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SCATTERING MATRIX AND SPECTRAL SHIFT OF THE NUCLEAR DISSIPATIVE SCATTERING THEORY

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1. Introduction

The present paper continues the investigations of the scattering matrix and the spectral shift function of a scattering theory of maximal dissipative operators.

The scattering theory of maximal dissipative operators was developed in [14,15]. In these papers the wave operators were introduced and a definition of the completeness of the wave operators of maximal dissipative operators was given. The scattering operator was defined in [15] and the investigation of this object was started there. A detailed representation of a dissipative scattering theory can be found in [19].

An attempt to define the notions of the scattering matrix and the spectral shift function as well as to clarify their interplay was undertaken in [17,20,18], where a maximal dissipative operator and a selfadjoint operator which differ by a nuclear dissipative operator were considered. The aim of the present paper is to generalize these results to a pair of operators $\{H_1, H_0\}$ consisting of a maximal dissipative operator H_1 and a selfadjoint operator H_0 both defined on a separable Hilbert space $\mathfrak{h}$ such that the resolvent difference belongs to the trace class, i.e.

$$(1.1) \qquad (H_1 - i)^{-1} - (H_0 - i)^{-1} \in \mathcal{L}_1(\mathfrak{h}),$$

where $\mathcal{L}_1(\mathfrak{h})$ denotes the class of trace operators on $\mathfrak{h}$.

To obtain such a generalization we start with a pair $\{T_1, T_0\}$ consisting of a contraction T_1 and a unitary operator T_0 both defined on $\mathfrak{h}$ such that their difference belongs to the trace class, i.e.

(1.2) $T_1 - T_0 \in \mathcal{L}_1(\mathfrak{H})$.

In the first section of chapter two we introduce the wave and scattering operator for such a situation and derive a formula of the family of scattering matrices. The second section is concerned with the definition of a spectral shift function and the verification of a trace formula for the pair $\{T_1,T_0\}$. It turns out that these considerations are independent of the assumption that T_0 is unitary. In such a way we assume through this section that T_0 is a contaction on $\mathfrak{H}$, too. The results of this section essentially are based on [21]. On account of the previous two sections we prove a certain Birman-Krein formula in the last section of this chapter.

In the third chapter we try to obtain similar results for a pair of operators consisting of a maximal dissipative operator and a selfadjoint operator. For this business the main tool will be the invariance principle of wave operators and the Cayley transform. Using the invariance principle and taking into account the Cayley transform we find a formula of the family of scattering matrices and we carry over the results of the second section of chapter two to a pair of maximal dissipative operators. The Birman-Krein formula follows then directly.

An attempt in the same direction was undertaken by A.V.Rybkin [22,23]. The results of A.V.Rybkin partially coincide with the results of [17,20,18]. Further publications of H.Langer [13] , R.V.Akopjan [2,3], V.M.Adamjan, B.S.Pavlov [1] and P.Jonas [8,9] are related to the subject of this paper.

2. Contractive case

2.1. Scattering matrix

First of all we remark that the condition (1.2) implies the existence of the wave operators W_+,

(2.1) $$W_+ = \operatorname*{s-lim}_{n\to+\infty} T_1^{*n} T_0^n P^{ac}(T_0),$$

and W_-,

(2.2) $$W_- = \operatorname*{s-lim}_{n\to+\infty} T_1^n T_0^{*n} P^{ac}(T_0),$$

where $P^{ac}(T_0)$ denotes the orthogonal projection from $\mathfrak{H}$ onto the absolutely continuous subspace $\mathfrak{H}^{ac}(T_0)$ of the unitary operator T_0. A Theorem of this contents can be nowhere found, but it is not hard to see that such a theorem should be the discret version of Theorem 2.1 of [16]. Consequently, transforming the considerations of Theorem 2.1 of [16] into a discrete language we obtain a proof of the above mentioned existence assertions. Moreover, following the same line we get the existence of the dilation wave operators $\underline{W}_\pm$,

(2.3) $$\underline{W}_\pm = \operatorname*{s-lim}_{n\to\pm\infty} U_1^{*n} T_0^n P^{ac}(T_0),$$

where U_1 denotes a minimal unitary dilation of T_1 defined on the dilation space $\mathcal{H}$, $\mathfrak{H} \subseteq \mathcal{H}$.

If the wave operators $W_\pm$ exist, then we call the triplet $\Lambda = \{T_1,T_0;I\}$ a scattering system in the following.

The scattering operator S of the scattering system Λ is defined by

(2.4) $$S = W_+^* W_-.$$

$$(2.11) \qquad \Gamma_\varsigma(e^{it}) = C - CB(T_1 - \varsigma e^{it})^{-1}BC,$$

$\varsigma > 1$. Taking into account Proposition 14 of [4,p.57] it is not hard to see that the limit $\Gamma(e^{it}) = \text{o-lim}_{\varsigma \downarrow 1} \Gamma_\varsigma(e^{it})$ exists for a.e. $t \in [0,2\pi]$ mod $|.|$.

Theorem 2.1. Let $L^2(\Delta, |.|; \mathfrak{h}_t, \mathfrak{Y})$ be a spectral representation of T_o^{ac} and let $\{T(e^{it})\}_{t\in\Delta}$ be the family of scattering amplitudes of the scattering system $\{T_1, T_o; I\}$ which obeys (1.2). Then there is a family of isometries $\{V(e^{it})\}_{t\in\Delta}$, $V(e^{it}):(\text{ima}(M(t)))^- \to \mathfrak{h}_t$, $t \in \Delta$, such that the representation

$$(2.12) \qquad T(e^{it}) = 2\pi V(e^{it})\sqrt{M(t)}\, e^{-it}\, \Gamma(e^{it})\sqrt{M(t)}\, V^*(e^{it})$$

holds for a.e. $t \in \Delta$ mod $|.|$.

Remark 2.2. It is quite possible that the set $\delta = \{t \in \Delta : M(t) = 0\}$ has a positive Lebesgue measure. In this case we set $V(e^{it}) = 0$ and $T(e^{it}) = 0$ for every $t \in \delta$.

The proof essentially follows the considerations of Theorem 2.15 and Corollary 2.17 of [17]. Therefore we omit the proof.

Corollary 2.3. If the assumptions of Theorem 2.1 are valid, then we have

$$(2.13) \qquad T(e^{it}) \in \mathcal{L}_1(\mathfrak{h})$$

for a.e. $t \in \Delta$ mod $|.|$.

Proof. The relation (2.13) immediately follows from Theorem 2.1. ■

2.2. Spectral shift function

In distinction from section one through this section T_o will denote a contraction on $\mathfrak{h}$, too.

The aim of this section is to define a spectral shift function for a pair of contractions $\{T_1, T_o\}$. An attempt in this direction was made in [17,20,18] for a dissipative situation. In the language of contractions these results can be expressed as follows. Let $\mathcal{G}_{\mathbb{T}^1}$ be a set of functions defined by the condition that their elements $\varphi(.)$ admit a Fourier decomposition

$$(2.14) \qquad \varphi(z) = \sum_{l=-\infty}^{+\infty} a_l\, z^l,$$

$z \in \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that the condition

$$(2.15) \qquad \sum_{l=-\infty}^{+\infty} |l a_l| < +\infty$$

is fulfilled. Introducing the functions $\varphi_+(.) \in \mathcal{G}_{\mathbb{T}^1}$,

$$(2.16) \qquad \varphi_+(z) = \sum_{l=0}^{+\infty} a_l\, z^l,$$

and $\varphi_-(.) \in \mathcal{G}_{\mathbb{T}^1}$,

$$(2.17) \qquad \varphi_-(z) = \sum_{l=-\infty}^{-1} a_l\, z^{-l},$$

$z \in \mathbb{T}^1$, we decompose $\varphi(.)$ into a sum of two functions,

$$(2.18) \qquad \varphi(z) = \varphi_+(z) + \varphi_-(\bar{z}),$$

$z \in \mathbb{T}^1$. The condition (1.2) yields

$$(2.19) \qquad \varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_o) - \varphi_-(T_o^*) \in \mathcal{L}_1(\mathfrak{h})$$

for every $\varphi(.) \in \mathcal{G}_{\mathbb{T}^1}$. If in addition to (1.2) the defect operators $D_{T_1} = \sqrt{I - T_1^*T_1}$, $D_{T_1^*} = \sqrt{I - T_1T_1^*}$,

$D_{T_o} = \sqrt{I - T_o^* T_o}$ and $D_{T_o^*} = \sqrt{I - T_o T_o^*}$ belong to the trace class, then there is a real measurable function $\mu(.) \in L^1([0,2\pi])$ such that the trace formula

$$(2.20) \qquad \mathrm{tr}\{\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_o) - \varphi_-(T_o^*)\} = \int_0^{2\pi} \mu(t) \frac{d}{dt}\varphi(e^{it})\, dt$$

holds for every $\varphi(.) \in \mathfrak{A}_{\mathbb{T}1}$. The function $\mu(.)$ is called a spectral shift function of the pair $\{T_1, T_o\}$ and is defined by (2.20) up to an additive constant. The function $\mu(.)$ admits the representation

$$(2.21) \qquad \mu(t) = -\lim_{\varsigma \downarrow 1} \frac{1}{\pi} \mathrm{Im} \log \det(I + (T_1 - T_o)(T_o - \varsigma e^{it})^{-1}) + \text{const.}$$

for a.e. $t \in [0,2\pi] \bmod |.|$, where we have assumed $\lim_{|z| \to +\infty} \log \det(I + (T_1 - T_o)(T_o - z)^{-1}) = 0$.

If the condition (1.2) is fulfilled but the defect operators do not belong to the trace class, then it is quite possible that the representation (2.21) makes sense but the spectral shift function defined by (2.21) is even not locally summable. Hence the trace formula (2.20) loses its meaning. But from standpoint of applications it is natural to demand that the defect operators belong to the Hilbert-Schmidt class and to have some trace formula. The following considerations give a solution of this problem.

The solution was obtained by taking into account ideas of L.S.Koplienko, who trys in [11] to define a generalized spectral shift function and to legitimate a

modified trace formula for selfadjoint operators which differ by a Hilbert-Schmidt operator. In [21] the results of L.S.Koplienko were extended to unitary operators and to pairs of selfadjoint operators such that the resolvent difference belongs to the Hilbert-Schmidt class.

In the following we apply these results to a pair of contractions which differ by a nuclear operator. In this connection we will see that the problem to define a spectral shift function for a pair of unitary operators under a Hilbert-Schmidt perturbation or for a pair of contractions which differ by a nuclear operator is essentially the same.

For further considerations we restrict the set $\mathfrak{A}_{\mathbb{T}1}$ to the set $\mathfrak{J}_{\mathbb{T}1}$ which elements are characterized by

$$(2.22) \qquad \sum_{l=-\infty}^{+\infty} l^2 |a_l| < +\infty.$$

<u>Theorem 2.4.</u> Let $\{T_1, T_o\}$ be a pair of contractions on $\mathfrak{H}$ such that the conditions

$$(2.23) \qquad I - T_1^* T_o \in \mathcal{L}_1(\mathfrak{H})$$

and

$$(2.24) \qquad I - T_1 T_o^* \in \mathcal{L}_1(\mathfrak{H})$$

are fulfilled. Then there is a real measurable function $\xi(.) \in L^1([0,2\pi])$ such that the formula

$$(2.25) \qquad \mathrm{tr}((T_1 - z)^{-1} - (T_o - z)^{-1}) = \int_0^{2\pi} \xi(t) \frac{e^{it} + z}{(e^{it} - z)^3} e^{it}\, dt$$

holds for every $z \in \mathbb{C}$ with $|z| > 1$. The function $\xi(.)$ is defined up to an additive constant by (2.25). For every $\varphi(.) \in \mathfrak{F}_{\mathbb{T}^1}$ the condition (2.19) is fulfilled and we have

$$(2.26) \qquad \operatorname{tr}\{\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_o) - \varphi_-(T_o^*)\} = = -\int_0^{2\pi} \xi(t) \frac{d^2}{dt^2}\varphi(e^{it})\, dt.$$

Proof. We prove Theorem 2.4 in several steps.

1. The conditions (2.23) and (2.24) imply that the contractions T_1 and T_o are Fredholm operators such that the condition (1.2) is fulfilled and the defect operators D_{T_1}, $D_{T_1^*}$, D_{T_o} and $D_{T_o^*}$ belong to the Hilbert-Schmidt class. To prove these assertions we use the formulas

$$(2.27) \qquad 2I - T_1^*T_o - T_o^*T_1 = = I - T_1^*T_1 + (T_1^*-T_o^*)(T_1-T_o) + I - T_o^*T_o \in \mathfrak{L}_1(\mathfrak{H})$$

and

$$(2.28) \qquad 2I - T_1T_o^* - T_oT_1^* = = I - T_1T_1^* + (T_1-T_o)(T_1^*-T_o^*) + I - T_oT_o^* \in \mathfrak{L}_1(\mathfrak{H}).$$

But (2.27) and (2.28) immediately imply $I - T_1T_1^* \in \mathfrak{L}_1(\mathfrak{H})$, $I - T_1^*T_1 \in \mathfrak{L}_1(\mathfrak{H})$, $I - T_oT_o^* \in \mathfrak{L}_1(\mathfrak{H})$ and $I - T_o^*T_o \in \mathfrak{L}_1(\mathfrak{H})$. Hence the defect operators belong to $\mathfrak{L}_2(\mathfrak{H})$. Taking into account Lemma 5.19 of [14] we find that T_1 and T_o are Fredholm operators. The formula $T_1 - T_o = T_1(I - T_o^*T_o) - (I - T_1T_o^*)T_o$ shows that we have $T_1 - T_o \in \mathfrak{L}_1(\mathfrak{H})$.

2. To apply [21] we introduce special unitary dilations U_1 and U_o of the contractions T_1 and T_o, respectively. It turns out that the matrix-construction of a unitary dilation described in the book of C.Foias and B.Sz.-Nagy [6] is very useful for our purposes. In accordance with this construction we introduce the dilation space $\mathfrak{K}$,

$$(2.29) \qquad \mathfrak{K} = \bigoplus_{j=-\infty}^{+\infty} \mathfrak{H}_j,$$

$\mathfrak{H}_j = \mathfrak{H}$, $j = 0, \pm1, \pm2, \dots$, where the original Hilbert space $\mathfrak{H}$ is identified with $\mathfrak{H}_o$, and define the unitary dilation U_j of T_j by

$$(2.30) \qquad U_j = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & I & 0 & 0 & 0 & 0 \\ & 0 & D_{T_j} & -T_j^* & 0 & 0 \\ & & T_j & D_{T_j^*} & 0 & 0 \\ & & & 0 & I & 0 \\ & & & & 0 & I \\ & & & & & \cdot \end{bmatrix},$$

$j = 0,1$. Taking into account the results of step one we get

$$(2.31) \qquad U_1 - U_o \in \mathfrak{L}_2(\mathfrak{H}).$$

In accordance with [21] we define $G = U_1U_o^{-1}$ and we use the representation $G = e^{iD}$, $D \in \mathfrak{L}_2(\mathfrak{H})$. Because of Corollary 2.4 and formula (2.69) of [21] we obtain a real measurable function $\eta(.) \in L^1([0,2\pi])$ such that the relation

$$(2.32) \qquad \operatorname{tr}\left\{(U_1-z)^{-1} - (U_o-z)^{-1} + (U_o-z)^{-1}\frac{e^{iD}-e^{-iD}}{2}U_o(U_o-z)^{-1}\right\} = \int_0^{2\pi} \eta(t)\frac{e^{it}+z}{(e^{it}-z)^3}e^{it}\, dt,$$

$|z| \neq 1$, holds.

3. Our next aim is to calculate the left-hand side expression of (2.32). Taking into account (2.30) a long but straightforward calculation proves

$$(2.33)\qquad \operatorname{tr}\left\{ (U_1-z)^{-1}-(U_o-z)^{-1}+(U_o-z)^{-1}\,\frac{e^{iD}-e^{-iD}}{2}\,U_o(U_o-z)^{-1}\right\} = \operatorname{tr}\left\{(T_1-z)^{-1}-(T_o-z)^{-1}+\tfrac{1}{2}(ET_o+FD_{T_o})(T_o-z)^{-2}\right\},$$

$|z|>1$, where E and F are given by

$$(2.34)\qquad E = T_1T_o^* - T_oT_1^* + D_{T_1^*}D_{T_o^*} - D_{T_o^*}D_{T_1^*}$$

and

$$(2.35)\qquad F = T_1D_{T_o} - D_{T_1^*}T_o - T_oD_{T_1} + D_{T_o^*}T_1.$$

We remark that we have $E \in \mathfrak{L}_1(\mathfrak{H})$ and $FD_{T_o} \in \mathfrak{L}_1(\mathfrak{H})$.

4. We assume that T_o is an isometry, i.e. $T_o = V_o$ with $V_o^*V_o = I$. From (2.32) - (2.35) we find

$$(2.36)\qquad \operatorname{tr}\left\{(T_1-z)^{-1} - (V_o-z)^{-1} + \tfrac{1}{2}EV_o(V_o-z)^{-2}\right\} = \int_0^{2\pi} \eta(t)\,\frac{e^{it}+z}{(e^{it}-z)^3}\,e^{it}\,dt,$$

$|z|>1$. The operator $Y = -\frac{1}{2}E$ is nuclear and selfadjoint. If $F_o(.)$ is the spectral measure of the unitary dilation of V_o, then we get

$$(2.37)\qquad i\,\operatorname{tr}(YV_o(V_o-z)^{-2}) = i\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}\,d\operatorname{tr}(YF_o(t)) = \int_0^{2\pi} \left(\frac{t}{2\pi}\operatorname{tr}(Y) - \operatorname{tr}(YF_o(t))\right)\frac{e^{it}+z}{(e^{it}-z)^3}\,e^{it}\,dt,$$

$|z|>1$. Defining $\xi(.)$ by

$$(2.38)\qquad \xi(t) = \eta(t) + \operatorname{tr}(YF_o(t)) - \frac{t}{2\pi}\operatorname{tr}(Y),$$

$t \in [0,2\pi]$ we have $\xi(.) \in L^1([0,2\pi])$ and

$$(2.39)\qquad \operatorname{tr}((T_1 - z)^{-1} - (V_o - z)^{-1}) = \int_0^{2\pi} \xi(t)\,\frac{e^{it}+z}{(e^{it}-z)^3}\,e^{it}\,dt,$$

$|z|>1$. Obviously, the function $\xi(.)$ is a real one.

5. We assume now that T_o is a co-isometry, i.e. $T_o = V_o$ with $V_oV_o^* = I$. Then the pair $\{T_1^*, V_o^*\}$ fulfils the assumptions of the previous step. Consequently, there is a real measurable function $\alpha(.) \in L^1([0,2\pi])$ such that the formula

$$(2.40)\qquad \operatorname{tr}((T_1^*-z)^{-1} - (V_o^*-z)^{-1}) = \int_0^{2\pi} \alpha(t)\,\frac{e^{it}+z}{(e^{it}-z)^3}\,e^{it}\,dt$$

$|z|>1$, holds. Taking the adjoint of (2.40) we get

$$(2.41)\qquad \operatorname{tr}((T_1-\bar{z})^{-1} - (V_o-\bar{z})^{-1}) = \int_0^{2\pi} \alpha(t)\,\frac{e^{-it}+\bar{z}}{(e^{-it}-\bar{z})^3}\,e^{-it}\,dt,$$

$|z|>1$. Setting $\bar{z} = z$ and $\xi(t) = \alpha(2\pi - t)$, $t \in [0,2\pi]$, we obtain a real measurable function $\xi(.) \in L^1([0,2\pi])$ obeying (2.39).

6. We solve the general case of two arbitrary contractions T_1 and T_o satisfying (2.23) and (2.24). To this end we introduce the polar decompositions $T_j = \operatorname{sign}(T_j)|T_j|$, $|T_j| = \sqrt{T_j^*T_j}$, $j = 0,1$. On account of step one T_1 and T_o are Fredholm operators. Hence we have $\operatorname{def}(\operatorname{sign}(T_j)) = \dim(\mathfrak{H} \ominus \operatorname{ima}(\operatorname{sign}(T_j))) < +\infty$ and $\operatorname{nul}(\operatorname{sign}(T_j)) = \dim(\ker(\operatorname{sign}(T_j))) < +\infty$, $j = 0,1$. In every case the

operators $\mathrm{sign}(T_1)$ and $\mathrm{sign}(T_o)$ are extendible to some isometries or co-isometries V_1 and V_o such that the representations $T_1 = V_1|T_1|$ and $T_o = V_o|T_o|$ are valid. Now the pairs $\{T_1, V_1\}$, $\{V_1, V_o\}$, $\{T_o, V_o\}$ fulfil the assumptions of step four and five. Summing up the corresponding formulas we prove (2.25).

7. Taking into account $T_1 - T_o \in \mathcal{L}_1(\mathfrak{H})$ we prove (2.19). From (2.25) we obtain

$$\text{(2.42)} \qquad \mathrm{tr}(T_1^k - T_o^k) = k^2 \int_0^{2\pi} \xi(t)\, e^{ikt}\, dt,$$

$k = 1,2,\ldots$. But this equality implies

$$\text{(2.43)} \qquad \mathrm{tr}(\varphi_+(T_1) - \varphi_+(T_o)) = -\int_0^{2\pi} \xi(t) \frac{d^2}{dt^2} \varphi_+(e^{it})\, dt,$$

$\varphi(.) \in \mathcal{F}_{\mathbb{T}^1}$. Taking the adjoint from (2.42) we get

$$\text{(2.44)} \qquad \mathrm{tr}(\varphi_-(T_1^*) - \varphi_-(T_o^*)) = -\int_0^{2\pi} \xi(t) \frac{d^2}{dt^2} \varphi_-(e^{-it})\, dt.$$

Summing up (2.43) and (2.44) we prove (2.26).

8. To prove the uniqueness of $\xi(.)$ it is sufficient to show that for every real measurable function $\delta(.) \in L^1([0,2\pi])$ the condition

$$\text{(2.45)} \qquad \int_0^{2\pi} \delta(t) \frac{e^{it} + z}{(e^{it} - z)^3} e^{it}\, dt = 0,$$

$|z| > 1$, implies $\delta(t) = \mathrm{const.}$. But from (2.45) we get

$$\text{(2.46)} \qquad \int_0^{2\pi} \delta(t)\, e^{ikt}\, dt = 0$$

for every $k = \pm 1, \pm 2, \ldots$. Hence we get $\delta(t) = \mathrm{const.}$. ∎

In the following we call a real measurable function $\xi(.) \in L^1([0,2\pi])$ obeying (2.25) or (2.26) an integrated

spectral shift function of the pair $\{T_1, T_o\}$. We note that the integrated spectral shift function is defined up to an additive constant.

Let $P_r(t,s)$ be the Poisson kernel,

$$\text{(2.47)} \qquad P_r(t,s) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t-s)},$$

$t,s \in [0,2\pi]$, and let $\xi(.) \in L^1([0,2\pi])$. If the limit $\xi^*(s)$,

$$\text{(2.48)} \qquad \xi^*(s) = \lim_{r \uparrow 1} -\int_0^{2\pi} \xi(t) \frac{d}{dt} P_r(t,s)\, dt$$

exists at $s \in (0,2\pi)$, then we call $\xi^*(s)$ the generalized derivative of $\xi(.)$ at the point $s \in (0,2\pi)$. It is possible to show that if the usual derivative $\xi'(s)$ of $\xi(.)$ exists at the point s, then the generalized derivative exists also at s and both derivatives coincide, i.e. $\xi^*(s) = \xi'(s)$.

<u>Theorem 2.5.</u> Let $\{T_1, T_o\}$ be a pair of contractions on $\mathfrak{H}$ such that the conditions (2.23) and (2.24) are fulfilled. If $\xi(.)$ denotes an integrated spectral shift function of $\{T_1, T_o\}$, then for a.e. $t \in [0,2\pi]$ mod $|.|$ the generalized derivative $\xi^*(t)$ exists and we have

$$\text{(2.49)} \qquad \xi^*(t) = -\lim_{r \uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + (T_1 - T_o)(T_o - r^{-1}e^{it})^{-1}),$$

where we have fixed a branch of the logarithm by the condition $\lim_{|z| \to +\infty} \log \det(I + (T_1 - T_o)(T_o - z)^{-1}) = 0$.

<u>Proof.</u> Because of $T_1 - T_o \in \mathcal{L}_1(\mathfrak{H})$, which follows from (2.23) and (2.24), the determinant $\det(I + (T_1 - T_o)(T_o - z)^{-1})$, $|z| > 1$, makes sense. We get

$$\text{(2.50)} \qquad \frac{d}{dz} \log \det(I + (T_1 - T_o)(T_o - z)^{-1}) =$$

$$= -\mathrm{tr}((T_1 - z)^{-1} - (T_o - z)^{-1}),$$

$|z| > 1$. Taking into account (2.25) we get

$$\log \det(I + (T_1 - T_o)(T_o - z)^{-1}) = \tag{2.51}$$

$$= -i \int_0^{2\pi} \xi(t) \frac{d}{dt} \frac{e^{it}}{e^{it} - z} dt,$$

$|z| > 1$. Hence we obtain

$$\frac{1}{\pi} \operatorname{Im} \log \det(I + (T_1 - T_o)(T_o - r^{-1}e^{is})^{-1}) = \tag{2.52}$$

$$= \int_0^{2\pi} \xi(t) \frac{d}{dt} P_r(t,s)\, dt,$$

$s \in [0,2\pi]$.

It remains to establish the existence of $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \det(I + (T_1 - T_o)(T_o - r^{-1}e^{is})^{-1})$ for a.e. $s \in [0,2\pi]$ mod $|.|$. To this end we use the notion of the regularized determinant $\widetilde{\det}(I + .)$, which is applicable to Hilbert-Schmidt operators. For a detailed presentation of this determinant the reader is refered to [7]. Taking into account the factorization (2.9) we get

$$\log \det(I + (T_1 - T_o)(T_o - r^{-1}e^{is})^{-1}) = \tag{2.53}$$

$$= \log \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B) +$$

$$+ \operatorname{tr}(CB(T_o - r^{-1}e^{is})^{-1}B),$$

$0 < r < 1$, $s \in [0,2\pi]$. From Proposition 14 of [4,p.57] we find that $\lim_{r\uparrow 1} CB(T_o - r^{-1}e^{is})^{-1}B$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$ in $\mathfrak{L}_2$. But the determinant $\widetilde{\det}(I + .\,)$ is continuous in the Hilbert-Schmidt norm. Consequently, the limit $\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$. For the same reason the limit $\lim_{r\uparrow 1} \widetilde{\det}(I - CB(T_1 - r^{-1}e^{is})B)$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$. Hence we obtain

$$\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B) \lim_{r\uparrow 1} \widetilde{\det}(I - CB(T_1 - r^{-1}e^{is})^{-1}B) \tag{2.54}$$

$$= \lim_{r\uparrow 1} \exp\{-\operatorname{tr}(CB(T_o - r^{-1}e^{is})^{-1}BCB(T_1 - r^{-1}e^{is})^{-1}B)$$

$$\neq 0$$

for a.e. $s \in [0,2\pi]$ mod $|.|$. But (2.54) implies $\lim_{r\uparrow 1} \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B) \neq 0$ for a.e. $s \in [0,2\pi]$ mod $|.|$. Consequently, the limit $\lim_{r\uparrow 1} \log \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$. It follows that $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \log \widetilde{\det}(I + CB(T_o - r^{-1}e^{is})^{-1}B)$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$. To show the existence of $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \operatorname{tr}(CB(T_o - r^{-1}e^{is})^{-1}B)$ we use Proposition 2 of [4,p.33]. Considering the transformation $\mathbb{R}^1 \ni \lambda \longrightarrow$ $\rightarrow 2 \operatorname{arc\,ctg} \lambda = t \in [0,2\pi]$ Proposition 2 of [4] can be formulated as follows. If $g(.): [0,2\pi] \longrightarrow \mathbb{C}$ is a function of bounded variation, then the limit

$$\lim_{r\uparrow 1} \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}} dg(t) \tag{2.55}$$

exists for a.e. $s \in [0,2\pi]$ mod $|.|$. Denoting by $F_o(.)$ the spectral measure of a unitary dilation of T_o we find

$$\operatorname{tr}(CB(T_o - r^{-1}e^{is})^{-1}B) = \tag{2.56}$$

$$= \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}} d(\operatorname{tr}(CBF_o(t)B)),$$

$0 < r < 1$, $s \in [0,2\pi]$. Setting $g(t) = \operatorname{tr}(CBF_o(t)B)$, $t \in [0,2\pi]$, it is not hard to see that $g(.)$ is of bounded variation. Hence the limit $\lim_{r\uparrow 1} \frac{1}{\pi} \operatorname{Im} \int_0^{2\pi} \frac{1}{e^{it} - r^{-1}e^{is}} d\operatorname{tr}(CBF_o(t)B)$ exists for a.e. $s \in [0,2\pi]$ mod $|.|$. ■

Theorem 2.5 shows us that the existence of the limit (2.21) does not require the additional conditions D_{T_1}, $D_{T_1^*}$, D_{T_o} and $D_{T_o^*} \in \mathcal{L}_1(\mathfrak{H})$. But if these conditions are fulfilled, then normalizing the spectral shift function $\mu(.)$ of $\{T_1,T_o\}$ by the condition $\int_0^{2\pi}\mu(t)\,dt = 0$ an integrated spectral shift function $\xi(.)$ of $\{T_1,T_o\}$ can be obtained by the formula $\xi(t) = \int_0^t \mu(s)\,ds$, $t \in [0,2\pi]$. Consequently, every integrated spectral shift function of $\{T_1,T_o\}$ is absolutely continuous. Hence the usual derivative $\xi'(t)$ exists for a.e. $t \in [0,2\pi]$ mod $|.|$ and the equality $\xi'(t) = \xi^*(t) = \mu(t)$ holds for a.e. $t \in [0,2\pi]$ mod $|.|$.

From this point of view it seems to be quite natural every function $\mu(.)$ which differs from $\xi^*(.)$ by a real constant to call a spectral shift function of the pair $\{T_1,T_o\}$, what we will do in the following.

We remember that the spectral shift function $\mu(.)$ can be even not local summable. To show this it is sufficient to consider the Cayley transform of the maximal dissipative operators of Example 3.10 of [20].

At the end of this section we note that introducing a disribution theory of functions over $C^\infty(\mathbb{T}^1) \subseteq \mathfrak{F}_{\mathbb{T}^1}$, regarding $\xi(.) \in L^1([0,2\pi])$ as a distribution ξ defined by $(\xi,\varphi) = \int_0^{2\pi}\xi(t)\varphi(t)\,dt$, $\varphi(.) \in C^\infty(\mathbb{T}^1)$ and considering the distribution derivative ξ' given by $(\xi',\varphi) = -(\xi,\varphi')$ the generalized trace formula (2.26) of Theorem 2.4 can be transformed into the form

$$(2.57)\qquad \mathrm{tr}\{\varphi_+(T_1) + \varphi_-(T_1^*) - \varphi_+(T_o) - \varphi_-(T_o^*)\} = (\xi',\varphi'),$$

$\varphi(.) \in C^\infty(\mathbb{T}^1)$. In accordance with the considerations of P.Jonas [9] the distribution derivative ξ' of the integrated spectral shift function $\xi(.)$ can be called the spectral shift functional or the spectral shift distribution of the pair $\{T_1,T_o\}$. Following this line the contents of Theorem 2.5 can be understood as the possibility to localize the spectral shift distribution at a.e. points of $[0,2\pi]$ mod $|.|$. Hence the spectral shift function $\xi^*(.)$ of the pair $\{T_1,T_o\}$ is none other than the a.e. mod $|.|$ localized spectral shift distribution of $\{T_1,T_o\}$. But it is in general impossible to restore ξ' or $\xi(.)$ from the spectral shift function $\xi^*(.)$.

2.3. Scattering matrix and spectral shift function

We return to the situation, where T_o is a unitary operator on $\mathfrak{H}$. Our next aim is to generalize the well-known Birman-Krein formula to our contractive situation.

To this end we remember if $\{U_1,T_o\}$ is a pair of unitary operators on $\mathfrak{H}$ such that the condition $U_1 - T_o \in \mathcal{L}_1(\mathfrak{H})$ is fulfilled, then a spectral shift function $\mu(.)$ of the pair $\{U_1,T_o\}$ exists, which belongs to $L^1([0,2\pi])$ and in accordance with (2.21) can be represented by

$$(2.58)\qquad \mu(t) = -\lim_{r\uparrow 1}\frac{1}{\pi}\,\mathrm{Im}\,\log\det(I + (U_1-T_o)(T_o-r^{-1}e^{it})^{-1}) + \mathrm{const.}$$

for a.e. $t \in [0,2\pi]$ mod $|.|$. The spectral shift function is only defined up to an additive constant. Usually a certain spectral shift function $\mu^o(.)$ is fixed by the condition

$$(2.59)\qquad \int_0^{2\pi}\mu^o(t)\,dt = -i\,\mathrm{tr}(\log(U_1T_o^*)),$$

where by $-i\log(U_1T_o^*)$ we denote that operator of $-i\,\mathrm{Log}(U_1T_o^*)$, which spectrum is situated in $(-\pi,\pi]$, and is called the mean value of the spectral shift functions of $\{U_1,T_o\}$. The family of scattering matrices $\{S_o(e^{it})\}_{t\in\Delta}$ of the scattering system $\Lambda_o = \{U_1,T_o;I\}$ defined in accordance with section one consits of unitary operators in this case. Taking into account Corollary 2.3 it makes sense to consider the function $\det(S_o(e^{it}))$, $t\in\Delta$. Now M.Š.Birman and M.G.Krein have established in [5], that the spectral shift function $\mu^o(.)$ and the family of scattering matrices $\{S_o(e^{it})\}_{t\in\Delta}$ are related by

(2.60) $$\det(S_o(e^{it})) = e^{2\pi i\,\mu^o(t)}$$

for a.e. $t\in\Delta$ $\mathrm{mod}|.|$.

To extend the Birman-Krein formula to our contractive situation it turns out necessary to introduce a new object which is called the characteristic function of a contraction and which was widely investigated by C.Foias and B.Sz.-Nagy in [6]. In the following we deal with contractions T_1 characterized by the condition $\dim\ker(T_1) = \dim\ker(T_1^*)$. Contractions of this structure allows a representation of the form $T_1 = U_1R$, where U_1 is a unitary operator on $\mathfrak{H}$ and $R = |T_1|$. For this restricted class of contractions the characteristic function $\Theta_{T_1}(.)$ can be defined by

(2.61) $$\Theta_{T_1}(z) = R - z\sqrt{I-R^2}\,U_1^*\,\frac{1}{I - zT_1}\sqrt{I-R^2},$$

$z\in\mathcal{D} = \{z\in\mathbb{C} : |z| < 1\}$, where the values of $\Theta_{T_1}(.)$ are considered as bounded linear operators acting from $(\mathrm{ima}(\sqrt{I-R^2}))^-$ into itself. It is not hard to see that the definition (2.61) is equivalent in the sense of

[6,chapter V, 2.4.] to the definition of the characteristic function given in [6,chapter V, 1.1].

The characteristic function is an analytic contractive one, which completely characterizes the contraction T_1. Demanding $I - R\in\mathcal{L}_1(\mathfrak{H})$ we obtain $\Theta_{T_1}(z) - I\in\mathcal{L}_1(\mathfrak{H})$ for every $z\in D$. Hence it makes sense to define the complex-valued function $\delta_{T_1}(.)$,

(2.62) $$\delta_{T_1}(z) = \det(\Theta_{T_1}(z)),$$

$z\in D$. The complex-valued function $\delta_{T_1}(.)$ is an analytic contractive one, too. But this fact implies that the limit $\delta_{T_1}(e^{it})$,

(2.63) $$\delta_{T_1}(e^{it}) = \lim_{r\uparrow 1}\delta_{T_1}(re^{it})$$

exists for a.e. $t\in[0,2\pi]$ $\mathrm{mod}|.|$.

<u>Theorem 2.6.</u> Let $L^2(\Delta,|.|;\mathfrak{h}_t,\mathfrak{I})$ be a spectral representation of T_o^{ac} and let $\{S(e^{it})\}_{t\in\Delta}$ be the family of scattering matrices of the scattering system $\Lambda = \{T_1,T_o;I\}$ which obeys (1.2). Then there is a spectral shift function $\gamma^o(.)$ of the pair $\{T_1,T_o\}$ such that

(2.64) $$\det(S(e^{it})) = \delta_{T_1}(e^{it})\,e^{2\pi i\gamma^o(t)}$$

holds for a.e. $t\in\Delta$ $\mathrm{mod}|.|$.

<u>Proof.</u> We prove this theorem in several steps.

1. On account of [10,chapter IV, §5] and condition (1.2) we find that T_1 is a Fredholm operator with the property $\mathrm{nul}(T_1) = \mathrm{def}(T_1)$. Consequently, the operator T_1 allows the representation $T_1 = U_1R$, where U_1 is a unitary operator on $\mathfrak{H}$ and $R = |T_1|$. Moreover, we find $U_1 - T_1\in\mathcal{L}_1(\mathfrak{H})$ and $U_1R - U_1\in\mathcal{L}_1(\mathfrak{H})$.

2. The considerations of the first step allow to divide the scattering system Λ into two new scattering system $\Lambda_1 = \{T_1,U_1;I\}$ and $\Lambda_o = \{U_1,T_o;I\}$ such that we have

$$\det(S(e^{it})) = \det(S_1(e^{it}))\,\det(S_o(e^{it})) \tag{2.65}$$

for a.e. $t \in \Delta$ mod $|.|$, where $\{S_1(e^{it})\}_{t\in\Delta}$ and $\{S_o(e^{it})\}_{t\in\Delta}$ are the families of scattering matrices of the scattering systems Λ_1 and Λ_o, respectively. The fact that the spectral cores of both families of scattering matrices are the same must be established but it can be easily done.

By $\mu^o(.)$ we denote a spectral shift function of the pair $\{U_1,T_o\}$ normalized by (2.59). If $\xi_1(.)$ denotes the integrated spectral shift function of the pair $\{T_1,U_1\}$, we choose the generalized derivative $\xi^*_1(.)$ for the spectral shift function of the pair $\{T_1,U_1\}$. Setting

$$\gamma^o(t) = \mu^o(t) + \xi^*_1(t), \tag{2.66}$$

$t \in [0,2\pi]$, we obviously define a spectral shift function of the pair $\{T_1,T_o\}$. Taking into account (2.65) and (2.60) it remains to show

$$\det(S_1(e^{it})) = \delta_{T_1}(e^{it})\, e^{2\pi i\,\xi^*_1(t)} \tag{2.67}$$

for a.e. $t \in \Delta$ mod $|.|$.

3. We prove the relation (2.67). To this end we apply Theorem 2.1 to the scattering system $\Lambda_1 = \{T_1,U_1;I\}$. Replacing T_o by U_1 the operator-valued function $e^{-it}M(t)$, $t \in [0,2\pi]$, can be represented by

$$e^{-it}M(t) = \lim_{r\uparrow 1} B\,\frac{1}{2\pi}\,\frac{(1-r^2)\,U_1^*}{1+r^2-rU_1^*e^{it}-rU_1e^{-it}}\,B \tag{2.68}$$

for a.e. $t \in [0,2\pi]$ mod $|.|$. The limit can be taken in the trace norm. Using this representation and the property $T_1 = U_1R$ we get

$$\det(S_1(e^{it})) = \tag{2.69}$$

$$= \lim_{r\uparrow 1} \det\Big(I + B\,\frac{(1-r^2)U_1^*}{1+r^2-rU_1^*e^{it}-rU_1e^{-it}}\,B\,\cdot$$

$$\cdot\Big\{C - CB\,\frac{1}{T_1-r^{-1}e^{it}}\,BC\Big\}\Big) =$$

$$= \lim_{r\uparrow 1} \det\Big(I - \sqrt{I-R}\,\frac{1-r^2}{1+r^2-rU_1^*e^{it}-rU_1e^{-it}}\sqrt{I-R}\,\cdot$$

$$\cdot\Big\{I + \sqrt{I-R}\,\frac{1}{T_1-r^{-1}e^{it}}\,U_1\sqrt{I-R}\Big\}\Big)$$

for a.e. $t \in \Delta$ mod $|.|$. We set

$$\Pi(z) = I + \sqrt{I-R}\,\frac{1}{T_1-z}\,U_1\sqrt{I-R}, \tag{2.70}$$

$|z| > 1$. We find

$$[\Pi(z)]^{-1} = I - \sqrt{I-R}\,\frac{1}{U_1-z}\,U_1\sqrt{I-R} \tag{2.71}$$

and

$$\det(\Pi(z)) = \left[\det(I + (T_1-U_1)(U_1-z)^{-1})\right]^{-1}, \tag{2.72}$$

$|z| > 1$. Consequently, we obtain

$$\det(S_1(e^{it})) = \lim_{r\uparrow 1} \det\Big(I - \sqrt{I-R}\,\frac{1}{U_1-r^{-1}e^{it}}\,U_1\sqrt{I-R} - \tag{2.73}$$

$$-\sqrt{I-R}\,\frac{1-r^2}{1+r^2-rU_1^*e^{it}-rU_1e^{-it}}\sqrt{I-R}\Big)\cdot$$

$$\cdot\det(\Pi(r^{-1}e^{it})),$$

for a.e. $t \in [0,2\pi]$ mod $|.|$. Hence we get

$$(2.74)\qquad \det(S_1(e^{it})) = \lim_{r\uparrow 1} \det(I - \sqrt{I-R}\,\frac{1}{1-re^{it}U_1^*}\sqrt{I-R})\cdot \det(\Pi(r^{-1}e^{it})),$$

for a.e. $t \in [0,2\pi]$ mod $|.|$.

A simple calculation proves the equality

$$(2.75)\qquad \det(\theta_{T_1}(re^{it})) = \det(I - \sqrt{I-R}\,(I+re^{it}U_1^*)\cdot (I - re^{it}T_1^*)^{-1}\sqrt{I-R}),$$

$0 \le r < 1$, $t \in [0,2\pi]$. Using (2.75) we find

$$(2.76)\qquad \det(\theta_{T_1}(re^{it}))[\det(\Pi(r^{-1}e^{it}))]^{-1} = \det(I - \sqrt{I-R}\,\frac{1}{1 - re^{it}U_1^*}\sqrt{I-R}),$$

$0 < r < 1$, $t \in [0,2\pi]$. Putting (2.76) into (2.74) we obtain

$$(2.77)\qquad \det(S_1(e^{it})) = \lim_{r\uparrow 1} \det(\theta_{T_1}(re^{it}))\frac{\det(\Pi(r^{-1}e^{it}))}{\det(\Pi(r^{-1}e^{it}))}$$

for a.e. $t \in [0,2\pi]$. From (2.72), (2.49) and (2.63) we obtain (2.67). ■

3. Dissipative case

In this chapter we transform the results of chapter two to a pair $\{H_1,H_0\}$ of operators on $\mathfrak{H}$ which consits of a maximal dissipative operator H_1 and a selfadjoint operator H_0 and which obeys the condition

$$(3.1)\qquad (H_1 - i)^{-1} - (H_0 - i)^{-1} \in \mathcal{L}_1(\mathfrak{H}).$$

We formulate the results and sketch the proofs only.

The main tool to obtain such a transformation is the Cayley transform

$$(3.2)\qquad T_j = (H_j + i)(H_j - i)^{-1},$$

$j = 0,1$. It is not hard to see that (3.1) implies the condition (1.2) for the Cayley transforms T_1 and T_0. In such a way the results of chapter one hold for the scattering system $\Lambda = \{T_1,T_0;I\}$. The problem is to carry over these results to the scattering system $\Xi = \{H_1,H_0;I\}$. Notice that under (3.1) the wave operators Ω_+

$$(3.3)\qquad \Omega_+ = \text{s-lim}_{t\to+\infty}\, e^{itH_1^*}\, e^{-itH_0}\, P^{ac}(H_0),$$

and Ω_-,

$$(3.4)\qquad \Omega_- = \text{s-lim}_{t\to+}\, e^{-itH_1}\, e^{itH_0}\, P^{ac}(H_0)$$

exist, where $P^{ac}(H_0)$ denotes the projection from $\mathfrak{H}$ onto the absolutely continuous subspace $\mathfrak{H}^{ac}(H_0)$ of the selfadjoint operator H_0. In section 2.1. it was remarked that under (1.2) behind $W_\pm$ the dilation wave operators $\widetilde{W}_\pm$ exist. The same can be said concerning the dilation wave operators of Ξ. See for instance [16]. Taking into account the invariance principle [4, Corollary 26 p.248] we obtain that the dilation wave operators of the scattering systems Λ and Ξ coincide. But from this fact we get the equalities $W_\pm = \Omega_\pm$. Hence the scattering operators S and $\Sigma = \Omega_+^*\Omega_-$ of scattering systems Λ and Ξ, respectively, coincide. Moreover, the family $\{\Sigma(\lambda)\}_{\lambda\in N}$ defined by

$$(3.5)\qquad \Sigma(\lambda) = S(2\operatorname{arc\,ctg}\lambda),$$

$\lambda \in N = \{\lambda \in \mathbb{R}^1 : \lambda = \mathrm{ctg}(t/2),\ t \in \Delta\}$ is a family of scattering matrices of the scattering system Σ. Obviously, we have

(3.6) $$\Sigma(\lambda) - I_{\mathfrak{h}_{2\,\mathrm{arc\,ctg}\lambda}} \in \mathcal{L}_1(\mathfrak{h})$$

for a.e. $\lambda \in N$ mod $|.|$.

By the transformation $\Psi(\lambda) = \varphi\left(\frac{\lambda + i}{\lambda - i}\right)$, $\lambda \in \mathbb{R}^1$, we obtain a new set of functions from $\mathcal{F}_{\mathbb{T}^1}$ which we denote by $\mathcal{F}_{\mathbb{R}^1}$. Similarly, we introduce the functions $\Psi_\pm(.)$. A simple calculation shows the validity of

(3.7) $$\lim_{\lambda \to \pm\infty} (1 + \lambda^2)\Psi'(\lambda) = -2 \frac{d}{dt}\varphi(e^{it})|_{t=0} = -2\varphi'(1).$$

Hence we get a certain subset $\mathcal{F}'_{\mathbb{R}^1}$ of $\mathcal{F}_{\mathbb{R}^1}$ setting

(3.8) $$\mathcal{F}'_{\mathbb{R}^1} = \{\Psi(.) \in \mathcal{F}_{\mathbb{R}^1} : \lim_{\lambda \to \pm\infty} (1 + \lambda^2)\Psi'(\lambda) = 0\}.$$

Supposing that H_o is also a maximal dissipative operator, Theorem 2.4 reads now as follows.

<u>Theorem 3.1.</u> Let $\{H_1, H_o\}$ be a pair of maximal dissipative operators on $\mathfrak{h}$ such that the conditions

(3.9) $$(H_1^*+i)^{-1} - (H_o-i)^{-1} + 2i(H_1^*+i)^{-1}(H_o-i)^{-1} \in \mathcal{L}_1(\mathfrak{h})$$

and

(3.10) $$(H_1-i)^{-1} - (H_o^*+i)^{-1} - 2i(H_1-i)^{-1}(H_o^*+i)^{-1} \in \mathcal{L}_1(\mathfrak{h})$$

are fulfilled. Then for every $\Psi(.) \in \mathcal{F}'_{\mathbb{R}^1}$ we have

(3.11) $$\Psi_+(H_1) + \Psi_-(H_1^*) - \Psi_+(H_o) - \Psi_-(H_o^*) \in \mathcal{L}_1(\mathfrak{h}).$$

Moreover, there is a real measurable function $\xi(.)$ belonging to $L^1(\mathbb{R}^1, (1+\lambda^2)^{-2}d\lambda)$ such that

(3.12) $$\mathrm{tr}(\Psi_+(H_1) + \Psi_-(H_1^*) - \Psi_+(H_o) - \Psi_-(H_o^*)) = = -\int_{-\infty}^{+\infty} \xi(\lambda)\Psi''(\lambda)d\lambda$$

holds for every $\Psi(.) \in \mathcal{F}'_{\mathbb{R}^1}$. The function $\xi(.)$ is defined by (3.12) up to a linear function.

The proof essentially follows the considerations of chapter 3 of [21].

In the following we call a real measurable function $\xi(.)$ belonging to $L^1(\mathbb{R}^1, (1+\lambda^2)^{-2}d\lambda)$ and satisfying (3.12) the integrated spectral shift function of $\{H_1, H_o\}$. We remark that the integrated spectral shift function is defined up to a linear function.

Let $\xi(.) \in L^1(\mathbb{R}^1, (1+\lambda^2)^{-2}d\lambda)$. If the limit $\xi^*(\lambda)$,

(3.13) $$\xi^*(\lambda) = -\lim_{y \to +0} \int_{-\infty}^{+\infty} \xi(x) \frac{d}{dx}\left\{\frac{1}{\pi}\frac{y}{(x-\lambda)^2+y^2} - \frac{1}{\pi}\frac{y}{1+x^2}\right\}dx,$$

$y > 0$, exists at the point $\lambda \in \mathbb{R}^1$, then we call $\xi^*(\lambda)$ the generalized derivative of $\xi(.)$ at the point $\lambda \in \mathbb{R}^1$. Let $\theta(.)$ be a smooth function on $\mathbb{R}^1$ obeying $0 \le \theta(x) \le 1$, $x \in \mathbb{R}^1$, $\theta(x) = 1$ for $x \in [-1,1]$ and $\theta(x) = 0$ for $|x| \ge 2$. Then we find that the generalized derivative $\xi^*(\lambda)$ can be expressed by

(3.14) $$\xi^*(\lambda) = -\lim_{y \to +0} \int_{-\infty}^{+\infty} \xi(x)\theta(x-\lambda) \frac{d}{dx}\frac{1}{\pi}\frac{y}{(x-\lambda)^2+y^2}\, dx,$$

$\lambda \in \mathbb{R}^1$. Using this representation we can show that if the usual derivative $\xi'(\lambda)$ exists at $\lambda \in \mathbb{R}^1$ of $\xi(\lambda)$, then the generalized derivative $\xi^*(\lambda)$ also exists at $\lambda \in \mathbb{R}^1$ and equals $\xi'(\lambda)$, i.e. $\xi'(\lambda) = \xi^*(\lambda)$. In such a way the generalized derivative of a linear function exists at every point and equals a constant.

Now Theorem 2.5 goes over into

Theorem 3.2. Let $\{H_1,H_0\}$ be a pair of maximal dissipative operators on $\mathfrak{H}$ such that the conditions (3.9) and (3.10) are fulfilled. If $\xi(.)$ denotes an integrated spectral shift function of the pair $\{H_1,H_0\}$, then the generalized derivative $\xi^*(\lambda)$ exists for a.e. $\lambda \in \mathbb{R}^1$ mod$|.|$.

The proof is based on the fact that for every integrated spectral shift function $\xi(.)$ of the pair of Cayley transforms $\{T_1,T_0\}$ there is a real constant such that we have

(3.15) $$\xi^*(t)|_{t=2\mathrm{arc\ ctg}\lambda} + \text{const.} = -\xi^*(\lambda)$$

for a.e. $\lambda \in \mathbb{R}^1$ mod$|.|$.

In accordance with the previous chapter we call the generalized derivative of the integrated spectral shift function of $\{H_1,H_0\}$ a spectral shift function of the pair $\{H_1,H_0\}$. We note that the spectral shift function is defined up to constant.

We return to the situation that H_0 is selfadjoint. Next we generalize the Birman-Krein formula to a pair $\{H_1,H_0\}$, where H_1 is a maximal dissipative operator and H_0 is a selfadjoint operator, obeying (3.1). Let $\theta_{T_1}(.)$ be the characteristic function of the Cayley transform T_1 of H_1. We call the operator-valued function $\theta_{H_1}(.)$ defined by

(3.16) $$\theta_{H_1}(z) = \theta_{T_1}\left(\frac{z+i}{z-i}\right),$$

Im $z<0$, the characteristic function of the maximal dissipative operator H_1. Obviously, the characteristic function $\theta_{H_1}(.)$ is a contractive analytic one on the lower half plane. Because of (3.1) we have $\theta_{H_1}(z) - I \in \mathcal{L}_1(\mathfrak{H})$ for every z of the lower half plane. Hence it makes sense to define the complex-valued function $\delta_{H_1}(z) = \det(\theta_{H_1}(z))$, Im $z<0$, which is a contractive analytic one, too. Consequently for a.e. $\lambda \in \mathbb{R}^1$ mod$|.|$ the limits $\delta_{H_1}(\lambda) = \lim_{y\to+0} \delta_{H_1}(\lambda - iy)$ exist. Obviously, we have $\delta_{H_1}(\lambda) = \delta_{T_1}\left(\frac{\lambda+i}{\lambda-i}\right)$ for a.e. $\lambda \in \mathbb{R}^1$ mod$|.|$.

Theorem 3.3. Let $\{\Sigma(\lambda)\}_{\lambda\in N}$ be the family of scattering matrices of the scattering system $\mathcal{S} = \{H_1,H_0;I\}$ obeying (3.1) with respect to some spectral representation of H_0^{ac}. Then there is an integrated spectral shift function $\xi(.)$ of the pair $\{H_1,H_0\}$ such that

(3.17) $$\det(\Sigma(\lambda)) = \delta_{H_1}(\lambda)\, e^{-2\pi i \xi^*(\lambda)}$$

holds for a.e. $\lambda \in N$ mod$|.|$.

The proof uses Theorem 2.6 and the relation (3.15).

We remark that the contents of Theorem 3.3 is very similar to the assertions of §4 of [12].

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Найдхарт X. E5-87-13

Матрица рассеяния и функция спектрального сдвига ядерной диссипативной теории рассеяния

Работа посвящена проблеме существования функции спектрального сдвига для пары сжимающих операторов, отличающихся друг от друга ядерным оператором. Известно, что функция спектрального сдвига для пары унитарных операторов на такой случай, в общем, не распространяется и она не обязана существовать как суммируемая функция. В частности, это влечет за собой то, что известная формула следов не имеет места в такой ситуации. Обобщается подходящим образом понятие функции спектрального сдвига так, что она будет суммируемая и удовлетворяет видоизмененной формуле следов. Показана модифицированная формула Бирамана-Крейна. Все результаты переносятся на пару максимальных диссипативных резольвентно-сравнимых операторов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Neidhardt H. E5-87-13

Scattering Matrix and Special Shift of the Nuclear Dissipative Scattering Theory

The paper is devoted to the problem of existence of the spectral shift function for a pair of contractions which differ by a nuclear operator. It is well-known that the usual spectral shift function for a pair of unitary operators does not allow an extension to this case in general and, moreover, the shift function cannot be obtained as a summable one. This fact yields for instance that the famous trace formula cannot be verified in this situation. In the paper the notion of the spectral shift function is generalized in an appropriate manner such that the shift is summable and satisfies a modified trace formula. Moreover, a modification of the Birman-Krein formula is established. All results are transformed to a pair of maximal dissipative operators whose resolvent difference belongs to the trace class.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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