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исследований
дубна

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H.Neidhardt

DISSIPATIVE LAX - PHILLIPS SCATTERING THEORY AND THE CHARACTERISTIC FUNCTION OF A CONTRACTION

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1. Introduction

In [5] P.D.Lax and R.S.Phillips generalize their scattering theory developed in [3,4] to include dissipative effects of the scattering process. Mathematically this generalization is reflected by the fact that instead of a selfadjoint operator to describe the scattering system now a maximal dissipative operator is used.

In [2] via the Cayley transform of a maximal dissipative operator the assumptions of P.D.Lax and R.S.Phillips [5] were necessarily and sufficiently carried over to contractions. For the convenience of the reader we repeat the assumptions of the dissipative Lax-Phillips scattering theory in terms of contractions. A triplet $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ consisting of a contraction T on a separable Hilbert space $\mathcal{H}$ and two subspaces $\mathcal{D}_\pm$ of $\mathcal{H}$ is called a dissipative Lax-Phillips scattering theory if the following assumptions are fulfilled.

(h1) $T\mathcal{D}_+ \subseteq \mathcal{D}_+$, $T^*\mathcal{D}_- \subseteq \mathcal{D}_-$,

(h2) $T\restriction\mathcal{D}_+$ and $T^*\restriction\mathcal{D}_-$ are isometries,

(h3) $\bigcap_{n\in\mathbb{Z}_+} T^n\mathcal{D}_+ = \{0\} = \bigcap_{n\in\mathbb{Z}_+} T^{*n}\mathcal{D}_-$,

(h4) $P^{\mathcal{H}}_{\mathcal{H}\ominus\mathcal{D}_+}T^n \to 0$, $P^{\mathcal{H}}_{\mathcal{H}\ominus\mathcal{D}_-}T^{*n} \to 0$ strongly for $n \to +\infty$.

Now every contraction T can be orthogonally decomposed into a unitary part T_0 acting on $\mathcal{H}_0$ and a completely nonunitary part T_1 acting on $\mathcal{H}_1$, i.e.

$$T = T_0 \oplus T_1. \tag{1.1}$$

The completely nonunitary part is completely characterized by the so-called characteristic function of the contraction T defined in [8].

Definition 1.1. We say the contraction T admits a dissipative Lax-Phillips scattering theory if and only if there are subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ such that the triplet $\{T, \mathcal{D}_+, \mathcal{D}_-\}$ obeys the conditions (h1) - (h4).

In connection with this definition we remark that it is not excluded that one of the subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ or both subspaces are zero. For instance a contraction T belonging to the class C_{00} admits a dissipative Lax-Phillips scattering theory. To show this we set $\mathcal{D}_+ = \mathcal{D}_- = \{0\}$.

Naturally, the question arises to describe those contractions admitting a dissipative Lax-Phillips scattering theory. In solving this problem it turns out that we can start with a completely nonunitary operator. For this class of operators we necessarily and sufficiently solve the problem in terms of the characteristic function. After that we describe those unitary operators which can be added to a completely nonunitary contraction admitting a dissipative Lax-Phillips scattering theory such that the sum admits such a scattering theory, too.

In the following the considerations are essentially based on a model developed in [6] in order to give an example of a dissipative Lax-Phillips scattering theory with a prescribed scattering matrix. As we will see this model can be regarded as new functional model for the class of contractions admitting a dissipative Lax-Phillips scattering theory. We begin with the description of this model.

2. A functional model

Let $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ be an analytical contraction-valued

function. We assume that there are analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ as well as a strongly measurable function $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ such that

$$(2.1) \qquad S'(t) = \begin{pmatrix} \Theta(e^{it})^* & C(e^{it})^* \\ C_*(e^{it})^* & S(t) \end{pmatrix} : \begin{matrix} \mathcal{L}_* \\ \oplus \\ \mathcal{N}_- \end{matrix} \longrightarrow \begin{matrix} \mathcal{L} \\ \oplus \\ \mathcal{N}_+ \end{matrix}$$

forms a unitary-valued function for a.e. $t \in [0, 2\pi)$.

Let $\hat{U}$ be the multiplication operator induced by e^{it} on $\mathcal{K} = L^2(Q_+)$, $Q_+ = \mathcal{L} \oplus \mathcal{N}_+$, and let S' be the multiplication operator acting between $L^2(Q_-)$, $Q_- = \mathcal{L}_* \oplus \mathcal{N}_-$, and $L^2(Q_+)$ and induced by the unitary-valued function $\{Q_-, Q_+, S'(t)\}$. Obviously, S' is an isometry from $L^2(Q_-)$ onto $L^2(Q_+)$.

Introducing the subspaces G_+,

$$(2.2) \qquad G_+ = H^2(\mathcal{L}),$$

and G_-,

$$(2.3) \qquad G_- = S'(L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*)),$$

as well as the subspace $\mathcal{H} = L^2(Q_+) \ominus (G_+ \oplus G_-)$ it is not hard to see that

$$(2.4) \qquad \hat{T} = P^{\mathcal{K}}_{\mathcal{H}} \hat{U} \restriction \mathcal{H}$$

defines a contraction on $\mathcal{H}$. Moreover, this contraction admits a dissipative Lax-Phillips scattering theory. To show this we introduce the subspaces $\mathcal{D}_+ = H^2(\mathcal{N}_+)$ and $\mathcal{D}_- = S'(L^2(\mathcal{N}_-) \ominus H^2(\mathcal{N}_-))$. Now taking into account the considerations of Theorem 4.1 of [6] it is not hard to see that the triplet $\{\hat{T}, \mathcal{D}_+, \mathcal{D}_-\}$ fulfils the assumptions (h1) - (h4).

Further we remark that the operator $\hat{U}$ is a unitary dilation of $\hat{T}$.

Our next step is to calculate the characteristic function of $\hat{T}$.

<u>Proposition 2.1.</u> If $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ is a purely analytical contraction-valued function, then the characteristic function of $\hat{T}$ coincides with $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$.

<u>Proof.</u> To prove this proposition we establish that $\hat{U}$ is a minimal unitary dilation of $\hat{T}$. We consider the subspaces $\mathfrak{L} = ((\hat{U} - \hat{T})\mathcal{H})^-$ and $\mathfrak{L}_* = ((I - \hat{U}^*\hat{T})\mathcal{H})^-$. It is easy to see that we have $\mathfrak{L} \subseteq H^2(\mathcal{L})$ and $\mathfrak{L}_* \subseteq S'(L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*))$. Since we obtain

$$(2.5) \qquad \mathfrak{L} \perp \hat{U} H^2(\mathcal{L})$$

we can identify $\mathfrak{L}$ with a subspace $\tilde{\mathcal{L}}$ of $\mathcal{L}$. Because of

$$(2.6) \qquad \mathfrak{L}_* \perp \hat{U}^* S'(L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*))$$

there is a subspace $\tilde{\mathcal{L}}_*$ of $\mathcal{L}_*$ such that we have

$$(2.7) \qquad \mathfrak{L}_* = \hat{U}^* S' \tilde{\mathcal{L}}_*,$$

where we have identified $\tilde{\mathcal{L}}_*$ with a subspace of the subspace of constants of $L^2(\mathcal{L}_*)$. We set

$$(2.8) \qquad \tilde{\mathcal{K}} = \tilde{G}_+ \oplus \mathcal{H} \oplus \tilde{G}_-$$

defining G_+ and G_- by

$$(2.9) \qquad \tilde{G}_+ = M_+(\mathfrak{L}) = H^2(\tilde{\mathcal{L}})$$

and

$$(2.10) \qquad \tilde{G}_- = M(\mathfrak{L}_*) \ominus M_+(\mathfrak{L}_*) = S'(L^2(\tilde{\mathcal{L}}_*) \ominus H^2(\tilde{\mathcal{L}}_*)).$$

On account of (2.8) - (2.10), the fact that $\hat{T}$ fulfils the assumptions (h1) - (h4), Lemma 3 of [2] and the structure theorem 2.1 of [8, chapter II] we obtain

$$(2.11) \qquad \tilde{\mathcal{K}} = L^2(\tilde{Q}_+) = S'L^2(\tilde{Q}_-),$$

where we have set $\tilde{Q}_+ = \mathcal{N}_+ \oplus \tilde{\mathcal{L}}$ and $\tilde{Q}_- = \mathcal{N}_- \oplus \tilde{\mathcal{L}}_*$. Because of $\mathcal{K} = L^2(Q_+) = S'L^2(Q_-)$ we find

$$(2.12) \qquad \mathcal{K} \ominus \tilde{\mathcal{K}} = L^2(\mathcal{L} \ominus \tilde{\mathcal{L}}) = S'L^2(\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*).$$

But (2.12) implies that $\{\mathcal{L} \ominus \tilde{\mathcal{L}}, \mathcal{L}_* \ominus \tilde{\mathcal{L}}_*, P^{\mathcal{L}_*}_{\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*} \Theta(\lambda) \upharpoonright \mathcal{L} \ominus \tilde{\mathcal{L}}\}$ is an inner function of both sides. Further, taking into account (2.8) - (2.10) and $L^2(Q_+) \ominus (H^2(\mathcal{L}) \oplus S'(L^2(\mathcal{L}_*) \ominus H^2(\mathcal{L}_*))) = \mathcal{H} \perp \mathcal{K} \ominus \tilde{\mathcal{K}}$ we obtain

$$(2.13) \qquad L^2(\mathcal{L} \ominus \tilde{\mathcal{L}}) = H^2(\mathcal{L} \ominus \tilde{\mathcal{L}}) \oplus S'(L^2(\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*) \ominus H^2(\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*)),$$

or, equivalently,

$$(2.14) \qquad L^2(\mathcal{L} \ominus \tilde{\mathcal{L}}) \ominus H^2(\mathcal{L} \ominus \tilde{\mathcal{L}}) = S'(L^2(\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*) \ominus H^2(\mathcal{L}_* \ominus \tilde{\mathcal{L}}_*)),$$

which implies that $\{\mathcal{L}\ominus\tilde{\mathcal{L}}, \mathcal{L}_*\ominus\tilde{\mathcal{L}}_*, P^{\mathcal{L}_*}_{\mathcal{L}_*\ominus\tilde{\mathcal{L}}_*}\Theta(\lambda)\upharpoonright\mathcal{L}\ominus\tilde{\mathcal{L}}\}$ is an outer function. Consequently, $\{\mathcal{L}\ominus\tilde{\mathcal{L}}, \mathcal{L}_*\ominus\tilde{\mathcal{L}}_*, P^{\mathcal{L}_*}_{\mathcal{L}_*\ominus\tilde{\mathcal{L}}_*}\Theta(\lambda)\upharpoonright\mathcal{L}\ominus\tilde{\mathcal{L}}\}$ is a unitary constant. But $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ is a purely analytical function. Consequently, we find $\mathcal{L} = \tilde{\mathcal{L}}$ and $\mathcal{L}_* = \tilde{\mathcal{L}}_*$, which shows that $\hat{U}$ is a minimal unitary dilation of $\hat{T}$.

Now taking into account Proposition 3.1 of [6] we conclude that $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ is the characteristic function of $\hat{T}$. ∎

Our next aim is to calculate the unitary part $\hat{T}_0$ of the contraction $\hat{T}$. In order to calculate this part we remark that the intersection of the residual and the $*$-residual subspace of the minimal unitary dilation of a contraction coincides with the unitary subspace of this contraction.

In the following we denote the multiplication operator induced by $\{\mathcal{L}, \mathcal{N}_-, C(e^{it})\}$ and acting between $L^2(\mathcal{L})$ and $L^2(\mathcal{N}_-)$ by C. Similarly, we denote the multiplication operator induced by $\{\mathcal{N}_+, \mathcal{L}_*, C_*(e^{it})\}$ by C_*.

<u>Proposition 2.2.</u> If $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ is a purely analytical contraction-valued function, then the unitary subspace $\mathcal{H}_0$ of $\hat{T}$ is given by

(2.15) $\mathcal{H}_0 = \ker(C_*) = S'\ker(C^*)$.

<u>Proof.</u> Using the previous remark and Lemma 3 of [2] we find

(2.16) $\mathcal{H}_0 = L^2(\mathcal{N}_+) \cap S'L^2(\mathcal{N}_-)$.

Consequently, $f \in L^2(\mathcal{N}_+)$ belongs to $\mathcal{H}_0$ if and only if there is an element $g \in L^2(\mathcal{N}_-)$ such that we have

(2.17) $\begin{bmatrix} 0 \\ f(t) \end{bmatrix} = \begin{bmatrix} \Theta(e^{it})^* & C(e^{it})^* \\ C_*(e^{it})^* & S(t) \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$

for a.e. $t \in [0,2\pi)$. Hence we obtain

(2.18) $\mathcal{H}_0 = S'\ker(C^*)$.

Taking into account (5.3) and (5.6) of [6] we obtain

(2.19) $S'(t)\ker(C(e^{it})^*) = \ker(C_*(e^{it}))$

for a.e. $t \in [0,2\pi)$, which implies $S'\ker(C^*) = \ker(C_*)$. ∎

From Proposition 2.2 we easily obtain that the operator $\hat{T}$ is completely nonunitary if and only if we have

(2.20) $\ker(C_*(e^{it})) = \{0\}$

or, equivalently,

(2.21) $\ker(C(e^{it})^*) = \{0\}$

for a.e. $t \in [0,2\pi)$ provided $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ is purely analytical.

<u>3. Dissipative Lax-Phillips scattering theory and characteristic function</u>

Let T be a contraction on a separable Hilbert space $\mathcal{H}$.

<u>Lemma 3.1.</u> If T admits a dissipative Lax-Phillips scattering theory, then the completely nonunitary part T_1 of T admits a dissipative Lax-Phillips scattering theory, too.

<u>Proof.</u> By $\mathcal{H}_1$ we denote the completely nonunitary subspace

of T. We introduce the subspaces $\mathfrak{A}_{\pm}$ of $\mathfrak{H}_1$ defined by

$$(3.1) \qquad \mathfrak{A}_{\pm} = (P^{\mathfrak{H}}_{\mathfrak{H}_1}\mathfrak{D}_{\pm})^{-}.$$

Next we show that $\{T_1, \mathfrak{A}_+, \mathfrak{A}_-\}$ forms a dissipative Lax-Phillips scattering theory. Obviously, the condition (h1) is fulfilled. Because of

$$(3.2) \qquad \|f\|^2 = \|Tf\|^2 = \|P^{\mathfrak{H}}_{\mathfrak{H}_0}Tf\|^2 + \|P^{\mathfrak{H}}_{\mathfrak{H}_1}Tf\|^2,$$

$\mathfrak{H}_0 = \mathfrak{H} \ominus \mathfrak{H}_1$, we obtain

$$(3.3) \qquad \|f\|^2 = \|P^{\mathfrak{H}}_{\mathfrak{H}_0}f\|^2 + \|T_1P^{\mathfrak{H}}_{\mathfrak{H}_1}f\|^2$$

or, equivalently,

$$(3.4) \qquad \|P^{\mathfrak{H}}_{\mathfrak{H}_1}f\|^2 = \|T_1P^{\mathfrak{H}}_{\mathfrak{H}_1}f\|^2$$

for every $f \in \mathfrak{D}_+$. Consequently, $T_1 \restriction \mathfrak{A}_+$ is an isometry. Similarly, we establish the second part of (h2). The condition (h3) follows from the fact that T_1 is completely nonunitary. To prove (h4) we note that $f \in \mathfrak{H}_1 \ominus \mathfrak{A}_+$ implies $f \in \mathfrak{H} \ominus \mathfrak{D}_+$. This yields $\mathfrak{H}_1 \ominus \mathfrak{A}_+ \subseteq \mathfrak{H} \ominus \mathfrak{D}_+$ or, equivalently, $P^{\mathfrak{H}}_{\mathfrak{H}_1 \ominus \mathfrak{A}_+} \leq P^{\mathfrak{H}}_{\mathfrak{H} \ominus \mathfrak{D}_+}$. Hence we get

$$(3.5) \qquad \lim_{n\to+\infty} P^{\mathfrak{H}_1}_{\mathfrak{H}_1 \ominus \mathfrak{A}_+}T_1^n f = \lim_{n\to+\infty} P^{\mathfrak{H}}_{\mathfrak{H}_1 \ominus \mathfrak{A}_+}P^{\mathfrak{H}}_{\mathfrak{H} \ominus \mathfrak{D}_+}T^n f$$
$$= 0$$

for every $f \in \mathfrak{H}_1$. Similarly, we prove $P^{\mathfrak{H}_1}_{\mathfrak{H}_1 \ominus \mathfrak{A}_-}T^{*n} \longrightarrow 0$ strongly for $n \to +\infty$. ∎

We note that it is quite possible that one of the subspaces $\mathfrak{A}_+$ and $\mathfrak{A}_-$ or both are zero.

Lemma 3.1 allows us to reduce the investigations to those completely nonunitary contractions admitting a dissipative Lax-Phillips scattering theory.

Theorem 3.2. The completely nonunitary contraction T admits a dissipative Lax-Phillips scattering theory if and only if there exist two analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ such that the characteristic function of T $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ obeys the conditions

$$(3.6) \qquad I = \Theta(e^{it})\Theta(e^{it})^* + C_*(e^{it})C_*(e^{it})^*$$

and

$$(3.7) \qquad I = \Theta(e^{it})^*\Theta(e^{it}) + C(e^{it})^*C(e^{it})$$

for a.e. $t \in [0, 2\pi)$.

Proof. Let us suppose that T admits a dissipative Lax-Phillips scattering theory. Applying Proposition 3.1, Theorem 3.3 and Proposition 5.1 of [6] we obtain the existence of analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ such that (3.6) and (3.7) are valid.

Let $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ be the characteristic function of a completely nonunitary contraction T which fulfils (3.6) and (3.7). We show that there is a strongly measurable contraction-valued function $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$ such that (2.1) forms a strongly measurable unitary-valued function for a.e. $t \in [0, 2\pi)$.

For this purpose we suppose without loss of generality that $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ is an outer function and

$\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ is an $*$-outer function. Both functions are uniquely determined by (3.6) and (3.7) in this case.

Next we establish that the relation

$$(3.8) \qquad S(t)C(e^{it}) = -C_*(e^{it})^* \theta(e^{it})$$

uniquely defines (mod$|.|$) a strongly measurable contraction-valued function $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$. To prove this it is sufficient to show that the inequality

$$(3.9) \qquad \theta(e^{it})^* C_*(e^{it}) C_*(e^{it})^* \theta(e^{it}) \leqslant C(e^{it})^* C(e^{it})$$

is valid for a.e. $t \in [0,2\pi)$. From $0 \leqslant (I - \theta(e^{it})^* \theta(e^{it}))^2$ for a.e. $t \in [0,2\pi)$ we obtain

$$(3.10) \qquad \theta(e^{it})^* (I - \theta(e^{it})\theta(e^{it})^*)\theta(e^{it}) \leqslant$$

$$\leqslant I - \theta(e^{it})^* \theta(e^{it})$$

for a.e. $t \in [0,2\pi)$. Taking into account (3.6) and (3.7) we obtain (3.9) for a.e. $t \in [0,2\pi)$. To verify that (2.1) is a unitary-valued function for a.e. $t \in [0,2\pi)$ it is necessary to prove that the relations (5.1) - (5.6) of [6] are valid. The relations (5.1), (5.4) and (5.5) of [6] coincide with (3.6), (3.7) and (3.8). To establish (5.2) of [6] we multiply (3.8) on the left by $C_*(e^{it})$. We find

$$(3.11) \qquad C_*(e^{it})S(t)C(e^{it}) = -C_*(e^{it})C_*(e^{it})^* \theta(e^{it})$$

for a.e. $t \in [0,2\pi)$. Using (3.6) and (3.7) we get

$$(3.12) \qquad C_*(e^{it})S(t)C(e^{it}) = -\theta(e^{it})C(e^{it})^* C(e^{it})$$

for a.e. $t \in [0,2\pi)$. But $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ is an outer function, which proves (5.2) of [6]. Further, we find from (3.8)

$$(3.13) \qquad C(e^{it})^* S(t)^* S(t) C(e^{it}) = \theta(e^{it})^* C_*(e^{it}) \cdot$$

$$\cdot C_*(e^{it})^* \theta(e^{it})$$

for a.e. $t \in [0,2\pi)$. Taking into account (3.6) we obtain

$$(3.14) \qquad C(e^{it})^* S(t)^* S(t) C(e^{it}) =$$

$$= \theta(e^{it})^* (I - \theta(e^{it})\theta(e^{it})^*)\theta(e^{it})$$

for a.e. $t \in [0,2\pi)$. Because of (3.7) we conclude

$$(3.15) \qquad C(e^{it})^* S(t)^* S(t) C(e^{it}) =$$

$$= C(e^{it})^* (I - C(e^{it}) C(e^{it})^*) C(e^{it})$$

for a.e. $t \in [0,2\pi)$. The function $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ is an outer one. Consequently, (3.15) implies (5.3) of [6]. Similarly we prove (5.6) of [6]. Hence $\{\mathcal{L}_* \oplus \mathcal{N}_-, \mathcal{L} \oplus \mathcal{N}_+, S'(t)\}$ is unitary-valued.

Now we consider the functional model of section 2. In accordance with Proposition 2.1 we obtain a contraction $\hat{T}$ characteristic function of which coincides with $\{\mathcal{L}, \mathcal{L}_*, \theta(\lambda)\}$. Further taking into account that $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ is an outer function and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ is an $*$-outer function we get that the relations (2.20) and (2.21) are fulfilled. Hence Proposition 2.2 implies that $\hat{T}$ is a completely nonunitary contraction. Consequently, the contractions T and $\hat{T}$ are unitarily equivalent. But $\hat{T}$ admits a dissipative Lax-Phillips scattering theory. Hence T admits such a scattering theory, too. ∎

Next we turn our attention to the unitary part. On account of Lemma 3.1 we assume that there is a completely nonunitary contraction T_1 admitting a dissipative Lax-Phillips scattering theory. Which unitary operators T_0 can be added such that $T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory, too? To answer to this question we need two lemmas.

Lemma 3.3. Let $\{P(t)\}_{t\in[0,2\pi)}$ be a strongly measurable family of projections acting on the separable Hilbert space $\mathcal{N}_+$ such that we have

$$(3.16) \qquad \dim(P(t)) = n \leq +\infty$$

for a.e. $t \in [0,2\pi)$. Then there is a separable Hilbert space $\mathcal{Q}$ of dimension n and a strongly measurable family of isometries $\{\mathcal{Q}, \mathcal{N}_+, V(t)\}$ such that we have

$$(3.17) \qquad P(t) = V(t)V(t)^*$$

for a.e. $t \in [0,2\pi)$.

We left the proof of Lemma 3.1 to the reader.

In the following we introduce two analytical contraction-valued functions $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$. In a natural way we associate with these functions two multiplication operators acting from $L^2(\mathcal{G})$ into $L^2(\mathcal{N}_-)$ and $L^2(\mathcal{N}_+)$ into $L^2(\mathcal{G}_*)$, which we denote by G and G_*, respectively. By R and R_* we denote multiplication operators induced by e^{it} on $L^2(\mathcal{N}_-)$ and $L^2(\mathcal{N}_+)$, respectively.

Lemma 3.4. Let T_0 be an absolutely continuous unitary operator. Then there is an inner function $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ and there is an $*$-inner function $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$ such that

$$(3.18) \qquad \dim(\ker(G_*(e^{it}))) = \dim(\ker(G(e^{it})^*))$$

for a.e. $t \in [0,2\pi)$ and, moreover, the unitary operators $R \restriction \ker(G^*)$ and $R_* \restriction \ker(G_*)$ are unitarily equivalent to T_0.

Proof. The proof is based on Satz 4.4.4 and Korollar 4.4.5 of [7]. Transforming the maximal dissipative operator H of these theorems to a contraction via the Cayley transform $H \longrightarrow \frac{H+i}{H-i}$ we find from Satz 4.4.4 and Korollar 4.4.5 that for every absolutely continuous operator T_0 there is a contraction D of class C_{10} and there is a contraction D_* of class C_{01} such that $*$-residual part of the minimal unitary dilation of D and the residual part of the minimal unitary dilation of D_* are unitarily equivalent to T_0. Let $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$ be the characteristic functions of D and D_*, respectively. In virtue of Proposition 3.5 of [8, chapter VI] $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ is an inner function and $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$ is an $*$-inner function. Taking into account the functional model of a contraction introduced by B.Sz.-Nagy and C.Foias [8, chapter VI, Theorem 2.3 and Theorem 2.3*] we find that $*$-residual part of the minimal unitary dilation of D is unitarily equivalent to $R \restriction \ker(G^*)$. Similarly, we get that the residual part of D_* is unitarily equivalent to $R_* \restriction \ker(G_*)$. Consequently, both operators $R \restriction \ker(G^*)$ and $R_* \restriction \ker(G_*)$ are unitarily equivalent to T_0. But this implies that the operators $R \restriction \ker(G^*)$ and $R_* \restriction \ker(G_*)$ are unitarily equivalent. Hence there is a partial isometry $V: L^2(\mathcal{N}_-) \to L^2(\mathcal{N}_+)$ such that $V^*V = P^{L^2(\mathcal{N}_-)}_{\ker(G^*)}$, $VV^* = P^{L^2(\mathcal{N}_+)}_{\ker(G_*)}$ and

$$R_*V = VR. \tag{3.19}$$

But R and R_* are multiplication operators induced by e^{it}. Consequently, V can be represented by a multiplication operator induced by a strongly measurable family $\{\mathcal{N}_-, \mathcal{N}_+, V(t)\}$ of partial isometries which fulfil $V(t)^*V(t) = I_{\mathcal{N}_-} - G(e^{it})G(e^{it})^*$ and $V(t)V(t)^* = I_{\mathcal{N}_+} - G_*(e^{it})^*G_*(e^{it})$ for a.e. $t \in [0,2\pi)$. Both relations imply (3.18). ∎

Corollary 3.5. Let T_0 be an absolutely continuous unitary operator. Then there is an inner function $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ and there is an $*$-inner function $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$ such that (3.18) and

$$\dim(\mathcal{G}) = \dim(\mathcal{G}_*) = +\infty \tag{3.20}$$

hold and, moreover, the unitary operators $R\restriction\ker(G^*)$ and $R_*\restriction\ker(G_*)$ are unitarily equivalent to T_0.

We left the proof to the reader. Now we come to the solution of the proposed problem.

Theorem 3.6. Let T_1 be a completely nonunitary contraction on $\mathcal{H}_1$ admitting a dissipative Lax-Phillips scattering theory $\{T_1, \mathfrak{A}_+, \mathfrak{A}_-\}$. Let T_0 be a unitary operator on $\mathcal{H}_0$.

(i) If one of the unilateral shifts $T_1\restriction\mathfrak{A}_+$ or $T_1^*\restriction\mathfrak{A}_-$ has a finite multiplicity, then $T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory if and only if T_0 is a bilateral shift.

(ii) If both unilateral shifts $T_1\restriction\mathfrak{A}_+$ and $T_1^*\restriction\mathfrak{A}_-$ have

an infinite multiplicity, then $T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory if and only if T_0 is absolutely continuous.

Proof. (i) Let $T_1\restriction\mathfrak{A}_+$ be a unilateral shift of finite multiplicity. By $\mathfrak{R}_{1*}$ we denote the $*$-residual subspace of the minimal unitary dilation U_1 of T_1. Let R_{1*} be the $*$-residual part of U_1, $R_{1*} = U_1\restriction\mathfrak{R}_{1*}$. Taking into account Lemma 3 of [2] we find that R_{1*} is a bilateral shift of finite multiplicity.

Let $T = T_0 \oplus T_1$. We denote the $*$-residual subspace and $*$-residual part of the minimal unitary dilation U of T by $\mathfrak{R}_*$ and R_*, respectively. Because of Lemma 3 of [2] R_* is a bilateral shift.

Regarding U_1 as a part of U we obtain $\mathfrak{R}_{1*} \subseteq \mathfrak{R}_*$. Moreover, the subspace $\mathfrak{R}_{1*}$ reduces R_* and we have $R_{1*} = R_*\restriction\mathfrak{R}_{1*}$. Representing the bilateral shift R_* as the multiplication operator induced by e^{it} on $L^2(\mathcal{N}_+)$ it is not hard to see that in this representation the projection $P^{\mathfrak{R}_*}_{\mathfrak{R}_{1*}}$ is represented as the multiplication operator induced by a strongly measurable family of projections $\{P(t)\}_{t\in[0,2\pi)}$. Obviously, we have $\dim(P(t)\mathcal{N}_+) = n < +\infty$ for a.e. $t \in [0,2\pi)$. Consequently, we find $\dim((I_{\mathcal{N}_+} - P(t))\mathcal{N}_+) = m \leq +\infty$ for a.e. $t \in [0,2\pi)$. Using now Lemma 3.3 we find that $R_*\restriction\mathfrak{R}_*\ominus\mathfrak{R}_{1*}$ is a bilateral shift, too. But $\mathfrak{R}_*\ominus\mathfrak{R}_{1*}$ coincides with $\mathcal{H}_0$ and $R_*\restriction\mathfrak{R}_*\ominus\mathfrak{R}_{1*}$ equals T_0. Hence T_0 is a bilateral shift. Similarly, we prove this assertion assuming $T_1^*\restriction\mathfrak{A}_-$ has a finite multiplicity.

To show that $T = T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory is obvious provided T_0 is a bilateral shift.

(ii) It is easy to see that T_0 is absolutely continuous if $T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory. To prove the converse we use the functional model developed in section 2. Let $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ be the characteristic function of T_1. On account of Theorem 3.2 there is an outer function $\{\mathcal{L}, \mathcal{G}, B(\lambda)\}$ and there is an $*$-outer function $\{\mathcal{G}_*, \mathcal{L}_*, B_*(\lambda)\}$ such that we have

$$(3.21) \qquad I = \Theta(e^{it})\Theta(e^{it})^* + B_*(e^{it})B_*(e^{it})^*$$

and

$$(3.22) \qquad I = \Theta(e^{it})^*\Theta(e^{it}) + B(e^{it})^*B(e^{it})$$

for a.e. $t \in [0,2\pi)$. In accordance with (3.8) the relation

$$(3.23) \qquad S_0(t)B(e^{it}) = -B_*(e^{it})^*\Theta(e^{it})$$

defines a strongly measurable contraction-valued function $\{\mathcal{G}, \mathcal{G}_*, S_0(t)\}$. Using the considerations of Theorem 3.2 and the contraction-valued functions $\{\mathcal{L}, \mathcal{L}_*, \Theta(e^{it})\}$, $\{\mathcal{L}, \mathcal{G}, B(e^{it})\}$, $\{\mathcal{G}_*, \mathcal{L}_*, B_*(e^{it})\}$ and $\{\mathcal{G}, \mathcal{G}_*, S_0(t)\}$ we perform a functional model of a completely nonunitary contraction $\hat{T}_{c.n.u.}$ which is unitarily equivalent to T_1.

The infinite multiplicity of the unilateral shifts $T_1 \restriction \mathfrak{A}_+$ and $T_1^* \restriction \mathfrak{A}_-$ yields that the Hilbert spaces $\mathcal{G}$ and $\mathcal{G}_*$ are infinite dimensional, i.e. $\dim(\mathcal{G}) = \dim(\mathcal{G}_*) = +\infty$.

In the following we modify the considerations of Theorem 3.2 to obtain not only a completely nonunitary contraction, but also a contraction with a prescribed unitary part which admits a dissipative Lax-Phillips scattering theory.

On account of Corollary 3.5 there is an inner function $\{\mathcal{G}, \mathcal{N}_-, G(\lambda)\}$ and there is an $*$-inner function $\{\mathcal{N}_+, \mathcal{G}_*, G_*(\lambda)\}$ such that (3.18) and $R \restriction \ker(G^*)$ and $R_* \restriction \ker(G_*)$ are unitarily equivalent to T_0.

We introduce the analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$,

$$(3.24) \qquad C(\lambda) = G(\lambda)B(\lambda),$$

and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$,

$$(3.25) \qquad C_*(\lambda) = B_*(\lambda)G_*(\lambda),$$

$\lambda \in \{z \in \mathbb{C} : |z| < 1\}$.

Because of (3.18) and Lemma 3.3 there is a strongly measurable family of partial isometries $\{\mathcal{N}_-, \mathcal{N}_+, V(t)\}$ obeying $V(t)^*V(t) = I_{\mathcal{N}_-} - G(e^{it})G(e^{it})^*$ and $V(t)V(t)^* = I_{\mathcal{N}_+} - G_*(e^{it})^*G_*(e^{it})$ for a.e. $t \in [0,2\pi)$. Introducing the strongly measurable contraction-valued function $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$,

$$(3.26) \qquad S(t) = \begin{pmatrix} V(t) & 0 \\ 0 & G_*(e^{it})^*S_0(t)G(e^{it})^* \end{pmatrix} :$$

$$: \begin{matrix} \ker(G(e^{it})^*) \\ \oplus \\ \mathrm{ima}(G(e^{it})) \end{matrix} \longrightarrow \begin{matrix} \ker(G_*(e^{it})) \\ \oplus \\ \mathrm{ima}(G_*(e^{it})^*) \end{matrix}$$

it is not hard to see that the strongly measurable operator-valued function $\{\mathcal{L} \oplus \mathcal{N}_-, \mathcal{L}_* \oplus \mathcal{N}_+, S'(t)\}$ performed by (2.1),(3.23),(3.24),(3.25) and (3.26) is unitary-valued. In such a way in accordance with Proposition 2.1 we obtain a contraction $\hat{T}$ characteristic function of which coincides with $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$. Hence the completely nonunitary part $\hat{T}_1$ of $\hat{T}$ is unitarily equivalent to T_1. It remains to calculate the unitary part $\hat{T}_0$ of $\hat{T}$.

On account of Lemma 3 of [2] the $*$-residual subspace of the minimal unitary dilation $\hat{U}$ of $\hat{T}$ coincides with $L^2(\mathcal{N}_+)$. Hence the $*$-residual part of $\hat{U}$ can be identified with R_*. On account of Proposition 2.2 we find $\hat{T}_0 = \hat{U}\restriction \ker(C_*)$. But a simple calculation shows $\ker(C_*) = \ker(G_*)$. Using $\hat{U}\restriction L^2(\mathcal{N}_+) = R_*$ we find $\hat{T}_0 = \hat{U}\restriction \ker(C_*) = R_*\restriction \ker(C_*) = R_*\restriction \ker(G_*)$. Consequently, the unitary part $\hat{T}_0$ of $\hat{T}$ is unitarily equivalent to T_0.

Summing up we find that the completely nonunitary part $\hat{T}_1$ of $\hat{T}$ and the unitary part $\hat{T}_0$ of $\hat{T}$ are unitarily equivalent to T_1 and T_0 of T, repsectively. But $\hat{T}$ admits a dissipative Lax-Phillips scattering theory. Hence $T_0 \oplus T_1$ admits a dissipative Lax-Phillips scattering theory, too. ∎

In connection with Theorem 3.6 we remark that if the characteristic function $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ of a contraction T fulfils the conditions (3.6) and (3.7) only for analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ acting between infinite dimensional Hilbert spaces, then the unitary part of T has no influence on the existence of a dissipative Lax-Phillips scattering theory with respect to T.

<u>Corollary 3.7.</u> Let T_1 be a completely nonunitary contraction on $\mathcal{H}_1$ admitting a dissipative Lax-Phillips scattering theory. There exists a unitary operator T_0 on $\mathcal{H}_0$ such that $T_0 \oplus T_1$ admits an orthogonal dissipative Lax-Phillips scattering theory if and only if the characteristic func-

tion $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ of T_1 possesses a Darlington synthesis in the sense of [1].

<u>Proof.</u> If $T = T_0 \oplus T_1$ possesses an orthogonal dissipative Lax-Phillips scattering theory, then the desired conclusion can be obtained from Proposition 3.1 and Corollary 3.4 of [6].

Conversely, if $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$ admits a Darlington synthesis, then there are analytical contraction-valued functions $\{\mathcal{L}, \mathcal{N}_-, C(\lambda)\}$, $\{\mathcal{N}_+, \mathcal{L}_*, C_*(\lambda)\}$ and $\{\mathcal{N}_+, \mathcal{N}_-, \Pi(e^{it})\}$ such that

$$S'(t) = \begin{pmatrix} \Theta(e^{it})^* & C(e^{it})^* \\ C_*(e^{it})^* & \Pi(e^{it})^* \end{pmatrix} : \begin{matrix} \mathcal{L}_* \\ \oplus \\ \mathcal{N}_- \end{matrix} \longrightarrow \begin{matrix} \mathcal{L} \\ \oplus \\ \mathcal{N}_+ \end{matrix} \tag{3.27}$$

forms a unitary-valued function for a.e. $t \in [0, 2\pi)$. Taking into account Corollary 4.2 of [6] we can regard $\{\mathcal{N}_-, \mathcal{N}_+, S(t)\}$, $S(t) = \Pi(e^{it})^*$, $t \in [0, 2\pi)$, as the scattering matrix of an orthogonal dissipative Lax-Phillips scattering theory $\{\hat{T}, \mathcal{D}_+, \mathcal{D}_-\}$, i.e. $\mathcal{D}_+ \perp \mathcal{D}_-$. Because of Proposition 2.1 the characteristic function of $\hat{T}$ coincides with $\{\mathcal{L}, \mathcal{L}_*, \Theta(\lambda)\}$. Hence the completely nonunitary part $\hat{T}_1$ of $\hat{T}$ is unitarily equivalent to T_1. But this yields the existence of a unitary operator T_0 such that $T_0 \oplus T_1$ admits an orthogonal dissipative Lax-Phillips scattering theory. ∎

Corollary 3.7 implies the following

<u>Corollary 3.8.</u> A completely nonunitary contraction T_1 can be orthogonally enlarged by a unitary operator such that the sum admits an orthogonal dissipative Lax-Phillips scattering theory if and only if the adjoint characteristic function of T_1 can be regarded as the scattering matrix of an orthogonal dissipative Lax-Phillips scattering theory.

We left the proof to the reader.

References

[1] R.G.Douglas, J.W.Helton, Inner dilations of analytic matrix functions and Darlington synthesis. Acta Sci. Math. 34(1973), 61-66.

[2] C.Foias, On the Lax-Phillips nonconservative scattering theory. J. Funct. Analysis 19(1975), 273-301.

[3] П.Д.Лакс, Р.С.Филлипс, "Теория рассеяния", Издательство "Мир", Москва, 1971.

[4] P.D.Lax, R.S.Phillips, Scattering theory for the acoustic equation in an even number of space dimensions. Indian Univ. Math. J. 22(1972), 101-134.

[5] P.D.Lax, R.S.Phillips, Scattering theory for dissipative hyperbolic systems. J. Funct. Analysis 14(1973), 172-235.

[6] H.Neidhardt, On the dissipative Lax-Phillips scattering theory. to appear.

[7] H.Neidhardt, Eine mathematische Streutheorie für maximal dissipative Operatoren. Report R-Math-03/86, AdW der DDR, Berlin 1986.

[8] Б.Секефальви-Надь, Ч.Фояш, Гармонический анализ операторов в гильбертовом пространстве, Издательство "Мир", Москва, 1970.

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Найдхардт Х. E5-87-400

Диссипативная теореия рассеяния Лакса-Филлипса и характеристическая функция сжимающего оператора

Рассматривается вопрос о характеристике всех тех сжимающих операторов, которые допускают диссипативную теорию рассеяния Лакса-Филлипса. Характеристика дана в терминах характеристической функции сжимающего оператора и его унитарной части. Более того, проблема поставлена и решена в описании всех тех вполне неунитарных сжимающих операторов, которые можно ортогональным образом расширить унитарным оператором так, что сумма допускает ортогональную диссипативную теорию рассеяния Лакса-Филлипса.

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Neidhardt H. E5-87-400

Dissipative Lax-Phillips Scattering Theory and the Characteristic Function of a Contraction

The paper deals with the problem to characterize all those contractions admitting a dissipative Lax-Phillips scattering theory. The characterization is given in terms of the characteristic function of a contraction and its unitary part. Moreover, the problem is considered and solved to describe all those completely nonunitary contractions which can be orthogonally enlarged by a unitary operator such that the sum admits an orthogonal dissipative Lax-Phillips scattering theory.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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