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V.P.Gerdt, A.B.Shabat*, S.I.Svinolupov*,
A.Yu.Zharkov**

COMPUTER ALGEBRA APPLICATION FOR INVESTIGATING INTEGRABILITY OF NONLINEAR EVOLUTION SYSTEMS

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*Bashkir Branch of the USSR Academy of Sciences, Ufa, USSR

**Saratov State University, Saratov, USSR

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1. At present an intensive work on classification of integrable non-linear partial differential equations with two independent variables is carried out. In a number of cases (/1/-/5/) the formulation of effective criteria of integrability has been achieved and the complete lists of the integrable systems have been obtained. For example, in /4/,/5/ the complete list of integrable systems of the Schrödinger type

$$u_t = u_{xx} + f(u, v, u_x, v_x), \quad v_t = -v_{xx} + g(u, v, u_x, v_x) \tag{1}$$

is obtained.

In frames of the symmetry approach (see /6/,/7/,/8/) the definition of integrability is based on internal properties of the equations and the conditions used for classification are the necessary conditions for existence of higher-order symmetries and conservation laws. General derivation scheme for this conditions (see /7/,/8/) covers the systems of the form

$$u_t = \Phi(x, u, \dots, u_N) = \Lambda u_N + F(x, u, \dots, u_{N-1}), \quad N \geqslant 2, \tag{2}$$

where

$$\begin{gathered} u = u(t,x), \quad u_k = \partial^k u / \partial x^k, \quad u = (u^1, \dots, u^M), \quad F = (F^1, \dots, F^M), \\ \Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_M), \quad \lambda_i \in \mathbb{C}, \quad \lambda_i \neq 0, \quad \lambda_i \neq \lambda_j \ (i \neq j). \end{gathered} \tag{3}$$

Evolution equations integrable by inverse spectral transform and linearizable like the Burgers equation satisfy the conditions of integrability arising in symmetry approach.

There are two types of problems solvable by means of the symmetry approach: 1) properly classification problems of obtaining the complete list of the fixed form systems and describing the most general transforms connecting this systems; 2) for a given concrete system testing the conditions of integrability and computing symmetries and conservation law densities. In present paper we discuss the problems of the second type. Note that such problems remain actual even after the complete lists of systems under consideration have been already obtained. For example, since the list of integrable systems (1) presented in /4/,/5/ is too large, it's more convenient for a given concrete system to check the integrability conditions directly rather than to identify it with one from the list.

The procedure of derivation and checking the necessary conditions of integrability demands tedious algebraic computations. To carry them out automatically it's worth while to use computer algebra systems (/9/).

In the case of scalar evolution equations the algorithm for checking up the integrability conditions for a given equation is already implemented in computer algebra systems REDUCE-2 (/10/) and FORMAC (/11/). In present paper we suggest the algorithm for checking the necessary conditions of integrability and its computer implementation in FORMAC for evolution systems (2)-(3). In addition our FORMAC program allows one to find symmetries and canonical conservation law densities.

2. Let us remind the basic concepts of the symmetry approach and the general derivation scheme for necessary integrability conditions.

The symmetry (generator of infinitesimal symmetry) of evolution system (2) is called the vector function $f=(f^1,\dots,f^M)$ of a finite number of dynamical variables from the infinite set $x, u, u_1, \dots$ such that

$$\frac{df}{dt}=\varphi_*(f), \tag{4}$$

where φ_* is a matrix differential operator

$$\varphi_*=\varphi_u+\varphi_{u_1}D+\dots+\varphi_{u_N}D^N \tag{5}$$

φ_{u_i} - Jacobi matrix, that is

$$[\varphi_{u_i}]_{kj}=\frac{\partial\varphi^k}{\partial u_i^j}.$$

The operators d/dt and $D=d/dx$ in (4)-(5) are the total differentiation operators with respect to variables t and x correspondingly, both acting on the functions of a finite number of dynamical variables:

$$\frac{d}{dt}=\sum_{i=1}^{M}\sum_{j=0}^{\infty}D^j(\varphi^i)\frac{\partial}{\partial u_j^i},$$

$$D=\frac{\partial}{\partial x}+\sum_{i=1}^{M}\sum_{j=0}^{\infty}u_{j+1}^i\frac{\partial}{\partial u_j^i}. \tag{6}$$

Equation (4) defining symmetries means that equation (2) and equation $u_\tau=f(x,u,u_1,\dots)$ are compatible or in other words it means the invariance of equation (2) under infinitesimal transformations $\bar t=t,\ \bar x=x,\ \bar u=u+\tau f(x,u,u_1,\dots)$. The linearization operator $*$ defined by the formula

$$f_*(v)=\frac{\partial}{\partial\varepsilon}f(x,u+\varepsilon v, D(u+\varepsilon v),\dots)\big|_{\varepsilon=0}$$

transforms equation (4) to the following operator relation

$$L_t+[L,\varphi_*]=(\varphi_*)_\tau, \tag{7}$$

where $L=f_*=f_u+f_{u_1}D+\dots$ (compare with (5)).

Let us define the n-th order formal symmetry of the system (2) as any differential operator of the n-th order

$$L=\sum_{k=0}^{n}A_kD^k,\quad \deg L=n \tag{8}$$

with matrix coefficients A_k depending on a finite number of dynamical variables such that

$$L_t-[\varphi_*,L]=Q, \tag{9}$$

where Q is a differential operator of the form $Q=\sum_{k<N}Q_kD^k$.

Substitution of (8) in (9) leads to a chain of equations arising after equating the coefficients of D^j, $j=N+n,\dots,N+1,N$. So, putting the coefficient of D^{N+n} equal to zero one gets

$$[\Lambda,A_n]=0. \tag{10}$$

Then for $j=N+n-1$ one can obtain

$$[\Lambda,A_{n-1}]+N\cdot\Lambda D(A_n)+[\varphi_{u_{N-1}},A_n]=0. \tag{11}$$

Next equations have the following general form

$$[\Lambda,A_{j-N}]+N\cdot\Lambda D(A_{j-N+1})+[\varphi_{u_{N-1}},A_{j-N+1}]+B_j=0,\quad j\geq N, \tag{12}$$

where B_j denotes matrices with components expressed in terms of the defferential operator (5) coefficients and the components of where $i>j-N+1$. One can find from equation (12) non-diagonal part of the matrix A_{j-N} and the diagonal part of the matrix A_{j-N+1}. From equations (10)-(11) it follows that

$$A_n=\mathrm{diag}(\mu_1,\dots,\mu_M),\quad \mu_i\in\mathbb{C}. \tag{13}$$

The formal symmetry (8) is called non-degenerated if

$$\det A_n=\prod_{k=1}^{M}\mu_k\neq 0.$$

Evidently the formal symmetry is defined up to addition the arbitrary diagonal matrix depending on a finite number of dynamical variables (this arbitrariness can be eliminated with normalization $diag\,(A_o)=0$).

The existence conditions of the non-degenerated formal symmetry being the conditions of solvability of eqs. (10)-(12) in terms of matrix-functions of dynamical variables are the criteria for constructing the lists of integrable equations. For example, the complete list of the second order integrable scalar equations of the general form $u_t=\varphi(x,u,u_1,u_2)$ presented in /3/ corresponds to the choice $n=5$. For equations of the form (2) with $M=1,\ N=3$ considered in paper /2/ it was supposed that n=9.

It's easy to verify that the existence of the n_1 -order symmetry (the order of the symmetry is called $deg\, f_*$) with $n_1\geqslant n>N$ leeds to existence of the n-th order formal symmetry (compare (7) and (9)). Thus the conditions of the n-th order formal symmetry existence are the necessary conditions of the n_1-th order symmetries existence, where $n_1\geqslant n$. One can prove (/7/,/8/) that from the existence of the pair of the local high-order conservation laws it follows the existence of the formal symmetry.

Let's remind that the <u>local conservation law</u> of system (2) is given by the pair of scalar functions (ρ,σ) of the dynamical variables such that

$$\frac{d}{dt}(\rho)=D(\sigma). \qquad (14)$$

The function ρ is the density of the conservation law (14) and the order of the conservation law is defined as a degree of the following polynomial in D (see /8/):

$$R=\left(\frac{\delta\rho}{\delta u}\right)_*=\sum_{i\geqslant 0}R_iD^i,\qquad \frac{\delta}{\delta u}=\sum_{k=0}^{\infty}(-1)^kD^k\frac{\partial}{\partial u_k}. \qquad (15)$$

Using the algorithms of manipulations with power series of the form

$$L=\sum_{k\geqslant m}A_kD^k=A_mD^m+\ldots+A_o+A_{-1}D^{-1}+\ldots, \qquad (16)$$

with negative powers of symbol D it's possible to generalize the above definition of the formal symmetry and to denote by the <u>formal symmetry of degree m and order n</u> every formal series (16) with matrix coefficients depending on dynamical variables and satisfying the relation

$$deg\,(L_t-[\varphi_*,L\,])<m+N-n. \qquad (17)$$

Multiplication of formal series (16) is defined by the formula

$$aD^l\cdot bD^k=a\sum_{i=0}^{\infty}\binom{l}{i}D^i(b)D^{l+k-i},$$

$$\binom{l}{i}=\frac{l(l-1)\ldots(l-i+1)}{1\cdot 2\ldots i} \qquad (18)$$

generalizing the well-known differential operator multiplication rule.

One of the principal propositions of the theory of formal symmetries is the following

<u>Theorem 1</u> (/7/,/8/). The conditions of the n-th order formal symmetry existence don't depend on the choice of the degree m of the formal symmetry (16), and on the choice of integration constants arising in the chain of eqs. (10)-(12) for the coefficients of the formal symmetry. □

Note that the system of equations for the first n coefficients of the m-th order formal symmetry obtained by equating the coefficients at $D^j,\ j=m+N,\ldots,m+N-n+1$ has the same form as the chain of eqs. (10)-(12) for the coefficients of the formal symmetry. In particular, for all $m=0,\pm1,\pm2,\ldots$ the leading coefficient of the formal symmetry (16) is a constant diagonal matrix, that's

$$A_m=diag\,(\mu_1,\ldots,\mu_M),\quad \mu_i\in\mathbb{C}. \qquad (19)$$

One can formulate the existence conditions of the formal symmetry of the order $n>N$ in terms of the lower-order formal symmetry coefficients as it's given below.

<u>Theorem 2</u> (/7/,/8/). Let the non-degenerated formal symmetry of the order $n=N+i,\ i\geqslant 0$ exists. Then the existence of the n-th order formal symmetry, where $n=N+i+1$ is equivalent to the following conditions

$$\frac{d}{dt}(R(i,j))\in Im\,D,\quad j=1,\ldots,M, \qquad (20)$$

where

$$R(i,j)=\begin{cases}\dfrac{\partial F}{\partial u^j_{N-1}}, & i=0\\[2mm] \dfrac{\partial}{\partial\mu_j}\,trace\,(res\,L), & i>0\end{cases} \qquad (21)$$

is a formal symmetry of the order $i+2$ and the degree i with the leading coefficient (19) depending on M arbitrary parameters $\mu_1,\ldots,\mu_M$. □

The condition $dR/dt \in \mathrm{Im}\, D$ (see (20)) means the existence of the function σ depending on dynamical variables $x, u, u_1, \ldots$ such that the pair (R, σ) specifies the conservation law (14) for the system (2). Formula (21) determines the algorithm for constructing the local conservation law densities from the infinite series which we shall call the canonical series. For all known examples of equations integrable by inverse spectral transform this canonical series coincides with the series of local conservation laws which can be constructed using the scattering matrix in frames of the inverse scattering method (/12/).

3. Algorithm for checking the integrability conditions for the system (2) (algorithm (I)) is the following. Firstly one tests conditions (20) for $i=0$, $j=1,\ldots,M$ which are equivalent (theorem 2) to existence of the N+1-order formal symmetry. At the next step one constructs the formal symmetry of the order 1 and the degree 3 and tests M conditions (20) for $i=1$, $j=1,\ldots,M$. The fulfilment of these conditions guarantees the existence of the $N+2$ - order formal symmetry and so on. The elements of matrix coefficients for the m-th degree formal symmetry (16) can be found from the following recurrence relations (see (11)-(13), (17)-(19)):

$$A_K = 0, \quad K > m$$

$$[A_m]_{ij} = \begin{cases} 0, & i \neq j \\ \mu_i, & i = j \end{cases}$$

$$[A_K]_{ij} = \begin{cases} \dfrac{1}{\lambda_i - \lambda_j} \left\{ \dfrac{d}{dt}([A_{N+K}]_{ij}) - [C_{N+K}]_{ij} \Big|_{A_K=0} \right\}, & i \neq j \\ \dfrac{1}{N\lambda_i} D^{-1} \left\{ \dfrac{d}{dt}([A_{N+K-1}]_{ii}) - [C_{N+K-1}]_{ii} \Big|_{\mathrm{diag}(A_K)=0} \right\} + \gamma_i^K, & i = j, \end{cases} \tag{22}$$

where $\mu_i, \gamma_i^K \in \mathbb{C}$ are arbitrary constants, C_i are the coefficients of the commutator

$$[\Phi_*, L] = \sum C_i D^i.$$

The main computational difficulty of the above algorithm is inversion of the operator D which is necessary for constructing the diagonal elements of A_K. This problem is reduced to solving the following equation

$$D(Q) = S, \tag{23}$$

where Q and S are the scalar functions of dynamical variables $x, u, u_1, \ldots$ and D is the total differentiation operator (6) with respect to x. Note that the equation (2) can be solved not for any right hand side S [x].

The algorithm for the operator D inversion (algorithm (II)) is given below. It allows to find the function Q as well as the set of relations for S which must be satisfied. The algorithm is based on the following conditions which must be hold for any function $S(x, u, \ldots, u_K) \in \mathrm{Im}\, D$:

$$\begin{aligned} &\frac{\partial^2 S}{\partial u_K^i \partial u_K^j} = 0, \quad \frac{\partial^2 S}{\partial u_K^i \partial u_{K-1}^j} - \frac{\partial^2 S}{\partial u_K^j \partial u_{K-1}^i} = 0, \quad K > 0, \\ &\frac{\partial S}{\partial u^i} = 0, \quad K = 0, \end{aligned} \tag{24}$$

where $i, j = 1, \ldots, M$.

If conditions (24) are satisfied then S can be written as:

$$S = D(q(x, u, \ldots, u_{K-1})) + \bar{S}(x, u, \ldots, u_{K-1}). \tag{25}$$

Determining q from (25) we find that the solution of (23) is represented in the form $Q = q + \bar{Q}$, where $\bar{Q}$ satisfies the equation $D(\bar{Q}) = \bar{S}$. Therefore the condition $S \in \mathrm{Im}\, D$ is reduced to the condition $\bar{S} \in \mathrm{Im}\, D$, where the order of $\bar{S}$ is lower than the order of S.

To find the function q from equation (25) it is necessary to compute indefinite integrals of the form

$$\int \frac{\partial S}{\partial u_K^i} \, du_{K-1}^i.$$

Our program allows to compute indefinite integrals $\int g(w)\, dw$ for the following class of integrands:

$$g(w) = \sum_j \left(P_j(w) e^{\lambda_j w} + (\alpha_j w + \beta_j)^{\gamma_j} \right), \tag{26}$$

where $\alpha_j, \beta_j, \gamma_j, \lambda_j$ are constants, $P_j(w)$ are polynomials in w. The integration constants are put equal to zero. The formal description of the above algorithm is the following

input: $S(x, u, \ldots, u_K)$, $K = \mathrm{ord}\, S$

output: $Q(x, u, \ldots, u_{K-1})$, Z

x) Note that the solvability of eq. (23) is equivalent to the equality $\dfrac{\delta S}{\delta u} = 0$ (/13/).

$Q := 0,\ Z := 0$

for $m := K$ to 1 step -1 do

for $i := 1$ to M do

for $j := i$ to M do

$$Y := \iint \frac{\partial^2 S}{\partial u^i_m \partial u^j_m}\, du^i_m\, du^j_m$$

$$S := S - Y$$

$$Z := Z + Y$$

for $j := 1$ to $i-1$ do

$$Y := \iint \frac{\partial^2 S}{\partial u^i_m \partial u^j_{m-1}}\, du^i_m\, du^j_{m-1}$$

$$S := S - Y \tag{27}$$

$$Z := Z + Y$$

$$Q := Q + \int \frac{\partial S}{\partial u^i_m}\, du^i_{m-1}$$

$$S := S - D\Big(\int \frac{\partial S}{\partial u^i_m}\, du^i_{m-1}\Big)$$

for $i := 1$ to M do

$$Y := \int \frac{\partial S}{\partial u^i}\, du^i$$

$$S := S - Y$$

$$Z := Z + Y$$

$$Q := Q + \int S\, dx .$$

Applying the algorithm (II) we have $S = D(Q) + Z$, where Z is "not integrable" expression. The equality $Z = 0$ is equivalent to $S \in \mathrm{Im}\, D$.

Example. Let $M = 2$, $u = (w, v)$,

$$S = w v_2 + \alpha w_1 v_1 + \beta w_1^2, \quad \text{where } \alpha, \beta \in \mathbb{C}.$$

Applying the algorithm (II) we get

$$Q = w v_1, \quad Z = (\alpha - 1) w_1 v_1 + \beta w_1^2 .$$

Thus $S \in \mathrm{Im}\, D$ is equivalent to $\alpha = 1, \beta = 0$.

The algorithm for computing the n -order symmetry (algorithm (III)) is based on formulae (7)-(12) and includes three steps.

At the first step one computes the n-th order formal symmetry of the form (8) applying the algorithm (I). The existence of the formal symmetry is ensured by the corresponding integrability conditions (theorems 1,2). Otherwise the algorithm informs about the absence of symmetry and terminates. Remind that recurrence relations (22) lead to the formal symmetry depending on parameters $\mu_1, \dots, \mu_M$ and integration constants γ_i^K .

The symmetry of the order n defined by (4) exists only if the equality $L = f_*$ holds with the n-th order formal symmetry L already found. The equation $L = f_*$ is equivalent to the following system of equations for the formal symmetry coefficients

$$\frac{\partial}{\partial u^m_\ell}[A_K]_{ij} - \frac{\partial}{\partial u^j_K}[A_\ell]_{im} = 0, \qquad \begin{array}{l} K, \ell = 0, 1, \dots, n \\ i, j, m = 1, \dots, M . \end{array} \tag{28}$$

At the second step one verifies the solvability of this system for obtaining concrete values of μ_i, γ_i^K . Note that conditions (28) are equivalent to existence of the function $H = H(x, u, \dots, u_n)$ $H = (H^1, \dots, H^M)$ satisfying the equation

$$D(H) - \frac{\partial}{\partial x}(H) - \Big(\frac{\partial H^1}{\partial u^1} u^1_1, \dots, \frac{\partial H^M}{\partial u^M} u^M_1\Big) = L(u_1). \tag{29}$$

If H is a solution of the equation (29) then the symmetry f can be represented in the form

$$f = H + h(x, u), \quad h = (h^1, \dots, h^M), \quad h^\ell = h^\ell(x, u^\ell) . \tag{30}$$

At the third step substituting (30) in (4) we derive the overdetermined system of equations for $h(x, u)$. For testing the conditions (28) and solving the equation (29) it is sufficient to modify the algorithm (II) slightly. Equation (29) is the set of M scalar equations

$$\hat{D}_\ell(Q) = S, \quad \hat{D}_\ell = \Big(D - \frac{\partial}{\partial x} - u^\ell_1 \frac{\partial}{\partial u^\ell}\Big), \quad \ell = 1, \dots, M. \tag{31}$$

To modify the algorithm (II) for solving the equation (31) one has to replace D by $\hat{D}_\ell$ in (27) and to modify conditions (31) for $K = \operatorname{ord} S = 0, 1$ which become the following

$$\frac{\partial S}{\partial u^\ell_1} = 0, \quad \frac{\partial^2 S}{\partial u^i_1 \partial u^j_1} = 0, \quad \frac{\partial^2 S}{\partial u^j_1 \partial u^i} - \frac{\partial^2 S}{\partial u^i_1 \partial u^j} = 0,$$

$$i, j = 1, \dots, M, \quad i \neq \ell, \ j \neq \ell$$

for $K = 1$ and $\frac{\partial S}{\partial u^i} = 0$, $i = 1, \dots, M$, $\frac{\partial S}{\partial x} = 0$

for $K=0$. Note that the general solution of the equation (31) containts an additive arbitrary function $g(x,u^l)$.

4. The above algorithms are implemented in the frame of computer algebra system FORMAC (/14/). The choice of FORMAC is caused by its high execution velocity, powerful tools for expression analysis and the possibility to extend the system by means of the PL/1 language. Our program includes two basic procedures - CONDS and SYM and about twenty auxiliary procedures. The input of the program includes M - the number of equations in system (2); N - the order of the system (2); $FF(i)$ $(i=1,\dots,M)$ - the components of the system (2) right-hand side. The variables u^i_j in the right hand-side of (2) are coded as $U(I,J)$. It's sufficient to use procedures CONDS and SYM to check up the integrability of concrete system (2) and to compute symmetries and canonical conservation law densities.

<u>Procedure CONDS</u> implements the algorithm (I). It allows to test the conditions of integrability (that is the existance of the formal symmetry) and to compute canonical conversation law densities (21).

The call for CONDS has the form CONDS (i_1,i_2,p) where i_1,i_2,p are integers such that $i_1\geq 0$, $i_2\geq i_1$, $p=0,1$. For an input

$$i_1=0,\quad i_2=n-N-1,\quad p=1$$

CONDS tests the conditions (20) of the n-th order formal symmetry existence for $n>N$. If any of these conditions are not satisfied the following relations

ZERO = <expression>

are printed out. For existence of the formal symmetry the right-hand sides of these relations must be put equal to zero. If it leads to contradiction then the evolution system under investigation has not any formal symmetry of the order greater or equal to n. Therefore it is not integrable.

For the integrable systems (2) CONDS computs and prints out the canonical conservation law densities (21). For an input i_1,i_2

with $i_1\leq i_2$ and $p=0$ the canonical densities $R(i,j)$ for

$$i=i_1,i_1+1,\dots,i_2,\quad j=1,\dots,M$$

are computed and printed out (for $p=0$ the corresponding conditions (20) are not tested).

<u>Procedure SYM</u> implements the algorithm (III). It allows to compute symmetries of the form (4) for the system (2). The procedure call is SYM(k), where n is a symmetry order. SYM computes and prints out the function H satisfying the equation (29). Moreover the system of equations for the constants μ_i, γ_i^k (coded as $MU(I)$, $G(I,K)$) and the system of equations for $h(x,u)$ (its components are coded as $H\emptyset(I).(X,U(I,\emptyset))$) are printed out as

ZERO = < expression >

which right-hand sides must be equal to zero.

Note that the current version of the program can be applied only to the systems (2) with right-hand sides being functions of u^i_j from the class (26). To extend the class of considered systems it is sufficient to modify the procedure INT implementing the indefinite integration.

5. Here we give some examples of program application.

1) Let us consider the following Nonlinear Shrödinger-like equation

$$u_t = u_4 + u^2 v,\quad -v_t = v_4 + v^2 u. \tag{32}$$

Using CONDS (0,6,1) we obtain that system (32) has the 10th order formal symmetries but has no 11th order non-degenerated formal symmetries, therefore it is not integrable.

2) The following system was obtained in /5/, where integrable systems (1) were classified:

$$u_t = u_2 + v_1^2 + \frac{\partial z}{\partial v}, \quad -v_t = v_2 + u_1^2 + \frac{\partial z}{\partial u}, \tag{33}$$

where

$$z = \beta_1 \exp(u+v) + \beta_2 \exp(\bar{\lambda} u + \lambda v) + \beta_3 \exp(\lambda u + \bar{\lambda} v),$$

$$\lambda = \exp(2\pi i/3), \quad \bar{\lambda} = \lambda^2,$$

β_i are arbitrary constants.

The system (33) is known to have the formal symmetry of the order $n=6$. By means of the program developed we found that (33) has the 4th order symmetry $f = (f^1, f^2)$, where

$$f^1 = u_4 + 2v_1 v_3 - 2v_1 u_1 u_2 + v_2^2 - \frac{4}{3} v_1^3 u_1 + \frac{1}{3} u_1^4 +$$

$$+ z(2u_2 + v_1^2) + \frac{\partial z}{\partial u}(v_2 + u_1^2) + \frac{1}{2} \frac{\partial z}{\partial v} \cdot z,$$

$$f^2 = -v_4 - 2u_1 u_3 + 2u_1 v_1 v_2 - u_2^2 + \frac{4}{3} u_1^3 v_1 - \frac{1}{3} v_1^4 -$$

$$- z(2v_2 + u_1^2) - \frac{\partial z}{\partial v}(u_2 + v_1^2) - \frac{1}{2} \frac{\partial z}{\partial u} \cdot z.$$

3) In classifying integrable third-order systems of two equations the following systems were obtained

$$u_t = u_3 + u u_1, \quad v_t = 4v_3 + u v_1 + \frac{1}{2} v u_1 \tag{34}$$

and

$$u_t = 5u_3 + \frac{5}{2} u u_1, \quad v_t = 8v_3 + \frac{3}{2} u v_1 + \frac{1}{4} v u_1 \tag{35}$$

which were known to have the 9th order formal symmetries only. Using the program developed we tested the existence of the 11th order formal symmetries. It turned out that the system (34) has and system (35) has no such formal symmetry. Moreover, we computed the 5-order symmetry (4) for the system (34) which is of the form $f = (f^1, f^2)$, where

$$f^1 = u_5 + \frac{5}{3} u u_3 + \frac{10}{3} u_1 u_2 + \frac{5}{6} u^2 u_1,$$

$$f^2 = 16 v_5 + \frac{20}{3} u v_3 + \frac{5}{2} v u_3 + 10 u_1 v_2 + \frac{25}{3} v_1 u_2 + \frac{5}{6} u^2 v_1 + \frac{5}{6} v u u_1.$$

All above examples take from 5 to 20 minutes of ES-1061 running time and about 300 K memory.

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Гердт В.П. и др. E5-87-40
Применение компьютерной алгебры для исследования
интегрируемости нелинейных эволюционных систем

Рассмотрен алгоритмический подход к исследованию условий интегрируемости систем нелинейных эволюционных уравнений, линейных относительно старших производных по пространственной переменной. Сформулированы математические основы симметрийного метода проверки условий интегрируемости и описан конструктивный алгоритм, реализующий данный метод для широкого класса нелинейных эволюционных систем. Этот алгоритм реализован в виде программы на языке системы компьютерной алгебры FORMAC. Дано описание основных процедур данной программы. Приведены примеры применения программы для исследования интегрируемости конкретных нелинейных эволюционных систем.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Gerdt V.P. et al. E5-87-40
Computer Algebra Application for Investigating
Integrability of Nonlinear Evolution Systems

An algorithmic approach is developed for investigating the integrability of nonlinear evolution systems linear on the highest derivatives with respect to spatial variable. The mathematical background of the symmetry approach to checking up the integrability conditions is formulated and the constructive algorithmic realization of this approach is proposed for the following wide class of nonlinear evolution systems. Our algorithm had been implemented on the basis of the computer algebra system FORMAC. The basic subroutines of the program are described. The concrete examples of evolution systems are considered.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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