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SOME PROPERTIES OF A GENERALIZED HENON SYSTEM

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1. Introduction

One of the most extensively studied discrete dynamical systems is the so-called Henon system /5/ given by the following pair of equations:

$$x_{n+1} = y_n + 1 - Ax_n^2$$
$$y_{n+1} = Bx_n \qquad (1)$$

This is one of the simplest nonlinear maps of the plane into itself and it reflects a lot of important properties of dynamical systems. In /5/ there is described a strange attractor for A = 1.4, B = 0.3. An exhaustive investigation of (1) was performed by Simó/8/. He described a lot of attractors (e.g. fixed points, periodic orbits, strange attractors), their shape, development and so on in dependence on the variation of A; B is fixed to be 0.3 . Most of the papers concerning (1) consider only strong dissipations (i.e. small $|B|$, for $|B| = 1$ the system is conservative, that means, the map is area-preserving). There are only few papers where the dynamical behaviour of (1) is investigated for larger $|B|$ (e.g. /2/, there the cases B = -1 and B = -0.85 are considered). While systems with strong dissipation are very appropriate to determine all kinds of attractors, systems with small dissipation show also interesting features, e.g. transient chaos. In /3/ and /4/ Chirikov and Izrailev studied some discrete dynamical system on the torus. They were the first who observed the following phenomenon which appears in course of the transition from the conservative to the dissipative regime of the system:

conservative regime: There exists a stochastic component (formed by one single trajectory) which has (small) islands of stability, i.e. trajectories which start within such an island cannot leave it and the motion is confined on some ellipse lying inside this island.

small dissipation: The stochastic component is completely destroyed, a periodic orbit appears. It is located in the former stable islands and attracts all points of the torus. Long transient chaos is possible.

Increasing dissipation: The period-doubling scenario appears and a strange attractor arises.

The point is, that there are systems in which the transition from stochastic motion in the conservative regime to stochastic motion in the dissipative regime (namely on a strange attractor) is discontinuous in the sense, that these two stochastic motions are separated by periodic motion.

Moreover, in these papers of Chirikov and Izrailev there was numerically established the following relation:

$$N \cdot S \cdot E \quad 1 \,.$$

Here the symbols have the following meaning:

N - life time of transient chaos

S - total area of stable islands

E dissipation,(i.e. $E = 1 - B$ for $B > 0$ if we consider the Henon system). For details see /6/.

Subsequently in the literature there was repeatedly discussed the question whether or not such a discontinuous transition is universal or not (cf. /1/,/7/,/11/). In /9/,/6/ and /10/ we discussed the results of /7/ where a continuous transition in a truncated Henon system was postulated. The trouble was that the truncation was of such a kind that in reality the supposed transition was not at all studied.

In the present paper we continue the investigation of the following generalized Henon system:

$$x_{n+1} = y_n + D - Ax_n^2 \qquad , 0 \quad D \leq 1$$
$$y_{n+1} = Bx_n . \tag{2}$$

In /6/ and /10/ we studied some properties of this generalized Henon system on a torus of period 4 (truncated in such a way that $- \;\leq x_n, y_n \;\; 2$). Let us denote the corresponding system by (2_t) (t means truncation). In these papers we found for (2_t) for $A = B = 1$, $D = 0.1, 0.2, \ldots, 1$

the stochastic component with stable islands and the phenomenon of discontinuous transition described above. Moreover, in all these cases we verified the relation $N \cdot S \cdot E \sim 1$.

Now we investigate and compare some properties of (2) and (2_t). Especially we give numerical results concerning the appearance of periodic and strange attractors, an interrupted period-doubling scenario and an example of period doubling into two directions (i.e. with increasing and decreasing parameter B starting with the same orbit). May be, this is new.

In section 3 we study the dependence of the number of unattracted trajectories in dependence on the number of iterations. It appears that for (2_t) this dependence is qualitatively the same (namely exponential decay) as for the dissipative Fermi map studied in /11/. Moreover we add some critical comments to some assertions in /11/ concerning the discontinuous transition of the stochastic motion in (2_t) in the course of changing from the conservative to the dissipative regime.

2. The generalized Henon system in the plane

We consider system (2) in the plane, i.e.

$$x_{n+1} = y_n + D - Ax_n^2 \qquad 0 < D \leq 1$$
$$y_{n+1} = Bx_n . \tag{3}$$

In all our considerations we fix $A = 1$.

The fixed points of the map (2) in its general form are given by:

$$x^F_{1,2} = (1/2A)((B-1) \pm ((B-1)^2+4AD)^{1/2}), \; y^F_{1,2} = Bx^F_{1,2} .$$

For $B = 1$ (conservative case) this means:

$$x^F_{1,2} = \pm (D/A)^{1/2} \quad , \quad y^F_{1,2} = x^F_{1,2} .$$

Moreover, it is easy to see, that the points $((D/A)^{1/2}, -(D/A)^{1/2})$ and $(-(D/A)^{1/2}, (D/A)^{1/2})$ form an orbit of period 2.

First let us consider the conservative system (B = 1) and let D vary from 1 to 0.1 (A is fixed to be 1). Among other things one observes the following:

- the fixed points are unstable and therefore cannot be found numerically. If one coordinate of the fixed point is a little bit perturbed from the true value (say by 10^{-10}), then this point escapes to infinity after a small number of iterations.
- the 2-orbit is stable (more exactly: transiently stable, see below), and each of its points is surrounded by small stable islands. Trajectories starting within remain there for ever and form ellipses.
  Clearly, the trajectory of a discrete system cannot form an ellipse in the strong sense of the word, but the points of the trajectory cover such an ellipse densely (i.e. without being periodic).
- if one takes the initial point near to the boundary of the stable regions, there can be observed more smaller stable islands surrounding the former stable region.
  Let us give some examples:
  D = 1: 2 x 7 (i.e. any of the two stable regions is surrounded by 7 smaller stable islands)
  D = .9 : 2 x 8 ; D =0.6 : 2 x 11 ; D = 0.4 : 2 x 5
  D = 0.3 : 2 x 6.

Fig.1 gives an impression of the location of one part of the stable region for A = B = 1, D = 0.3 for the truncated system (2_t). The 6 pieces of the 12-harmonics surrounding the stable island are very well seen. This figure is formed by 3 trajectories: two in the inner part of the island and one (take as initial point e.g. (0.778,- 0.548)) which forms the stochastic component which has these 6 (resp. 12) small stable islands. In the plane one would get almost the same picture but with the difference that the initial point indicated above is divergent. Nevertheless, in the course of iteration this point has yet time to indicate clearly these 12 islands before it escapes to infinity.

Because it may be not so easy to find these small surrounding islands numerically, we collect some information in the following table. The second column contains initial points which lead to ellipses within the islands around the 2-orbit but already close to the boundary. The third column indicates initial points which trajectories form the surrounding small islands. Finally in the last column we have the number of these islands. For some values of D we did not found data because we did not seek long enough.

Table 1: A = B = 1, system (2) or (2_t)

D	initial point near boundary	initial point for surrounding isl.	number of islands
1	(1, -0.975)	(1, -0.955)	2 x 7
0.9	(1.013,-0.948)	(1.023,-0.948)	2 x 8
0.8	(0.939,-0.894)	?	?
0.7	(0.947,-0.837)	?	?
0.6	(0.925,-0.775)	(0.935,-0.775)	2 x 11
0.5	(0.817,-0.707)	?	?
0.4	(0.835,-0.632)	(0.370,-0.370)	2 x 5
0.3	(0.758,-0.548)	(0.240,-0.480)	2 x 6
0.2	(0.747,-0.447)	(0.767,-0.447)	2 x 9
0.1	(0.576,-0.316)	?	?

Next we consider small dissipations. Let us mention only two cases in detail, further informations are contained in table 2 below.

A = 1, D = 1

As already mentioned above (see also /6/), the 2-orbit is stable, but this stability is just a transient one. More exactly, the eigenvalues of the linearized system at the 2-orbit are both equal to -1. Thus,already the weakest dissipation leads immediately to a bifurcation and a stable 4-orbit arises. We give some data of the corresponding period-doubling scenario:

B = 0.999 999 ... 0.86 4-orbit
0.85 ... 0.835 8-orbit
0.830 ... 0.827 16-orbit
0.8269 ... 0.8262 32-orbit
0.826 ... 0.825 86 64-orbit
0.825 85 ... 0.825 82 128-orbit
0.825 81 ... 0.825 808 256-orbit
0.825 806 512-orbit
0.825 strange attractor with Lyapunov exponent 0.038.

Let us remark that for 0.5 < B < 0.825 we could not find any non-divergent point.

A = 1 , D = 0.95

Here we find for small dissipations the 2-orbit, and then the period-doubling goes as usual, e.g.

B = 0.95 2-orbit
0.9 ... 0.8 4-orbit
....
0.716 066 512-orbit
0.7 strange attractor with Lyapunov exponent 0.086.

In the following table we collect some information about the whole range of D. The entries of this table indicate the most dominant periodic orbits/attractors, i.e. those to which most of the non-divergent initial points are attracted.
The symbols have the following meaning: a - strange attractor, d - only divergent points are observed in our calculations.

Let us comment a little bit on this table 2.
i) The first rows (D = 1 ... D=0.94) express the usual period-doubling scenario for increasing B (e.g. 0.1 ... 0.4) as well as for decreasing B (e.g. 0.95 ... 0.8). The corresponding strange attractors evolve further and change their shape and size. As already remarked, there is a range of B-values where we found only divergent points.
ii) An interesting phenomenon is indicated in the first row for B = 0.5. There an 8-orbit and a strange attractor coexist. And this 8-orbit undergoes period-doubling for increasing as well as for decreasing B. Such a bifurcation of one orbit into two directions of parameter changes seems to be very seldom.

Table 2: A = 1, system (2)

D \ B	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	0.45	0.4	0.35	0.3	0.25	0.2	0.15	0.1
1.00	4	4	8	d	d	d	d	d	d	8/a	a	a	8	4	4	4	2	2
0.98	4	4	4	8	8	d	d	d	a	a	a	8	4	4	4	2	2	2
0.96	4	4	4	4	16	d	d	a	a	a	8	4	4	4	4	2	2	2
0.95	2	4	4	4	8	a	a	a	a	a	8	4	4	4	2	2	2	2
0.94	2	4	4	4	8	8	a	a	a	8	8	4	4	4	2	2	2	2
0.92	2	4	4	4	4	4	8	8	8	4	4	4	4	4	2	2	2	2
0.9	2	2	4	4	4	4	4	4	4	4	4	4	4	2	2	2	2	2
0.8	2	...	...				2	4	2		...	...						2
0.7	2	...	...								...	...						2
0.6	2	...	...								...	...						2
0.5	2	...	...								...	...			2	2	1	1
0.4	2	...	...								...	...		2	1	1	1	1
0.3	2	...	...						...	...		2	1	...	...		1	1
0.2	2	...	...						2	2	1	...	...	...			1	1
0.1	2	...	...	...		2	2	1	...	...	...						1	1

Table 3: A = 1 , system (2_t)

D \ B	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	0.45	0.4	0.35	0.3	0.25	0.2	0.15	0.1
1.00	4	4	8	a	a	a	a	a	a	8/a	a	a	8	4	4	4	2	2
0.98	4	4	4	8	a	a	a	a	a	a	a	8	4	4	4	2	2	2
0.96	4	4	4	4	16	a	a	a	a	a	8	4	4	4	4	2	2	2
0.95	2	4	4	4	8	a	a	a	a	a	8	4	4	4	2	2	2	2
0.94	2	4	4	4	8	8	a	a	a	8	8	4	4	4	2	2	2	2
0.92	2	4	4	4	4	4	8	8	8	4	4	4	4	4	2	2	2	2

We cannot give any other example from the literature. We give some details of these bifurcations:

B = 0.503 16-orbit, B = 0.5037 32-orbit, B = 0.50411 64-orbit

on the other hand:

B = 0.46 16-orbit, B = 0.4598 32-orbit, B = 0.4596 64-orbit

iii) With decreasing D (starting approximately with D = 0.92) the period-doubling scenario for the indicated periodic orbits is interrupted. A similar phenomenon we already described in /9/. But there the reason for the interruption was another one. Namely, the orbit wanders to the boundary of the square and then falls into the domain of attraction of another orbit. More exactly, any period-doubling was prevented even by this fact.
Here we find quite another reason for this interruption. Let us consider the case of increasing B. If - say for D = 0.92 - the 8-orbit arises from the 4-orbit, we have two groups of four points. Now in the course of increasing B we can observe, that the change (of the coordinates)of the points of the two groups takes place with different velocity. The effect is, that in the course of the evolution of this 8-orbit, these two groups get closer and closer, and finally they fuse before the 8-orbit succeeds to bifurcate. Just this difference can be very clearly seen if one follows the development of the mentioned 8-orbit for D = 0.92 and 0.94.
Let us yet remark that the fixed point which appears, e.g. for D = 0.5 and B = 0.1 can be observed also for larger D if one admits also negative B-values.

3. The generalized Henon system on a torus of period 4

Let us now consider system (2_t), i.e. the generalized Henon system confined onto a torus of period 4. One of the advantages of such truncated systems is the fact that there are non divergent points. The stable islands and the surrounding harmonics were already mentioned in section 2 and some overview was given in table 1. The same table is also valid for the truncated system. Moreover, one can give some impression about the relative area of the stable islands (relatively to the square $-2 \leq x,y \leq 2$). To calculate this relative area one can

proceed as follows. Take a lattice of initial points on the torus (i.e.in the square $-2 \leq x,y \leq 2$) and then calculate with the help of system (2) (i.e. the non-truncated system!) the portion of initial points which do not escape to infinity. Then this portion is a good measure of the relative area.
Let us give the results in the following table. In the columns the numbers indicate the % of the square which gives the relative area of the stable regions. In the first column we give the different D-values, the second column contains the relative area for a lattice consisting of 200 x 200 points, and always 1000 iterations are done to decide whether or not the point escapes (if it does not give overflow already earlier). The third column contains the results for a 100 x 100 lattice and 10 000 iterations.

D	200x200 points, 1000 it.	100x100 points,10000 it.
1	0.5	0.5
0.9	0.68	0.69
0.8	0.013	0.02
0.7	1.9	1.8
0.6	2.55	2.5
0.5	3.4	3.0
0.4	3.3	3.2
0.3	5.8	4.7
0.2	7.6	6.6
0.1	6.6	6.3

As it can be seen the results coincide very well.
Next let us consider system (2_t) in more detail. Table 3 contains the related results for the same parameter ranges as in table 2. It is enough to write down only the parameter range $0.94 \leq D \leq 1$, because for the other values of D the tables coincide. Most of the remarks related to table 2 can be repeated here word for word. But there are some necessary changes which are caused by the truncation. For example, there are of course no divergent points. Thus, in the first row of table 3 we have instead of divergent points strange attractors, but of very different shape, size and origin. There are such strange attractors, which are obtained in the course of the period-doubling.For example, for B = 0,825, the strange attractor has a

Lyapunov exponent equal to 0.038. It develops further, the Lyapunov exponent increases a little bit, but then, say for B = 0.824 there is an abrupt change of the size of the strange attractor. This is a kind of explosion. The former shape is yet well seen, but the Lyapunov exponent is now already 0.155. Similar remarks could be made also for other parameter values.
In Figs.2 ... 4 we give some impression of other attractors. While those, shown in figures 2 and 3 can also be observed in the plane (cf. table 2), the large attractor in Fig.4 is only present on the torus. It corresponds to divergent points in the plane.
Now let us return to the phenomenon of transient chaos. This means - roughly speaking - the following. Some initial points in systems with weak dissipation lead to trajectories which are for a long time apparently chaotic but finally they are attracted by some periodic orbit. This eratic wandering of the trajectory can extend over several hundred thousand iterations.
In /11/ there was considered the following problem for the dissipative Fermi map. Take 100 randomly chosen initial points from the stochastic component of the conservative system. Then, for different very small dissipations study the dependence of the number of unattracted trajectories on the number of iterations. Here "unattracted" means, that for a given number of iterations the trajectory is not yet in some ε -neighbourhood of a periodic orbit. For the Fermi map it appears that this dependence is exponential (exponential decay).

In Fig. 5 ... 8 we give some results for system (2_t). The vertical axis is divided logarithmic and indicates the number N_u of unattracted trajectories.
The numerical calculations were performed for the following parameter values: A = 1, D = 1, 0.95 , 0.9 , 0.6 and for small dissipations: B = 0,9995, 0.999 0.995 and 0.99.
We calculated 50 x 4000 iterations, and after any 4000 iterations we tested whether or not the trajectory is in an ε -neighbourhood of a periodic orbit, namely, a 4-orbit for D = 1, a 2-orbit for D = 0.95, 0.9, 0.6 ; ε was taken to be 0.001.
The deviation from a straight line is partially caused by the fact that we took into account only <u>one</u> periodic orbit, but

for B = 0.995 and 0.99 there coexist yet other periodic attractors which therefore perturb the behaviour of the curves. Moreover, the horizontal parts of the curves in the figures indicate long transient chaos or/and also the coexistence of other periodic attractors. Be that as it may, also in system (2_t) with sufficient precision the decay is exponential.

Finally we comment on an assertion made in /11/ concerning the discontinuous transition from chaotic motion in the conservative regime of (2_t) to motion on a strange attractor in the dissipative regime. In /11/ the authors found a period six attractor for A = 1.4, B = 0.9 and D = 1. For these parameters there are initial values with very long transient chaos. The pictures obtained in the course of the iteration of such initial value seem to represent a strange attractor, but in reality this is transient chaos. The authors conclude that this is numerical evidence for the fact that chaos do not exist persistent when the Hamiltonian system is smoothly transformed into a strongly dissipative system. But there is some trouble with this conclusion. The point is, that for a smooth tranformation one must investigate B-values very close to 1 (cf. the results of section 2 for the 4-orbit at B = 0.999 999...). In that sense, B = 0.9 is clearly too far from 1! Indeed, if one increases B, this 6-orbit undergoes the usual period-doubling scenario, namely: B = 0.9 6-orbit, B = 0.9009 12-orbit,
B = 0.9008 24-orbit, B = 0.900 004 48-orbit, ...
... B = 0.901 010 1312 768 orbit.
Then one observes a strange attractor.
Let us remark, that we could not find the typical features for a discontinuous transition, namely stable islands for B = 1.
The conclusion may be the following (cf. also /10/): <u>if</u> for B = 1 (or more general, for the conservative case) there exist stable regions, then the transition is non-continuous. But there does not seem to exist any method to decide a priori whether or not stable islands exist at all. In practice they can be so small, that it is impossible (or at least hopeless) to find them by simple computer experiments.

Fig.1. Generalized Henon system on a torus of period 4, A = B = 1, D = 0.3. Part of the stochastic component (0 = x = 1.2, -1.2 = y = 0) with stable regions and 2 trajectories within the large stable island.

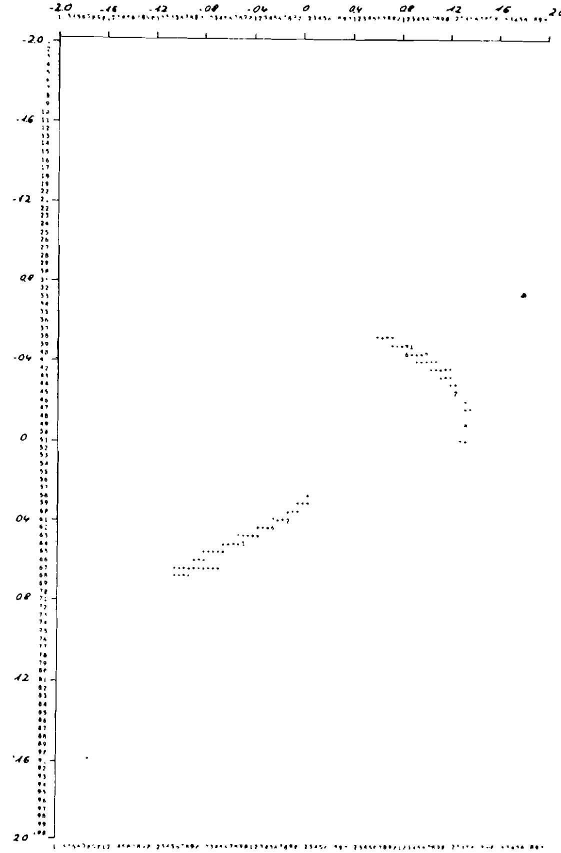

Fig.2. Generalized Henon system on a torus of period 4, A = 1, B = 0.5, D = 0.98. Strange attractor with Lyapunov exponent 0.188, initial value (1.309,0.o12)

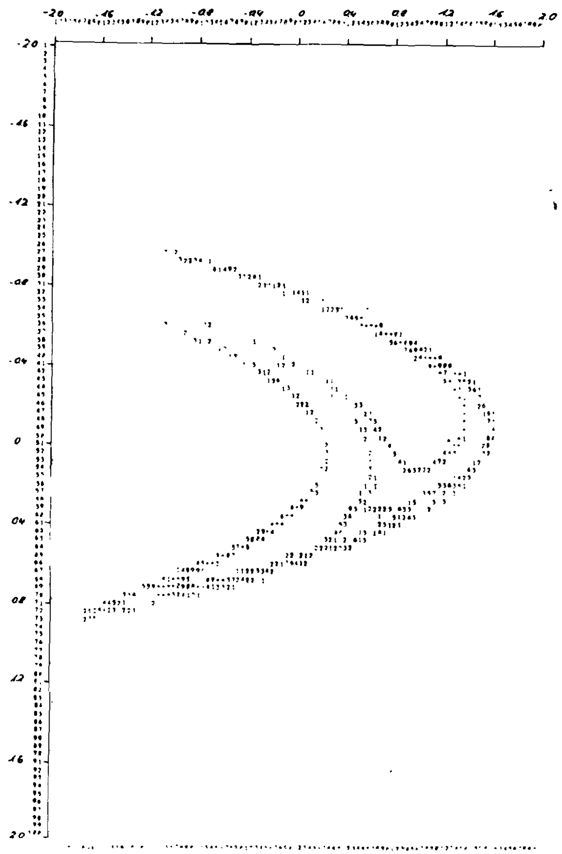

Fig.3. Generalized Henon system on a torus of period 4, A = 1, B = 0.55, D = 0.98. Developed strange attractor of Fig.2, Lyapunov exponent 0.232, initial value (1.175, -0.302).

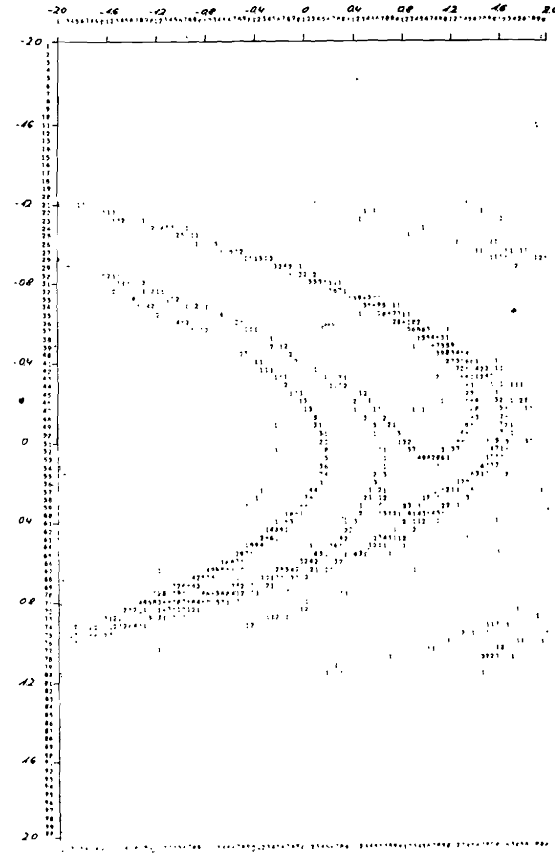

Fig.4. Generalized Henon system on a torus of period 4, A = 1, B = 0.6, D = 0.98. New strange attractor, Lyapunov exponent 0.315, initial value (-0.287,0.488).

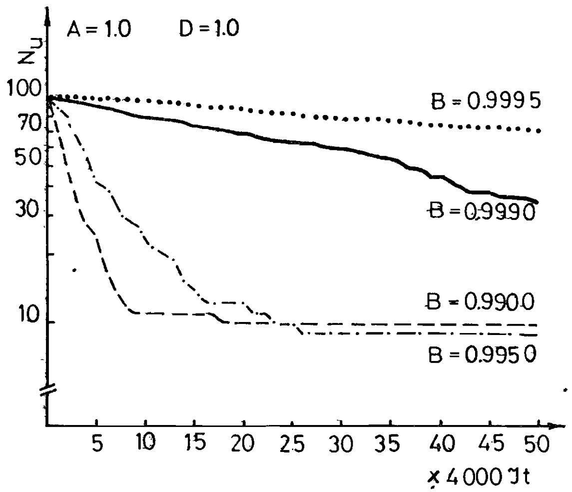

Fig. 5. Number of unattracted trajectories N_u versus number of iterations for the generalized Henon system on a torus of period 4 for different values of D and B.

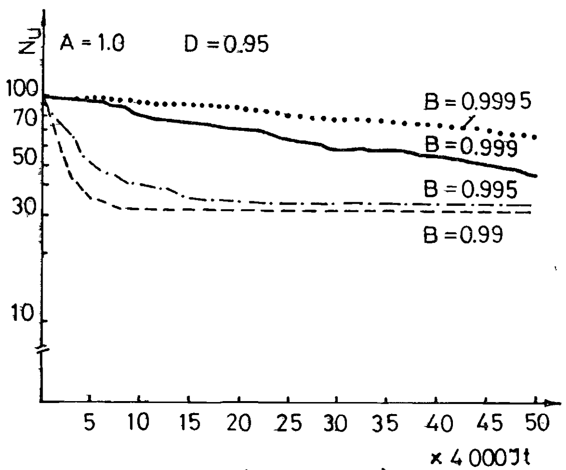

Fig. 6 (see Fig. 5).

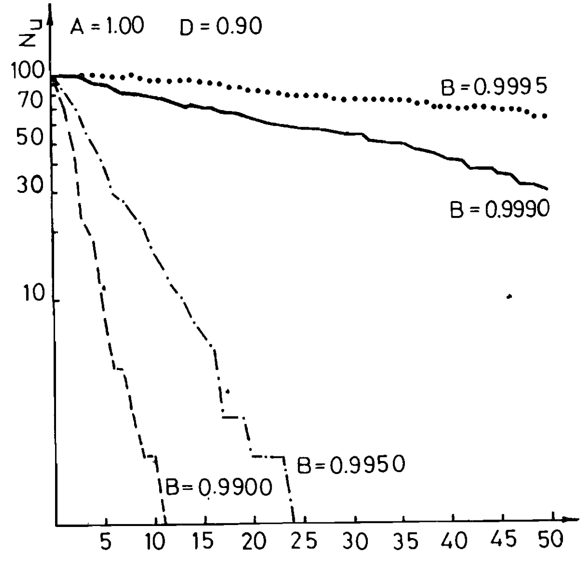

Fig. 7 (see Fig. 5).

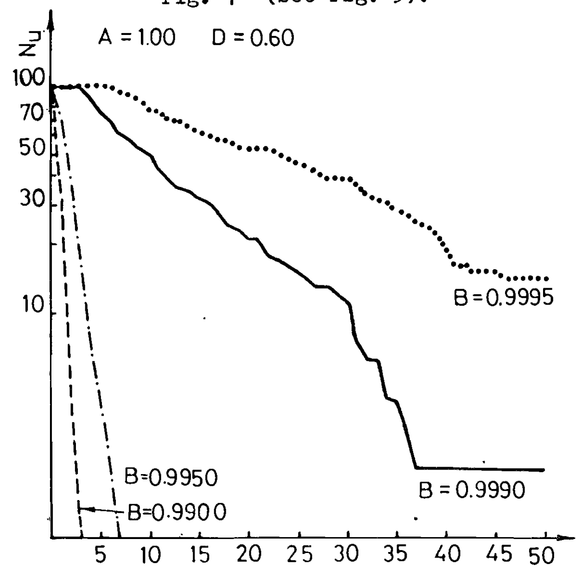

Fig. 8 (see Fig. 5).

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Тиммерманн Б., Тиммерманн В. E5-87-458
Некоторые свойства обобщенной системы Хенона

Исследуются некоторые свойства обобщенной системы Хенона на плоскости и на торе. Отмечено несколько интересных совйств картины удвоения периода. Обсужден переход хаотического движения в консервативной системе в движение на странном аттракторе в диссипативной системе. Для слабодиссипативной системы число непритягивающих траекторий убывает экспоненциально в зависимости от числа итераций.

Работа выполнена в Лаборатории теоретической фиэики ОИЯИ.

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Timmermann B., Timmermann W. E5-87-458
Some Properties of a Generalized Henon System

Some properties of a generalized Henon system in the plane and on a torus are studied. There are mentioned some interesting aspects of the period doubling scenario. The transition from chaotic motion in the conservative regime to motion on a strange attractor in the dissipative regime is discussed. For the weakly dissipative system the number of unattracted trajectories decays exponential in dependence of the number of iterations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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