сообщения
объединенного
института
ядерных
исследований
дубна

E5-87-480

M.Znojil

THE DOUBLE ANHARMONIC POTENTIAL $ax^2 + bx^4 + cx^6$ AND THE EXTENDED CONTINUED FRACTIONS

1987

1. INTRODUCTION AND SUMMARY

All the salient features of the so-called Hill determinant method [1] may be illustrated on the sextic anharmonic oscillator

$$[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V_1(r)]\psi(r) = E_1\psi(r), \quad \ell = 0,1,\ldots .$$

$$V_1(r) = g_1 r^2 + g_2 r^4 + g_3 r^6, \quad g_3 > 0. \qquad (1.1)$$

This was proposed by Singh at al.[2] and the problem (1.1) has attracted much attention during the last few years (cf., e.g., the review by Hautot [3]). In fact, a simple change of variables

$$r \to r^{1/p}, \quad \psi \to r^{\text{const}}\psi, \quad \ell + \frac{1}{2} \to (\ell + \frac{1}{2})/p, \quad p = 2,3,4 \qquad (1.2)$$

may be used to convert the interaction $V = V_1$ in (1.1) into the other three equivalent forces V_p,

$$V_2(r) = \frac{1}{2} g_0 r^{-1} + 2g_2 r + 4g_3 r^2,$$

$$V_3(r) = g_0(3r)^{-4/3} + g_1(3r)^{-2/3} + g_3(3r)^{2/3}, \qquad (1.3)$$

$$V_4(r) = \frac{1}{8} g_0 r^{-3/2} + \frac{1}{4} g_1 r^{-1} + \frac{1}{2} g_2 r^{-1/2}$$

with the new coupling $g_0 = -E_1$ and with the new respective definitions of energies $E = E_p$,

$$E_2 = -g_1, \quad E_3 = -g_2, \quad E_4 = -g_3 \ (<0). \qquad (1.4)$$

Hence, a number of the physically interesting situations may be represented by a re-interpretation (1.2)-(1.3) of eq.(1.1) as well as of the corresponding power series ansatz

$$\psi(r) = \exp P(r) \sum_{n=0}^{\infty} h_n r^{2n+\ell+1}, \quad P(r) = \mu r^4 + \nu r^2. \qquad (1.5)$$

In accord with Hautot [3], the best numerical results may be obtained with certain optimal values of μ and ν in the polynomial $P(r)$. In the alternative, less numerically oriented approach of Singh et al.[2] emphasizing the simplicity of formulas, we have to consider $g_2 > 0$ and choose $P(r)$ as an exact WKB asymptotic estimate of $\psi(r)$. In the spirit of our preceding paper [4], we shall analyse here a compromise, with the WKB value of μ and with an arbitrary parameter at r^2,

$$P(r) = -(\frac{1}{2}\alpha)^2 r^4 - \frac{1}{2}\beta r^2, \quad \alpha^4 = g_3. \tag{1.6}$$

Such a choice of $P(r)$ has also been made in the purely numerical study by Killingbeck[5].

A genuine merit of our ansatz (1.5) + (1.6) lies in the straightforward numerical tractability of the eigenvalue problem (1.1) by means of the extended continued fractions (ECF[6]). As an alternative to the divergent perturbation series, the ECP formalism will be described here in detail, with a particular emphasis laid upon a systematic acceleration of its convergence. This will be our main result. Indeed, without such an improvement, the Hill determinants seem to give a slowly convergent results when compared with the standard variational procedures. In this respect, we may recall the extensive numerical analyses by Hautot[3], Killingbeck[5] and Tater[7] or Znojil[8] which motivated the present completion of the standard ECF construction[9].

2. THE ASYMMETRIC MATRIX SCHRÖDINGER EQUATION AND ITS TRUNCATION

An insertion of (1.5) and (1.6) in (1.1) leads to our basic set of linear equations

$$Q\begin{pmatrix} h_0 \\ h_1 \\ \dots \end{pmatrix} = E\begin{pmatrix} h_0 \\ h_1 \\ \dots \end{pmatrix}, \tag{2.1}$$

where

$$Q_{nn} = A_n = (4n + 2\ell + 3)\beta, \quad -Q_{nn+1} = B_n = (2n+2)(2n + 2\ell + 3),$$

$$Q_{n+1\,n} = C_{n+1} = (4n + 2\ell + 5)\alpha^2 + g_1 - \beta^2, \tag{2.2}$$

$$Q_{n+2\,n} = D = 2\alpha^2(\beta_0 - \beta), \quad \beta_0 = g_2/(2\alpha^2)$$

and $n = 0, 1, \dots$. Obviously, this is a non-variational eigenvalue problem. For $D = 0$, it coincides with the relations of Ref.[2], applicable for $g_2 > 0$ and $\beta_0 > 0$ only. Hence, we shall assume that $D \neq 0$ in what follows.

In a phenomenological context, the "inadmissible" couplings

$$g_2 \le 0 \tag{2.3}$$

correspond to the more interesting (e.g., double-well) interactions. Hence, after a discovery of a surprising failure of the Singh's analytic continued-fractional construction[2,3], various attempts of its improvement may be quoted, ranging from an analytic continuation[10] up to the fixed-point expansions[7] or non-WKB choices of $P(r)$[3]. In accord with our study [4], the choice (1.6) represents the simplest solution of the above inadmissibility puzzle: We proved that the restriction of β's,

$$\beta + \beta_0 > 0 \tag{2.4}$$

is sufficient for a possibility of truncating eq.(2.1). This will be a starting point here.

3. THE FACTORIZATION OF Q-E AND BINDING ENERGIES

For a large index $M \gg 1$, we may replace all the coefficients $Q_{M+m,\,M+n}$, $m,n \ge 1$ by zeros and obtain the truncated matrix form of our basic eq.(2.1),

$$Q^{(M)}\begin{pmatrix} h_0 \\ h_1 \\ \dots \\ h_M \end{pmatrix} = E^{(M)}\begin{pmatrix} h_0 \\ h_1 \\ \dots \\ h_M \end{pmatrix}. \tag{3.1}$$

Due to Ref.[4], we obtain the exact results in the limit $M \to \infty$.

The basic assumption of the present paper reads

$$Q^{(M)} - E = \begin{pmatrix} 1 & -B_0F_1^{(M)} & 0 & \dots & 0 \\ 0 & 1 & -B_1F_2^{(M)} & \dots & 0 \\ & \dots & & & \\ 0 & \dots & & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1/F_0^{(M)} & 0 & \dots & & 0 \\ G_1^{(M)} & 1/F_2^{(M)} & \dots & & 0 \\ D & G_2^{(M)} & \dots & & 0 \\ \dots & & & & \\ 0 & \dots & 0 & D & G_M^{(M)} \quad 1/F_M^{(M)} \end{pmatrix}. \tag{3.2}$$

With an abbreviation

$$G_k^{(M)} = C_k + B_k F_{k+1}^{(M)} D, \quad k = 1, 2, \dots, M, \tag{3.3}$$

this becomes an algebraic identity whenever the ECF auxiliary sequence $F_k^{(M)}$ may be defined by the recurrences

$$F_k^{(M)} = 1/(A_k - E + B_k F_{k+1}^{(M)} G_{k+1}^{(M)}), \quad k = 0,1,\dots,M \tag{3.4}$$

from the initial values

$$F_{M+1}^{(M)} = F_{M+2}^{(M)} = 0 \tag{3.5}$$

in the limit $M \to \infty$.

The standard secular equation related to eq.(3.1)

$$\det(Q^{(M)} - E^{(M)}) = 0 \tag{3.6}$$

may be given an alternative ECF form

$$\prod_{k=0}^{M} \frac{1}{F_k^{(M)}} = 0 \tag{3.7}$$

since determinant of the first factor in (3.2) is equal to one. Moreover, an existence of the ECF quantities $F_k^{(M)}$, $k \geq k_0$ necessitates a regularity of the inversion (3.4),

$$\frac{1}{F_k^{(M)}} \neq 0, \quad k \geq k_0 + 1 > 0. \tag{3.8}$$

Hence, we may re-write our secular equation in the simplified form

$$\prod_{k=0}^{k_0} \frac{1}{F_k^{(M)}} = 0 \tag{3.9}$$

which is more suitable for an analysis of the $M \to \infty$ limit.

In particular, we may often succeed in guaranteeing the regularity condition (3.8) with $k_0 = 0$, e.g., via a small change of the free parameter β. Then, we obtain the simplest ECF eigenvalue condition in the form

$$\lim_{M \to \infty} 1/F_0^{(M)} = 0. \tag{3.10}$$

Below, its formal aspects are to be analysed in more detail.

4. A REMOVAL OF SINGULARITIES IN THE AUXILIARY ECF SEQUENCE

On a set of measure zero, the values of couplings and energies may imply a non-existence of our basic factorization (3.2) in the limit $M \to \infty$. This may be handled as follows.

In a straightforward formulation, we may contemplate a direct limiting transition $M \to \infty$, $E \to E_{critical}$, or a similar limit for some of the coupling constants. The violation of (3.10) (characterized by an appearance of a spurious zero

$$1/F_{k_0+1} \doteq 0$$

at some index $k_0 \geq 0$) would lead to a singularity in the matrix element $B_{k_0} F_{k_0+1}$. Then, from (3.4), it is necessary to derive the explicit regularization rules of the type

$$F_{k_0+1} F_{k_0} \doteq 1/(B_{k_0} G_{k_0+1}).$$

In the discussion it is in fact sufficient to distinguish between the following two cases.

(i) we assume that

$$F_{k_0+2} \neq -C_{k_0+1}/(B_{k_0+1} D),$$

i.e., $G_{k_0+1} \neq 0$. Then, we have also $1/(F_{k_0} F_{k_0+1}) \neq 0$. The auxiliary ECF quantities remain well defined since $F_{k_0} = 0$. Equation (3.4) remains applicable for all $k < k_0$ in principle.

(ii) provided that

$$F_{k_0+2} = -C_{k_0+1}/(B_{k_0+1} D)$$

we get $G_{k_0+1} = 0$ and $1/F_{k_0} = A_{k_0}$. Now, equation (3.4) implies that $F_{k_0-1} = 0$ and the rest (F_{k_0-2}, etc.) may be evaluated without difficulties. In the indeterminate ECF product, we have to omit the ill-defined quantities and employ the regularization rule

$$1/(F_{k_0+1} F_{k_0} F_{k_0-1}) = -B_{k_0-1} B_{k_0} D \neq 0.$$

Again, the rest of our recurrences remains regular and well defined.

In the light of the preceding analysis, we may expect that k_0 is always smaller than some maximal value dependent on a range of couplings. Thus, an alternative and quite universal remedy of the singularity appearance may be found in a re-partitioning of the factorization. In the simplest case, we may eliminate a need of one singular matrix $B_{k_0} F_{k_0+1}$ by mere re-partitioning of the critical 3x3-dimensional submatrices in our factorization prescription - putting

$$\begin{pmatrix} \dots & A_{n-1} & -B_{n-1} & 0 & \dots \\ \dots & C_n & A_n & -B_n & \dots \\ \dots & D & C_{n+1} & A_{n+1} & \dots \end{pmatrix} =$$

$$\begin{pmatrix} \dots 1 & \overset{\cdots}{-B_{n-1}F_{\tilde{n}}} & -\tilde{\phi}_n & \dots \\ \dots 0 & 1 & 0 & \dots \\ \dots 0 & 0 & 1 & \dots \end{pmatrix} \begin{pmatrix} \dots 1/F_{n-1} & \overset{\cdots}{0} & 0 & \dots \\ \dots C_n & A_n & -B_n & \dots \\ \dots D & G_{n+1} & 1/F_{n+1} & \dots \end{pmatrix}$$

we may interpret the off-triangular matrix element

$$\tilde{\phi}_n = \frac{B_{n-1}}{G_{n+1} + A_n/(B_n F_{n+1})}, \quad n = k_0$$

as the needed regularization factor. The same trick may be repeated at a smaller singular index or extended to more dimensions.

5. AN ACCELERATION OF THE ECF CONVERGENCE AND AN OUTLINE OF ITS RIGOROUS PROFF

In the $n \gg 1$ asymptotic region, we may write

$$B_n = 4n^2(1 + O(1/n)),\ A = 4\beta n(1 + O(1/n)),\ C_n = 4\alpha^2 n(1 + O(1/n)) \quad (5.1)$$

and, omitting the superscipts, replace the exact ECF recurrences (3.4) by their asymptotic approximation of the leading-order form

$$\frac{1}{F_n} = 4\beta n + 16\alpha^2 n^3 F_{n+1} + 32\alpha^2(\beta_0 - \beta) n^4 F_{n+1} F_{n+2},\ n \gg 1. \quad (5.2)$$

In fact, this is equivalent to considering the parameters as accompanied by the implicit error-estimate factors,

$\alpha^2 = \alpha^2 (1 + O(1/n)), \quad \beta = \beta(1 + O(1/n)), \dots .$

Such a notation is very useful - we may solve exactly the asymptotic ECF recurrences (5.2) now,

$$F_{N-2k} = 1/\{[2k(\beta + \beta_0) + 4\beta]N\},$$

$$F_{N-2k-1} = [2k(\beta + \beta_0) + 4\beta]/(16\alpha^2 N^2), \quad k = 0, 1, \dots, k_1. \quad (5.3)$$

The limitation $k \le k_1$ in (5.3) stems from the fact that we must also take into account some (linear) propagation of errors. Hence, the large and decreasing quantity F_{N-2k} should not become smaller than the small and increasing values of F_{N-2k-1}. They become comparable just in the domain of indices

$k \approx k_1$. There, the estimate (5.3) ceases to be reliable and we arrive at

$$F_{N-2k} \approx F_{N-2k-1} \approx \frac{1 + O(1/\sqrt{N})}{4\alpha N^{3/2}}, \quad (5.4)$$

where we have to put $\text{sign}(\beta + \beta_0) = \text{sign}\,\alpha$. When we insert (5.3) in the left-hand side here, we obtain also the explicit range of the admissible indices

$$k_1 = \frac{2}{\beta + \beta_0}[-\beta + (\alpha + O(k_1/N))\sqrt{N}] = N^{1/2}/\gamma + O(1),$$

$$\gamma = \frac{1}{2\alpha}(\beta + \beta_0) > 0.$$

In the $k > k_1$ domain of indices, we may use a natural reparametrization of the present ECF quantities

$$F_n^{(N)} = \frac{1}{4\alpha n^{3/2}}(1 + f_n/\sqrt{n}), \quad n \ge 1, \quad N \le \infty. \quad (5.5)$$

An insertion of this formula converts our $n \gg 1$ ECF recurrences (5.2) into equivalent relations

$$-\frac{f_n}{1 + \lambda f_n} = \gamma + f_{n+1} + \lambda \frac{\beta_0 - \beta}{2\alpha}(f_{n+1} + f_{n+2} + \lambda f_{n+1} f_{n+2}),$$

$$n = n_0, n_0 - 1, \dots, n_1, \quad (5.6)$$

$$n_0 = N - 2k_1 \gg n_1 \gg 1, \quad \lambda = 1/\sqrt{n_0}(1 + O(\epsilon)), \quad \gamma = \gamma(1 + O(\epsilon)), \quad \epsilon \ll 1.$$

Here, a new error estimate $\epsilon = O(\lambda)$ propagates again with the change of indices n. In the rough approximation, we may expect that our subtracted sequence f_n (which changed sign in the domain of indices $n = N - 2k$, $k < k_1$) will be small for $k \gg k_1$, i.e.,

$f_n/\sqrt{n} = O(\epsilon), \quad 1 \gg \epsilon \gtrsim O(\lambda), \quad n \gtrsim n_1.$

Thus, we may drop all the higher-order corrections in (5.6) and get a new simplification of the original ECF mapping,

$$\frac{-f_n}{1 + \lambda f_n} = \gamma + f_{n+1}, \quad n \in (n_1, n_0). \quad (5.7)$$

This already has a sufficiently trivial geometric interpretation: when we iterate it once, it loses its oscillatory character and may be written as a simple rational mapping

$$z' = \frac{\mu z + \nu}{1 + \lambda z}, \quad z = \gamma + f_{n+1}, \; z' = \gamma + f_{n-1}, \quad \mu = 1 - (2 - \lambda\gamma)\nu, \quad \nu = \lambda\gamma/(1 + \lambda\gamma). \tag{5.8}$$

On the present $O(\epsilon)$ level of precision, the convergence of the mapping $z \to z'$ is obvious - the initial values $z = O(\epsilon/\lambda)$ lie sufficiently far from the unstable fixed point $z^{(-)} = \gamma - 2/\lambda$ of this mapping. Hence, a simple geometry of (5.8) implies an accumulation of our ECF values near $z \approx z^{(+)} \approx \gamma/2$, i.e.,

$$f_n \approx f_{n+1} \approx -\frac{1}{2}\gamma, \quad n \approx n_1. \tag{5.9}$$

This becomes a good approximation after a few iterations of (5.8).

From a purely algebraic point of view, a contractive behaviour (convergence $z \to z^{(+)}$) of our mapping is based on the standard geometric-series majorization with

$$\left|\frac{\partial z'}{\partial z}\right| < 1, \quad z > -1/\lambda\nu.$$

A decrease of index n implies also a decrease of our new error estimate up to a new level of precision neglected in the preceding considerations. We may write

$$F_n^{(\infty)} = \frac{1}{4\alpha n^{3/2}} - \frac{\gamma}{8\alpha n^2} + O\left(\frac{1}{n^{5/2}}\right). \tag{5.10}$$

and linearise the ECF mapping. The rest of the proof is trivial /9/.

6. A SYSTEMATIC ACCELERATION OF CONVERGENCE

The idea of the preceding paragraph, namely, a systematic subtraction of the fixed point approximations, may be employed also in the re-formulation of the ECF formalism itself. For example, assuming that

$$F_n \approx F_{n+1} \approx F_{n+2} \approx P^{(0)} \tag{6.1}$$

say, in the $n \gg 1$ asymptotic region, we may replace the auxiliary ECF sequence $F_n = F_n^{[0]}$ by a sequence of ECF differences $F_n^{[1]} = F_n^{[0]} - P_n^{(0)}$.

In general, the prescription

$$F_n^{[k+1]} = F_n^{[k]} - P_n^{(k)} \tag{6.2}$$

and its efficiency depends on our specification of the subsequent approximants $P_n^{(k)}$. For example, when we recall (6.1) and demand, in the light of eq.(3.4),

$$P^3 D B_{n+1} B_n + P^2 C_{n+1} B_n + P(A_n - E) - 1 = 0 \tag{6.3}$$

we may define $P_n^{(0)}$ by the Cardano formulas /11/.

In the latter construction, a representation of $P_n^{(0)}$ in the form of Taylor series leads to a use of $\rho = n^{-1/2}$ as a natural variable, while the numerical considerations suggest a replacement of (6.1) by similar relations for a new quantity $B_{n-1}F_n$. Summarising these two indications, we shall replace here the recurrences (6.2) and equations of the type (6.3) by a simpler though equally general ansatz to be used in (3.4),

$$B_{n-1} F_n = \frac{1}{\alpha\rho} \sum_{m=0}^{M} \phi_m \rho^m + B_{n-1} F_n^{\{M\}}, \quad \rho = 1/\sqrt{n} \ll 1, \tag{6.4}$$

where $\phi_0 = 1$ and, presumably, $B_{n-1} F_n^{\{M\}} = O(\rho^M)$.

A detailed derivation of the coefficients ϕ_m is a little bit nontrivial. It may significantly be simplified, when we proceed as follows.

(1) We employ the Taylor formula

$$(n + j)^{-m/2} = (\rho')^m = \rho^m \sum_{k=0}^{\infty} (-j\rho^2)^k \, w_k^{(m)},$$

$$w_k^{(m)} = \Gamma\left(k + \frac{1}{2}m\right) / \left(k!\, \Gamma\left(\frac{1}{2}m\right)\right)$$

and complement (6.4) by the expansions with shifted indices

$$B_{n+j-1} F_{n+j} \approx \frac{1}{\alpha\rho} \sum_{m=0}^{\infty} \phi_m \rho^m \sum_{k=0}^{\infty} w_k^{(m-1)} (-j\rho^2)^k.$$

Of course, the variable $\rho = 1/\sqrt{n}$ remains the same.

(2) We avoid the multiple multiplications in (3.4) by means of the inversion formula

$$(B_{n-1} F_n)^{-1} = \alpha\rho \sum_{m=0}^{\infty} \psi_m \rho^m, \quad \psi_0 = 1, \; \psi_1 = -\phi_1, \ldots .$$

Here, the tilded quantities $\tilde{\psi}_{m-1} = \tilde{\psi}_{m-1}(\phi_1, \phi_2, \ldots, \phi_{m-1}) = \psi_m + \phi_m$ denote the m-dimensional determinants

$$\tilde{\psi}_{m-1} = (-1)^m \det \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_{m-1} & 0 \\ 1 & \phi_1 & \dots & \phi_{m-2} & \phi_{m-1} \\ 0 & 1 & \dots & \phi_{m-3} & \phi_{m-2} \\ & & \dots & & \\ 0 & \dots & 0 & 1 & \phi_1 \end{pmatrix}, \quad m = 2,3,\dots$$

and $\tilde{\psi}_0 = 0$.

(3) We re-write the coefficients as polynomials

$$\rho^4 B_{n-1} = 4 + 4\delta_1\rho^2, \quad \delta_1 = \ell + \frac{1}{2},$$

$$\rho^3 A_n = 4a(\delta_2\rho + \delta_3\rho^3), \quad \delta_2 = \beta/a, \quad \delta_3 = \frac{1}{4a}[(2\ell+3)\beta - E],$$

$$\rho^2 C_n = 4a^2(1 + \delta_4\rho^2), \quad \delta_4 = \frac{1}{4a^2}[(2\ell+1)a^2 - \beta^2 + g_1],$$

$$\rho D_n = 4a^3\delta_5\rho, \quad \delta_5 = \frac{1}{2a}(\beta_0 - \beta).$$

(4) We convert (3.4) into a power-series condition

$$(1+\delta_1\rho^2)\sum_m \psi_m\rho^m = (\delta_2 + \delta_3\rho^2)\rho + (1+\delta_4\rho^2) \times$$

$$\times \sum_{m,k} (-1)^k \phi_m w_k^{(m-1)} \rho^{m+2k} +$$

$$+ \delta_5\rho \sum_{\substack{m,m' \\ k,k'}} (-1)^{k+k'} 2^{k'} \phi_m \phi_{m'} w_k^{(m-1)} w_{k'}^{(m'-1)} \rho\, \rho^{m+m'+2k+2k'}$$

and re-arrange it in such a way that

$$-2\sum_{m=1}^{\infty} \phi_m\rho^m = \delta_2\rho + \delta_3\rho^3 + \delta_4\rho^2 \sum_{m,k} (\dots) +$$

$$+ \delta_5\rho \sum_{\substack{mm' \\ kk'}} (\dots) - \delta_1\rho^2 \sum_{m=0}^{\infty} \psi_m\rho^m - \sum_{m=2}^{\infty} \tilde{\psi}_{m-1}\rho^m +$$

$$\rho + \sum_{t=1}^{\infty} \sum_{m=0}^{t-1} (-1)^{t-m} [\rho^{2t} w_{t-m}^{(2m-1)} \phi_{2m} + \rho^{2t+1} w_{t-m}^{(2m)} \phi_{2m+1}],$$

i.e., the left-hand side coefficient ϕ_m at ρ^m becomes defined by its right-hand side counterparts.

(5) We employ the power-by-power independence of these requirements and get the explicit formulas

$$-2\phi_1 = \delta_2 + \delta_5,$$

$$-2\phi_2 = \delta_4 + 2\phi_1\delta_5 - \delta_1 - \phi_1^2 + \frac{1}{2},$$

$$-2\phi_3 = \delta_3 + \phi_1\delta_4 + (\phi_1^2 + 2\phi_2 + \frac{3}{2})\delta_5 + \phi_1\delta_1 + \phi_1^3 - 2\phi_1\phi_2,$$

$$\dots$$

re-expressing them at ρ^1, ρ^2, ρ^3, etc., respectively.

(6) Finally, we convert the recurrent definitions $\phi_m = \phi_m(\phi_{m-1}, \phi_{m-2}, \dots, \phi_1)$ into the explicit formulas

$$\phi_1 = -\frac{\beta + \beta_0}{4a},$$

$$\phi_2 = \frac{1}{16a^2}[(4\ell - 2)a^2 + (\beta + \beta_0)^2 + 4\beta_0^2 - 4g_1],$$

etc, by the repeated insertions.

The resulting formulas seem rather cumbersome. We may recommend their computer generation: The definitions are linear, so that the manipulations leading to the final results are straightforward.

REFERENCES

1. Biswas S.N. et al. - Phys.Rev., 1971, D4, p.3617. Ginsburg C.A. - Phys.Rev.Lett., 1982, 48, p.839.
2. Singh V., Biswas S.N., Datta K. - Phys.Rev., 1978, D18, p.1901; M.Znojil M. - Phys.Rev., 1982, D26, p.3750.
3. Hautot A. - Phys.Rev., 1986, D33, p.437.
4. Znojil M. - Phys.Rev., 1986, D34, p.1224.
5. Killingbeck J. - Phys.Lett., 1986, A115, p.301.
6. Znojil M. - J.Phys., 1976, A9, p.1.
7. Tater M. - J.Phys., 1987, to appear.
8. Znojil M. - Lett. Math.Phys., 1981, 5, p.405.
9. Znojil M. - J. Math.Phys., 1983, 24, p.1136.
10. Masson D. - J. Math.Phys., 1983, 24, p.2074.
11. Chaudhuri R.N., Tater M., Znojil M. - J.Phys., 1987, A20, p.1401.

Received by Publishing Department
on June 26, 1987.

Зноил М. E5-87-480

Двойной ангармонический потенциал

$ax^2 + bx^4 + cx^6$ и расширенные цепные дроби

Рассматривается проблема связанных состояний сэкстического ангармонического осциллятора. Решение /Singh et al., Phys.Rev., 1978, D18, p.1901/ с $b > 0$ и цепными дробями расширяется на все константы связи. Начиная с хилловского построения /Znojil,Phys.Rev., 1986, D34, p.1224/, используется рекуррентная факторизация и получается полное решение в терминах так называемых расширенных цепных дробей. Некоторые математические вопросы /сингулярности, сходимость/ разъясняются детально и, в частности, предлагается систематическое ускорение сходимости. Достигается пополнение и улучшение оригинальных конструкций.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Znojil M. E5-87-480

The Double Anharmonic Potential

$ax^2 + bx^4 + cx^6$ and the Extended Continued Fractions

The sextic anharmonic-oscillator bound-state problem is considered, with an intention to extend its nonperturbative continued-fractional $b > 0$ solution to all couplings. Starting from its specific Hill-determinant treatment we employ a recurrent factorization of the related non-hermitean Hamiltonian and arrive at a complete nonperturbative solution represented in terms of the so-called extended continued fractions. The underlying less standard mathematical questions (singularities, convergence) are also clarified and, in particular, a systematic algebraic acceleration of convergence is proposed. In this way, both complection and improvement of the original constructions is achieved.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987