcal{S}[\phi_0;\varrho,\nu](t)}$$

$$= \sup_{t \in R_+} \frac{t^{1-a}(1+t)^{a-b}\, |\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)|}{t^{1-a}(1+t)^{a-b}\, \mathcal{S}[\phi_0;\varrho,\nu](t)}$$

(5.2)

$$\leq C_1 \sup_{t \in R_+} t^{1-a}(1+t)^{a-b} \, |\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)|$$

$$= C_1\, \mu_{a,b,0}(\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu]) .$$

By using inequality (3.5), (5.2) may be transformed into

$$(5.3) \quad \sup_{t \in R_+} \frac{|\mathcal{S}[L_{n,\varrho,\nu}\phi;\varrho,\nu](t)|}{\mathcal{S}[\phi_0;\varrho,\nu](t)} \leq C_2\, \tilde{\mu}_{a,b,1}(\phi) .$$

Because $\mathcal{S}[f;\nu,\varrho]$ has a quasiasymptotic behaviour at infinity the second term of (5.1) for $k \geq k_0$, k_0 sufficient large,

$$\frac{1}{\varkappa(k)} \langle f(kt), \mathcal{S}[\phi_0;\varrho,\nu](t) \rangle$$

(5.4)

$$= \frac{1}{k^{2-\varrho-\nu}\,\varkappa(k)} \langle \mathcal{S}[f;\nu,\varrho](kx), \phi_0(x) \rangle \leq C(\phi_0)$$

is bounded. Consequently, from inequalities (5.1) , (5.3) and (5.4) it follows that for $k \geq k_0$

$$\left| \frac{1}{\varkappa(k)} \langle (L_{n,\nu,\rho}\, \varphi[f;\nu,\rho])(kx), \phi(x) \rangle \right|$$

$$= \left| \frac{1}{k^{2-\rho-\nu}\varkappa(k)} \langle \varphi[f;\nu,\rho](kx), L_{n,\rho,\nu,x}\phi(x) \rangle \right|$$

$$\leq C\, \tilde{\mu}_{a,b,1}(\phi)$$

for every $\phi \in M_{a,b}$ where the constant C depends only on f, k_0 and ϕ_0. This means that in $M'_{a,b}$ the set $\mathfrak{M}$

$$\mathfrak{M} = \bigcup_{k \geq k_0} \left\{ \frac{1}{\varkappa(k)} (L_{n,\nu,\rho}\, \varphi[f;\nu,\rho])(kx) : n \in Z_+ \right\}$$

is bounded.

The theorem is proved.

6. Examples

1. Consider $\rho = \nu = 1$. Put $\varphi[f;1,1](x) = x^{\gamma-1} \ln x$, $x > 0$ and $0 < \gamma < 1$. Choose α, β such that $0 < \beta < 1-\gamma < \alpha < 1$. Let $a = \alpha$, $b = \beta$ (hence $\rho = \nu = 1$). We have $\theta(x)\, x^{\gamma-1} \ln x \in M'_{\alpha,\beta}$. Let further $\varkappa(k) = k^{\gamma-1}\ln(1+k)$. Obviously, $\theta(x)\, x^{\gamma-1} \ln x$ has in $M'_{\alpha,\beta}$ a quasiasymptotic behaviour at infinity with respect to $k^{\gamma-1}\ln(1+k)$. By direct computation we have for $x > 0$

$$L_{n,1,1}\, \varphi[f;1,1](x) = \frac{1}{(\Gamma(\gamma)\Gamma(1-\gamma))^2} \left\{ A_n x^{\gamma-1} \ln x + B_n x^{\gamma-1} \right\}$$

where

$$A_n = \left(\frac{\Gamma(n+\gamma)\Gamma(n+1-\gamma)}{n!\,(n-1)!} \right)^2$$

and B_n is a sum of Γ-functions.

From Stirling's formula it follows that A_n converges to 1 as n goes to infinity. By careful estimates and Stirling's formula it can be proved that B_n is bounded (the bound depends on γ) as n goes to infinity. Note further that

$$\frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} = \frac{\sin \pi\gamma}{\pi}.$$

We have for $k \geq 1$

$$\frac{1}{k^{\gamma-1}\ln(1+k)} \left| \langle L_{n,1,1}\, \varphi[f;1,1](kx), \phi(x) \rangle \right|$$

$$= \left(\frac{\sin \pi\gamma}{\pi}\right)^2 \frac{1}{k^{\gamma-1}\ln(1+k)} \left| \langle A_n (kx)^{\gamma-1} \ln kx + B_n (kx)^{\gamma-1}, \phi(x) \rangle \right|$$

$$\leq \left(\frac{\sin \pi\gamma}{\pi}\right)^2 A_n \left| \langle x^{\gamma-1}, \phi(x) \rangle \right| +$$

$$+ \left(\frac{\sin \pi\gamma}{\pi}\right)^2 \frac{1}{\ln(1+k)} \left| \langle A_n x^{\gamma-1} \ln x + B_n x^{\gamma-1}, \phi(x) \rangle \right| .$$

Hence condition (4.1) is valid. Thus the S_2-original f has in $M'_{a,b}$ a quasiasymptotic behaviour at infinity.

Remarks:

i) f is not non-negative.

ii) In this example f can be calculated explicitly. Look for $f(t)$ in the form $\theta(t)\, t^{\gamma-1}(A \ln t + B)$.

We have by (3.4) for $\phi \in M_{\alpha,\beta}$

$$\langle x^{\gamma-1} \ln x, \phi(x) \rangle = \langle t^{\gamma-1}(A \ln t + B), \varphi[\phi;1,1](t) \rangle$$

$$= \int_0^\infty t^{\gamma-1}(A \ln t + B) \int_0^\infty \frac{\ln x - \ln t}{x - t} \phi(x)\, dx\, dt$$

$$= \int_0^\infty \int_0^\infty \frac{\ln x - \ln t}{x - t} t^{\gamma-1}(A \ln t + B)\, dt\, \phi(x)\, dx$$

$$= \int_0^\infty \int_0^\infty \frac{\ln u}{u - 1} (ux)^{\gamma-1}(A \ln ux + B)\, du\, \phi(x)\, dx$$

$$= \int_0^\infty \left\{ x^{\gamma-1} \ln x\, A \int_0^\infty \frac{u^{\gamma-1}\ln u}{u-1} du + x^{\gamma-1} \int_0^\infty \frac{u^{\gamma-1}\ln u}{u-1}(A \ln u + B) du \right\} \phi(x)\, dx .$$

Choosing

$$A = \left(\int_0^\infty \frac{\ln u}{u-1} u^{\gamma-1}\, du \right)^{-1} = \left(\frac{\sin \pi\gamma}{\pi}\right)^2$$

and

$$B = - A^2 \int_0^\infty \frac{\ln u}{u-1} u^{\gamma -1} \ln u \, du = \frac{\sin 2\pi\gamma}{\pi}$$

we get the desired result.

2. The following example shows that the conditions (3.7) and (3.8) can't be essentially weakened.
Let $0 < \varrho < 1$, $\nu = 1$. Consider

$$f(t) = \Theta(t) \sqrt{t}\, e^{-\sqrt[4]{t}} \cos \sqrt[4]{t} \in M'_{1,b} \qquad 1-\varrho < b < 1 \quad .$$

Then for $\phi \in M_{1,b}$, $k \geq 1$

$$(6.1) \quad \left| \frac{1}{k^{-1}} \langle f(kt), \phi(t) \rangle \right| = \int_0^\infty \sqrt{t}\, e^{-\sqrt[4]{t}} \cos \sqrt[4]{t}\, \phi\left(\frac{t}{k}\right) dt$$
$$\leq \mu_{1,b,0}(\phi) \int_0^\infty \sqrt{t}\, e^{-\sqrt[4]{t}} \left(1 + \frac{t}{k}\right)^{b-1} dt \leq C\, \mu_{1,b,0}(\phi) \quad .$$

From the Lebesgue theorem it follows that

$$\lim_{k \to \infty} \int_0^\infty \sqrt{t}\, e^{-\sqrt[4]{t}} \cos \sqrt[4]{t}\, \phi\left(\frac{t}{k}\right) dt =$$
$$= \phi(0) \int_0^\infty \sqrt{t}\, e^{-\sqrt[4]{t}} \cos \sqrt[4]{t}\, dt = 0 \quad .$$

Hence if f has in $M'_{1,b}$ a quasiasymptotic behaviour of order γ , then $\gamma \leq -1$.
Further for all φ of the subspace S_+ of $M_{1,b}$ it can be proved that for every $\gamma \in R$

$$(6.2) \quad \lim_{k \to \infty} k^{\gamma} \langle f(kt), \varphi(t) \rangle = 0 \quad .$$

Really we have

$$\int_0^\infty \sqrt{kt}\, e^{-\sqrt[4]{kt}} \cos \sqrt[4]{kt}\, \varphi(t)\, dt =$$
$$(6.3) \quad = \sqrt{k} \cdot 4 \int_0^\infty u^5 e^{-\sqrt[4]{k}\, u} \cos\left(\sqrt[4]{k}\, u\right) \varphi(u^4)\, du \quad .$$

Because

$$e^{-at} \cos at = - \frac{(e^{-at} \cos at)^{(iv)}}{a^4}$$

by integrating (6.3) by parts we have

$$\int_0^\infty e^{-\sqrt[4]{k} \cdot u} \cos(\sqrt[4]{k}\, u) u^5 \varphi(u^4)\, du =$$
$$= \frac{1}{k} \int_0^\infty e^{-\sqrt[4]{k} \cdot u} \cos \sqrt[4]{k} \cdot u \left(\sum_{j=0}^{4} a_j u^{4j+1} \varphi^{(j)}(u^4)\right) dt$$

so that

$$\int_0^\infty \sqrt{kt}\, e^{-\sqrt[4]{kt}} \cos \sqrt[4]{kt}\, \varphi(t)\, dt =$$
$$(6.4) \quad = \frac{1}{k} \int_0^\infty \frac{e^{-\sqrt[4]{kt}} \cos \sqrt[4]{kt}}{\sqrt{kt}} \left(\sum_{j=0}^{4} a_j t^j \varphi^{(j)}(t)\right) dt \quad .$$

Iterating (6.4) as many times as desired we get

$$\int_0^\infty \sqrt{kt}\, e^{-\sqrt[4]{kt}} \cos \sqrt[4]{kt}\, \varphi(t)\, dt =$$
$$= \frac{1}{k^n} \int_0^\infty \frac{e^{-\sqrt[4]{kt}} \cos \sqrt[4]{kt}}{\sqrt{kt}} \sum_{j \leq 4n} P_j(t)\, \varphi^{(j)}(t)\, dt \quad .$$

From (6.1) , (6.2) follows. Hence for an arbitrary regular varying function f can't have in $M'_{a,b}$ a quasiasymptotic behaviour.
Because f is absolute summable, the generalized S_2-transform of f can be calculated as follows

$$\mathcal{S}[f;\varrho,1](x) = \int_0^\infty \mathcal{K}(x,t;\varrho,1)\, f(t)\, dt =$$
$$= \int_0^\infty \frac{dy}{(x+y)^{\varrho}} \int_0^\infty \frac{f(t)}{t+y}\, dt \quad .$$

Using formula 2.5.33.4 from /3/ we get

$$\mathcal{S}[f;\varrho,1](x) = \pi \int_0^\infty \frac{\sqrt{y}\, e^{-\sqrt[4]{4y}}}{(x+y)^{\varrho}}\, dy \quad .$$

Obviously $\mathcal{S}[f;\varrho,1](x) \in M'_{\alpha,\beta}$ if $\alpha > \varrho$, $\beta \leq 0$.

Let $\phi(x)\in M_{\alpha,\beta}$, $\nu(k) = k^{2-\varrho-\nu}\cdot k^{-1} = k^{-\varrho}$. Then

$$\lim_{k\to\infty} \frac{1}{k^{-\varrho}} \langle \mathcal{S}[f;\varrho,1](kx),\phi(x)\rangle =$$

$$= \lim_{k\to\infty} \pi k^{\varrho}\int_0^\infty \phi(x)\,dx \int_0^\infty \frac{\sqrt{y}\, e^{-\sqrt[4]{4y}}}{(kx+y)^{\varrho}}\,dy .$$

Because

$$\left| k^{\varrho}\int_0^\infty\int_0^\infty \frac{\sqrt{y}\, e^{-\sqrt[4]{4y}}}{(kx+y)^{\varrho}}\,\phi(x)\,dx\,dy\right| \leq$$

$$\leq \mu_{\alpha,\beta,0}(\phi)\int_0^\infty\int_0^\infty \frac{\sqrt{y}\, e^{-\sqrt[4]{4y}}}{(x+k^{-1}y)^{\varrho}}\, x^{\alpha-1}(1+x)^{\beta-\alpha}\,dx\,dy$$

$$\leq \mu_{\alpha,\beta,0}(\phi)\int_0^\infty\int_0^\infty \sqrt{y}\, e^{-\sqrt[4]{4y}}\, x^{\alpha-1-\varrho}(1+x)^{\beta-\alpha}\,dx\,dy$$

by the Lebesgue theorem, for arbitrary $\phi\in M_{\alpha,\beta}$

$$\lim_{k\to\infty} \frac{1}{k^{-\varrho}} \langle \mathcal{S}[f;\varrho,1](kx),\phi(x)\rangle$$

$$= \pi\int_0^\infty \sqrt{y}\, e^{-\sqrt[4]{4y}}\,dy \int_0^\infty \frac{\phi(x)}{x^{\varrho}}\,dx = c\,\langle\theta_{1-\varrho}(x),\phi(x)\rangle .$$

Hence $\mathcal{S}[f;\varrho,1](x)$ has in every $M'_{\alpha,\beta}$, $\alpha>\varrho$, $\beta<0$, a quasiasymptotic behaviour at infinity with respect to $k^{-\varrho}$.

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Received by Publishing Department
on June 30, 1987.

Трегер Г.

E5-87-489

Одна Тауберова теорема для обобщенного S_2-преобразования обобщенных функций

Асимтотическое поведение обобщенных функций играет существенную роль в исследовании сингулярных точек интегральных преобразований. С использованием техники квазиасимптотики описываются асимптотические соотношения для S_2-преобразования обобщенных функций. Условие Тауберова типа, которое сформулировано в данной работе, является более общим по сравнению с предположением о том, что S_2-прообраз есть положительная мера.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

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E5-87-489

A Tauberian Theorem for the Generalized S_2-Transform of Distributions

Asymptotic behaviour of distributions plays a fundamental role in the analysis of singularities of integral transforms. Using the technique of quasiasymptotics, we describe the asymptotic relations for the generalized S_2-transform of distributions. The Tauberian condition given here is more general than the assumption by which the S_2-original is a non-negative measure.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987