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ON THE ITERATED STIELTJES TRANSFORM OF GENERALIZED FUNCTIONS

Submitted to "Mathematische Nachrichten"

1986

1. Introduction

This paper is concerned with the iteration of the generalized Stieltjes transform

$$(1.1)\quad \mathcal{S}[f;\varrho](z) = \int_0^\infty (z+t)^{-\varrho} f(t)\,dt,$$

where $\varrho \in R$ is fixed, $-\pi < \arg z < \pi$, the principal value of $(z+t)^{-\varrho}$ is taken and $(1+t)^{-\varrho} f(t) \in \mathcal{L}(0,\infty)$ is assumed. Then it is known /10/ that (1.1) defines an analytic function in $\mathbb{C}\setminus(-\infty,0]$. The transform (1.1) can be inverted by use of a differential operator of infinite order. In the case $\varrho = 1$ it is additional known that (1.1) can be inverted by a complex integral formula. Benedetto /1/, Erdélyi /5/, Pandey /6/, Pathak /7/, Stancovič /9/ and Zemanian /10/ extended the transform (1.1) to generalized functions following different approaches.

When the transform (1.1) is iterated, one leads to the transform

$$(1.2)\quad \mathcal{S}[\mathcal{S}[f,\nu];\varrho](z) = \int_0^\infty (z+y)^{-\varrho} \int_0^\infty (y+t)^{-\nu} f(t)\,dt$$

or, if one can change the order of integration, to the transform

$$(1.3)\quad \mathcal{S}[f,\varrho,\nu](z) = \int_0^\infty K(z,t;\varrho,\nu)\, f(t)\,dt,$$

where

$$(1.4)\quad K(z,t;\varrho,\nu) = \int_0^\infty (z+y)^{-\varrho}(y+t)^{-\nu}\,dy .$$

To distinguish between these two cases, we refer to (1.2) as the generalized iterated Stieltjes transform and to (1.3) as the generalized S_2-transform.

Boas and Widder /2/ studied the transform (1.2) and (1.3) in the case $\varrho = \nu = 1$, Dube /4/ in the case $\varrho = \nu > 1/2$. A distributional extention of (1.3) has recentl- been given by Dube /3,4/ for the cases $\varrho = \nu = 1$ and $\varrho = \nu > 1/2$. Dube proved that for fixed $z > 0$ and $\varrho = \nu > 1/2$ the kernel (1.4) of the transform (1.3) can be embedded in a test function space. Then the generalized S_2-transform of elements of the dual space is defined by

$$(1.5) \quad \mathcal{S}[f;\varrho,\nu](z) = \langle f, K_{z;\varrho,\nu} \rangle,$$

where

$$K_{z;\varrho,\nu}(t) = K(z,t;\varrho,\nu) \, .$$

According to Zemanian /11/ this method is called the direct approach to the integral transform of generalized functions.
There is a basically different approach to generalize integral transforms which may be called the method of adjoint. The idea is the following. Consider a test function space A which is mapped by the integral transform continuously into another test function space B . The adjoint mapping then defines the generalized integral transform for elements of the dual space of B . The transform thus defined is no longer a function but a distribution, an element of the dual space of A . For usual functions with suitable integrability properties the double integral

$$(1.6) \quad \int_0^\infty \int_0^\infty f(x)\, K(x,t;\varrho,\nu)\, g(t)\, dt$$

can be evaluated in two different ways showing that

$$(1.7) \quad \langle \mathcal{S}[f;\nu,\varrho], g \rangle = \langle f, \mathcal{S}[g;\varrho,\nu] \rangle$$

and this relation is the basis for the application of the method of adjoints to the S_2-transform. One example for the method of adjoints is the well-known theory of Fourier transforms of tempered distributions. For the transform (1.1) this way was presented in /5/.
The aim of the present paper is further to develop these two approaches for the transform (1.3). At first the properties of the kernel (1.4) are discussed. Dube's approach is improved to the case $\varrho, \nu > 0$, $\varrho + \nu > 1$. Then we discuss the method of adjoints for the transform (1.3). It is proved that the inversion operator is again a linear differential operator of infinite order. Examples are given.

2. The test function space

Following Erdélyi /5/ we define for infinitely differentiable complex-valued functions $\phi(t)$ on R_+ and $a, b \in R$ the set of seminorms

$$(2.1) \quad \mu_{a,b,k}(\phi) = \sup_{t \in R_+} t^{1-a+k} (1+t)^{a-b} \, |\phi^{(k)}(t)| \, .$$

The test function space $\mathcal{M}_{a,b}$ is given by

$$(2.2) \quad \mathcal{M}_{a,b} = \{ \phi \in C^\infty(R_+) : \mu_{a,b,k}(\phi) < \infty \ \text{for all} \ k \in Z_+ \},$$

where Z_+ is the set of all non-negative integers. $\mathcal{M}_{a,b}$ is equipped with the topology generated by the seminorms (2.1). A sequence $\{\phi_n(t)\}$, $\phi_n(t) \in \mathcal{M}_{a,b}$, converges in $\mathcal{M}_{a,b}$ to $\phi(t)$ if $\mu_{a,b,k}(\phi_n - \phi)$ tends to zero as n goes to infinity for each $k \in Z_+$. It can be proved that $\mathcal{M}_{a,b}$ is a Fréchet-space. Note that $\mathcal{M}_{a,b}$ is not nuclear. The dual space $\mathcal{M}'_{a,b}$ consists of all continuous linear functionals on $\mathcal{M}_{a,b}$ and is equipped with the usual weak topology.
Let $f(t)$ be a function defined on R_+ such that $t^{a-1}(1+t)^{b-a} f(t) \in \mathcal{L}(0,\infty)$. Then it generates an element of $\mathcal{M}'_{a,b}$ according to

$$\langle f, \phi \rangle = \int_0^\infty f(t)\, \phi(t)\, dt \ , \quad \phi \in \mathcal{M}_{a,b} \, .$$

These elements of $\mathcal{M}'_{a,b}$ will be called regular elements, and we use the same notation for the function $f(t)$ and the regular element of $\mathcal{M}'_{a,b}$ which it generates.

3. The kernel

At first we study the properties of the kernel (1.4) of our generalized S_2-transform.
Suppose once for all in this paper $\varrho, \nu > 0$, $\varrho + \nu > 1$.

Lemma 1:
Let $x, y > 0$ and consider $K(x,t;\rho,\nu)$ defined by (1.4). Then the following properties hold:

i) smoothness:

$$K(x,t,\rho,\nu) \in C^{\infty}(R_+ \times R_+),$$

ii) symmetry:

$$K(x,t;\rho,\nu) = K(t,x;\rho,\nu),$$

iii) homogeneity:

$$K(\alpha x,\alpha t;\rho,\nu) = \alpha^{1-\rho-\nu} K(x,t;\rho,\nu),$$

iv) partial-differential equation:

$$(x\partial_x + t\partial_t + \rho+\nu-1)\, K(x,t;\rho,\nu) \equiv 0,$$

v) positivity:

$$(-1)^{m+n} (\partial_x)^m (\partial_t)^n K(x,t;\rho,\nu) > 0 \quad \text{for all } m,n \in Z_+$$

vi) asymptotic behaviour:

$$K(1,t;\rho,\nu) = O(t^{\min(0,1-\nu)}) \quad \text{as} \quad t \to 0 \quad (\nu \neq 1)$$

$$K(1,t;\rho,\nu) = O(t^{-\nu+\max(0,1-\rho)}) \quad \text{as} \quad t \to \infty \quad (\rho \neq 1)$$

with a logarithmic correction factor on the right-hand side if $\nu = 1$ or if $\rho = 1$.

Proof:
The properties i) and ii) are obvious. Properties iii) to vi) follow by direct computations.

Now we show that (1.4) for fixed $z \in \mathbb{C}\setminus(-\infty,0]$ is an element of $M_{a,b}$ for special a and b.

Theorem 1:
For fixed $z \in \mathbb{C}\setminus(-\infty,0]$ the kernel $K(z,t;\rho,\nu)$ belongs to $M_{a,b}$ if

$$a \leq 1+\min(0,1-\nu) \quad \text{and} \quad a < 1 \quad \text{if} \quad \nu = 1$$

(3.1) $$b \geq 1-\nu+\max(0,1-\rho) \quad \text{and} \quad b > 1-\nu \quad \text{if} \quad \rho = 1$$

Proof:
Consider

$$\mu_{a,b,k}(K(z,t;\rho,\nu)) =$$

$$= \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \sup_{t\in R_+} t^{1-a+k}(1+t)^{a-b} \,|K(z,t;\rho,\nu+k)|$$

$$\leq \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \sup_{t\in R_+} t^{1-a}(1+t)^{a-b} \int_0^\infty |z+y|^{-\rho}(y+t)^{-\nu}\left(\frac{t}{y+t}\right)^k dy .$$

Note that $(t/(y+t)) < 1$ if $t,y>0$ and for $z = |z|e^{i\arg z}$, $-\pi < \arg z < \pi$ we have

$$\cos\frac{\arg z}{2} \leq \frac{|z+y|}{|z|+y} \leq 1$$

so that

$$|z+y|^{-\rho} \leq \left(\cos\frac{\arg z}{2}\right)^{-\rho}(|z|+y)^{-\rho}$$

and consequently

$$\mu_{a,b,k}(K(z,t,\rho,\nu)) \leq$$

(3.2) $$\leq \frac{\Gamma(\nu+k)}{\Gamma(\nu)}\left(\cos\frac{\arg z}{2}\right)^{-\rho} \mu_{a,b,0}(k(|z|,t,\rho,\nu)) .$$

It is sufficient to investigate $\mu_{a,b,0}(k(x,t;\rho,\nu))$ for $x>0$. Because of the homogeneity of the kernel we have

$$\mu_{a,b,0}(K(x,t,\rho,\nu)) = \sup_{t\in R_+} t^{1-a}(1+t)^{a-b} K(x,t;\rho,\nu)$$

$$= x^{2-\rho-\nu-b} \sup_{t\in R_+} \left(\frac{t}{x}\right)^{1-a}\left(\frac{1}{x}+\frac{t}{x}\right)^{a-b} K\left(1,\frac{t}{x};\rho,\nu\right)$$

$$= x^{2-\rho-\nu-a} \sup_{t\in R_+} \left(\frac{x}{t}\right)^{\rho+\nu-2+b}\left(x+\frac{x}{t}\right)^{a-b} K\left(\frac{x}{t},1;\rho,\nu\right).$$

Having in mind that $x > 0$ is up to now fixed after substitution of $\frac{t}{x}$ and $\frac{x}{t}$ by q and r respectively we get

$$\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right) =$$

$$(3.3)\qquad = x^{2-\rho-\nu-b} \sup_{q\in\mathbb{R}_+} q^{1-a}\left(\tfrac{1}{x}+q\right)^{a-b} K(1,q;\rho,\nu)$$

and

$$\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right) =$$

$$(3.4)\qquad = x^{2-\rho-\nu-a} \sup_{r\in\mathbb{R}_+} r^{\rho+\nu-2-b}(x+r)^{a-b} K(1,r;\nu,\rho) .$$

Since

$$(3.5)\qquad \min(1,x) \le \frac{x+t}{1+t} \le \max(1,x) \quad \text{if } t,x>0$$

it follows from (3.3) that

$$\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right) \le C\ \mu_{a,b,0}\left(K(1,t,\rho,\nu)\right),$$

where C depends on a,b,ρ,ν and x. The use of property vi) of Lemma 1 gives immediately that

$$\mu_{a,b,0}\left(K(1,t;\rho,\nu)\right) \le C'$$

if a,b satisfy conditions (3.1) where C' depends on a,b,ρ and ν. This completes the proof of the theorem.

Corollary:
Let $z\in\mathbb{C}\setminus(-\infty,0]$. If a,b satisfy conditions (3.1), then (1.4) is an analytic function of z in the topology of $\mathcal{M}_{a,b}$.

Further we need the behaviour of $\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right)$ for $x\to+0$ and $x\to\infty$, respectively. First let $x\le 1$. For $b\le a$ it is $t^{a-b}\le(x+t)^{a-b}\le(1+t)^{a-b}$ and from (3.4) we have $x^{\rho+\nu-2+a}\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right)$ has upper and lower nonzero bounds. If $b>a$, then

$$\left(\tfrac{1}{x}\right)^{a-b}(1+t)^{a-b} \le \left(\tfrac{1}{x}+t\right)^{a-b} \le (1+t)^{a-b}$$

and from (3.3) we have that $x^{\rho+\nu-2+b}\mu_{a,b,0}\left(K(x,t;\rho,\nu)\right)$ has upper and lower nonzero bounds. Thus there exist positive constants C_1 and C_2 independent of x but dependent on a,b,ρ,ν such that for $x\le 1$ we have

$$C_1 x^{2-\rho-\nu-\max(a,b)} \le \mu_{a,b,0}\left(K(x,t,\rho,\nu)\right) \le C_2 x^{2-\rho-\nu-\max(a,b)} .$$

Similarly for $x>1$ there exist positive constants C_3 and C_4 such that for $x>1$

$$C_3 x^{\min(0,2-\rho-\nu-\min(a,b))} \le \mu_{a,b,0}\left(K(x,t;\rho,\nu)\right) \le C_4 x^{\min(0,2-\rho-\nu-\min(a,b))} .$$

We summarize what we shall need in the following lemma:

Lemma 2:
Let a,b satisfy conditions (3.1), then

$$\mu_{a,b,0}\left(K(x,t,\rho,\nu)\right) = O\left(x^{2-\rho-\nu-\max(a,b)}\right) \quad \text{as } x\to+0$$

$$(3.6)\qquad \mu_{a,b,0}\left(K(x,t,\rho,\nu)\right) = O\left(x^{\min(0,2-\rho-\nu-\min(a,b))}\right) \quad \text{as } x\to\infty .$$

4. Dube's approach

First let us define the generalized S_2-transform of distributions by the direct approach. The definition is similar to Dube's definition in /4/ for the case $\rho=\nu>\frac{1}{2}$.
Suppose a,b satisfy conditions (3.1) and suppose $z\in\mathbb{C}\setminus(-\infty,0]$. Then $K_{z,\rho,\nu}(t) = K(z,t,\rho,\nu)\in\mathcal{M}_{a,b}$ and (1.5) defines the generalized S_2-transform for $f\in\mathcal{M}'_{a,b}$. From the Corollary of Theorem 1 it follows that $\mathcal{S}[f;\rho,\nu](z)$ is an analytic function of $z\in\mathbb{C}\setminus(-\infty,0]$. From (3.1) and (3.6) we have

$$\mathcal{S}[f;\rho,\nu]^{(k)}(z) = O\left(|z|^{2-\rho-\nu-\max(a,b)}\right) \quad \text{as } |z|\to+0$$

$$(4.1)\qquad \mathcal{S}[f,\rho,\nu]^{(k)}(z) = O\left(|z|^{\min(0,2-\rho-\nu-\min(a,b))}\right) \quad \text{as } |z|\to\infty$$

uniformly in any sector $0 < |z| < \infty$, $|\arg z| \leq \pi - \delta < \pi$. If f is a regular element of $\mathcal{M}'_{a,b}$, then this definition coinsides with that for the classical case.
Additionally we have from (3.2)

$$(4.2) \quad \left| \mathcal{S}[f,\varrho,\nu]^{(k)}\left(e^{i \arg z}\right) \right| \leq C \left(\cos \frac{\arg z}{2} \right)^{-\varrho-k}$$

5. The method of adjoints

We prove the following theorem:

Theorem 2:
Let

$$(5.1) \quad \alpha > \max(0, \nu - 1), \quad \beta < \nu + \min(0, \varrho - 1).$$

Then the generalized S_2-transform maps $\mathcal{M}_{\alpha,\beta}$ continuously into $\mathcal{M}_{a,b}$ if

$$(5.2) \quad \begin{array}{llll} a \leq 1 + \min(0, 1-\varrho) & \text{and} & a < 1 & \text{if } \varrho = 1 \\ a \leq 2 - \varrho - \nu + \alpha & \text{and} & a < 2 - \varrho & \text{if } \nu = \alpha \\ b \geq 1 - \varrho + \max(0, 1-\nu) & \text{and} & b > 1 - \varrho & \text{if } \nu = 1 \\ b \geq 2 - \varrho - \nu + \beta & \text{and} & b > 2 - \varrho - \nu & \text{if } \beta = 0. \end{array}$$

Proof:
Let $\phi(t) \in \mathcal{M}_{\alpha,\beta}$. Then $\mathcal{S}[\phi;\varrho,\nu](x)$, defined as in (1.3) is a smooth function on $\mathbb{R}_+$ and

$$\mathcal{S}[\phi;\varrho,\nu]^{(k)}(x) = (-1)^k \frac{\Gamma(\varrho+k)}{\Gamma(\varrho)} \mathcal{S}[\phi;\varrho+k,\nu](x)$$

It follows that

$$\left| \mathcal{S}[\phi;\varrho,\nu]^{(k)}(x) \right| \leq$$

$$\leq \frac{\Gamma(\varrho+k)}{\Gamma(\varrho)} \mu_{a,b,0}(\phi) \, \mathcal{S}[t^{\alpha-1}(1+t)^{\beta-\alpha};\varrho,\nu](x)$$

so that

$$\mu_{a,b,k}(\mathcal{S}[\phi;\varrho,\nu]) \leq$$

$$(5.3) \quad \leq \mu_{a,b,0}(\phi) \, \mu_{a,b,k}(\mathcal{S}[t^{\alpha-1}(1+t)^{\beta-\alpha};\varrho,\nu]).$$

From the asymptotic behaviour of the hypergeometric function it is known /5/ that

$$(5.4) \quad \int_0^\infty (y+t)^{-p} t^{q-1} (1+t)^{-r} dt = \mathcal{O}\left(y^{\min(0,q-p)} (1+y)^{\max(0,q-r)-\min(p,q)}\right)$$

with a logarithmic correction factor on the right-hand side if $p = q$ or $q = r$. Because the integrand in the double integral $\mathcal{S}[t^{\alpha-1}(1+t)^{\beta-\alpha};\varrho,\nu](x)$ is positive, the order of integration can be changed and we have

$$\int_0^\infty \frac{t^{\alpha-1}(1+t)^{\beta-\alpha}}{(y+t)^\nu} dt = \mathcal{O}\left(y^{\min(0,\alpha-\nu)} (1+y)^{\max(0,\beta)-\min(\alpha,\beta)}\right), \quad \alpha \neq \nu, \ \beta \neq 0,$$

and therefore

$$\mathcal{S}[t^{\alpha-1}(1+t)^{\beta-\alpha};\varrho+k,\nu](x) =$$

$$= \int_0^\infty \frac{dy}{(x+y)^{\varrho+k}} \int_0^\infty \frac{t^{\alpha-1}(1+t)^{\beta-\alpha}}{(y+t)^\nu} dt$$

$$(5.5) \quad = \mathcal{O}\left(x^{\min(0, \min(0,\alpha-\nu)+1-\varrho-k)} (1+x)^{\max(0, \max(0,\beta)+1-\nu-\min(1,\alpha+1-\nu,\varrho+k))}\right)$$

$$\varrho \neq 1 \text{ if } \alpha > \nu, \quad 0 \leq \nu - 1 \neq \beta.$$

Taking into account (5.1) and (5.2), (5.5) gives us

$$(5.6) \quad \mu_{a,b,k}(\mathcal{S}[t^{\alpha-1}(1+t)^{\beta-\alpha};\varrho,\nu]) \leq C,$$

where C depends on $\alpha, \beta, a, b, \varrho, \nu$ and k. Thus, the theorem is proved.

Now suppose that α, β, a, b satisfy conditions (5.1) and (5.2) and let $\phi \in \mathcal{M}_{\alpha,\beta}$ and $f \in \mathcal{M}'_{a,b}$. From Theorem 2 we have $\mathcal{S}[\phi;\varrho,\nu](x) \in \mathcal{M}_{a,b}$ and so (1.7) defines a generalized S_2-transform of $f \in \mathcal{M}'_{a,b}$.

6. The inversion operator

In /5/ it is proved that the generalized Stieltjes transform of distributions can be inverted by a sequenee of differential operators.

Let $L_{n,\varrho}$ be an operator which acts on functions $\Phi(t)\in C^{\infty}(\mathbb{R}_+)$ as follows

$$L_{n,\varrho}\,\Phi(t) = L_{n,\varrho,t}\,\Phi(t) =$$

$$(6.1)\quad = \frac{(-1)^n\,\Gamma(\varrho)}{n!\,\Gamma(n+\varrho-1)}\left(\frac{d}{dt}\right)^n t^{2n+\varrho-1}\left(\frac{d}{dt}\right)^n \Phi(t),\quad n\in\mathbb{Z}_+ .$$

Erdélyi proved that the sequence $\{L_{n,\varrho}\,\mathcal{S}[\Phi;\varrho](t)\}$ converges in $\mathcal{M}_{\alpha,\beta}$ to $\Phi(t)$. In this section we shall prove a similar statement for the trânsform (1.3).

Let $L_{n,\varrho,\nu}$ be an operator which acts on functions as follows

$$L_{n,\varrho,\nu}\,\Phi(t) = L_{n,\varrho,\nu,t}\,\Phi(t) =$$

$$= L_{n,\nu}\circ L_{n,\varrho}\,\Phi(t)$$

$$(6.2)\quad = \frac{\Gamma(\varrho)\,\Gamma(\nu)}{(n!)^2\,\Gamma(n+\varrho-1)\,\Gamma(n+\nu-1)}\left(\frac{d}{dt}\right)^n t^{2n+\nu-1}\left(\frac{d}{dt}\right)^{2n} t^{2n+\varrho-1}\left(\frac{d}{dt}\right)^n\Phi(t).$$

Note that $L_{n,\varrho,\nu}$ maps $\mathcal{M}_{\alpha+2-\varrho-\nu,\,\beta+2-\varrho-\nu}$ continuously into $\mathcal{M}_{\alpha,\beta}$. For the (formal) adjoint operator we have

$$L^{*}_{n,\varrho,\nu} = L_{n,\nu,\varrho}\,.$$

Let us compute $L_{n,\varrho,\nu,x}\,K(x,t;\varrho,\nu)$. Because /10/

$$(6.3)\quad L_{n,\varrho,x}\,(x+t)^{-\varrho} = \frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\;\frac{x^{n+\varrho-1}\,t^n}{(x+t)^{2n+\varrho}}$$

it follows

$$L_{n,\varrho,\nu,x}\,k(x,t;\varrho,\nu) = L_{n,\nu,x}\circ L_{n,\varrho,x}\,k(x,t;\varrho,\nu) =$$

$$(6.4)\quad = \frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\;L_{n,\nu,x}\int_0^\infty \frac{x^{n+\varrho-1}\,y^n}{(x+y)^{2n+\varrho}\,(y+t)^{\nu}}\,dy\,.$$

Now we show that (6.4) may be written as

$$(6.5)\quad \frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\int_0^\infty \frac{x^{n+\varrho+\nu-2}\,y^{n+1-\nu}}{(x+y)^{2n+\varrho}}\left(L_{n,\nu,y}\,(y+t)^{-\nu}\right)dy\,.$$

The substitution $y=u\cdot x$ in (6.5) leads to

$$\frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\int_0^\infty \frac{u^{n+1-\nu}}{(1+u)^{2n+\varrho}}\left(L_{n,\nu,ux}\,(ux+t)^{-\nu}\right)du$$

$$= \frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\int_0^\infty \frac{u^{n+1-\nu}}{(1+u)^{2n+\varrho}}\,u^{\nu-1}\left(L_{n,\nu,x}\,(ux+t)^{-\nu}\right)du$$

$$= \frac{\Gamma(2n+\varrho)}{n!\,\Gamma(n+\varrho-1)}\;L_{n,\nu,x}\int_0^\infty \frac{u^n}{(1+u)^{2n+\varrho}}\,(ux+t)^{-\nu}\,du\,.$$

Replace ux by y in the last integral, (6.5) gets to (6.4).
From (6.3), (6.4) and (6.5) we have

$$L_{n,\varrho,\nu,x}\,K(x,t;\varrho,\nu) =$$

$$(6.6)\quad = \frac{\Gamma(2n+\varrho)\,\Gamma(2n+\nu)}{(n!)^2\,\Gamma(n+\varrho-1)\,\Gamma(n+\nu-1)}\int_0^\infty \frac{x^{n+\varrho+\nu-2}\,y^{2n}\,t^n}{(x+y)^{2n+\varrho}\,(y+t)^{2n+\nu}}\,dy\,.$$

Next we compute

$$\int_0^\infty L_{n,\varrho,\nu,x}\,K(x,t;\varrho,\nu)\,dt\,.$$

Using the formula for the beta-function

$$(6.7)\quad \int_0^\infty \frac{y^{\alpha-1}}{(t+y)^{\beta}}\,dy = t^{\alpha-\beta}\,\frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)}$$

we get for $n+2>\nu$, $n+1>\varrho$

$$\int_0^\infty L_{n,\varrho,\nu,x}\,k(x,t,\varrho,\nu)\,dt =$$

$$= \frac{\Gamma(n+1+1-\nu)\,\Gamma(n+\varrho-1+\nu-1)}{\Gamma(n+1)\,\Gamma(n+\varrho-1)} = \Theta_n(\varrho,\nu) = \Theta_n\,.$$

By Stirling's formula it follows that the sequence $\{\Theta_n\}$ converges to 1 as n goes to infinity.

In the following we use the operator

$$(6.8)\qquad \bar{L}_{n,\varrho,\nu} = (\theta_n)^{-1} L_{n,\varrho,\nu}$$

instead of $L_{n,\varrho,\nu}$ so that for $n \in \mathbb{Z}_+$, $n+2>\nu$, $n+1>\varrho$ we have

$$1 = \int_0^\infty \bar{L}_{n,\varrho,\nu,x} K(x,t;\varrho,\nu) = \gamma_{n,\varrho,\nu} \int_0^\infty \frac{x^{n+\varrho+\nu-2}\, y^{2n}\, t^n}{(x+y)^{2n+\varrho}\,(y+t)^{2n+\nu}}\, dy .$$

We want to show that the sequence $\{\bar{L}_{n,\varrho,\nu,x} K(x,t;\varrho,\nu)\}$ tends to $\delta(x-t)$ in a weak sense. Because of the homogeneity it is sufficient to consider $x=1$.

Let $t=a+1$, $a \geq -1$, suppose $a \neq 0$ and put

$$\bar{L}_{n,\varrho,\nu,x} K(x,1+a,\varrho,\nu)\Big|_{x=1} = \bar{L}_{n,\varrho,\nu} K(1,1+a;\varrho,\nu) .$$

Then we have

$$\bar{L}_{n,\varrho,\nu} K(1,1+a;\varrho,\nu) =$$

$$= \gamma_{n,\varrho,\nu} \int_0^\infty \frac{y^{2n}\,(1+a)^n}{(1+y)^{2n+\varrho}\,(y+1+a)^{2n+\nu}}\, dy$$

$$= \gamma_{n,\varrho,\nu} (1+a)^{n+\lambda_a(\nu)} \int_0^\infty \frac{y^{2n}}{[(1+y)(y+1+a)]^{2n-1/2}} \frac{(1+a)^{-\lambda_a(\nu)}}{(1+y)^{\varrho-1/2}\,(y+1+a)^{\nu-1/2}}\, dy ,$$

where

$$\lambda_a(\nu) = \begin{cases} 1/2-\nu & \text{if} \quad -1 \leq a < 0 \\ 0 & \text{if} \quad a > 0 \end{cases}$$

Because

$$0 < \frac{(1+a)^{-\lambda_a(\nu)}}{(1+y)^{\varrho-1/2}\,(y+1+a)^{\nu-1/2}} < C(\varrho,\nu)$$

for all $y>0$ and taking into account formula 2.2.6.23 from /8/ we conclude

$$\bar{L}_{n,\varrho,\nu} K(1,1+a,\varrho,\nu) \leq$$

$$\leq \frac{\sqrt{\pi}}{2} C(\varrho,\nu)\cdot\gamma_{n,\varrho,\nu} \cdot \frac{\Gamma(2n-2)}{\Gamma(2n-1/2)} \cdot$$

$$(6.9)\qquad \cdot \frac{(1+a)^{n+\lambda_a(\nu)}\,[2+a+2(2n-1)\sqrt{1+a}\,]}{(1+\sqrt{1+a}\,)^{4n-2}}$$

<u>Lemma 3:</u>

If $\varphi(t)$ is a function continuous in $(0,1]$ and there exists a real number λ such that

$$\int_0^1 t^\lambda \varphi(t)\, dt \qquad \text{converges,}$$

then

$$\int_0^{1-\varepsilon} \varphi(t)\, \bar{L}_{n,\varrho,\nu} K(1,t;\varrho,\nu)\, dt$$

tends to zero if n goes to infinity.

<u>Proof:</u>

From (6.9) we have for $n > \lambda+\nu$, $n \in \mathbb{Z}_+$

$$\left| \int_0^{1-\varepsilon} \varphi(t)\, \bar{L}_{n,\varrho,\nu} K(1,t;\varrho,\nu)\, dt \right| =$$

$$= \left| \int_{-1}^{-\varepsilon} \varphi(1+a)\, \bar{L}_{n,\varrho,\nu} K(1,1+a,\varrho,\nu)\, da \right| \leq$$

$$\leq \frac{\sqrt{\pi}}{2} C(\varrho,\nu)\cdot\gamma_{n,\varrho,\nu} \frac{\Gamma(2n-2)}{\Gamma(2n-\frac{1}{2})} \int_{-1}^{-\varepsilon} |\varphi(1+a)|\,(1+a)^{n+\frac{1}{2}-\nu} \frac{[2+a+2(2n-1)\sqrt{1+a}\,]}{(1+\sqrt{1+a}\,)^{4n-2}}\, da$$

$$(6.10)\quad \leq C_1 \frac{(2n-1)\,\Gamma(2n+\varrho)\,\Gamma(2n+\nu)\,\Gamma(2n-2)}{\Gamma(n+1)\,\Gamma(n+\nu-1)\,\Gamma(n+1+1-\nu)\,\Gamma(n+\varrho-1+\nu-1)\,\Gamma(2n-\frac{1}{2})} \left[\frac{1-\varepsilon}{(1+\sqrt{1-\varepsilon}\,)^4}\right]^{n+1-\nu-\lambda}$$

where we have used that

$$\frac{1+a}{(1+\sqrt{1+a}\,)^4}$$

increases in the interval $(-1,0)$. C_1 depends on ϱ, ν and φ.

Note that

$$\frac{2^4(1-\varepsilon)}{(1+\sqrt{1-\varepsilon}\,)^4} \leqslant (1-\delta) \quad , \quad \delta = \delta(\varepsilon) > 0$$

and using Stirling formula we can continue the estimate (6.10) and get

$$0 \leqslant \left| \int_0^{1-\varepsilon} \varphi(t)\, \bar{L}_{n,\varsigma,\nu}\, k(1,t;\varsigma,\nu)\, dt \right| \leqslant C_2\,(1-\delta)^{n+1-\nu-\lambda}$$

as $n \to \infty$.

Thus, the lemma is proved.

Lemma 4:

If $\varphi(t)$ is a function continuous in $[1,\infty)$ and there exists a real number λ such that

$$\int_1^\infty t^\lambda \varphi(t)\, dt \qquad \text{converges,}$$

then

$$\int_{1+\varepsilon}^\infty \varphi(t)\, \bar{L}_{n,\varsigma,\nu}\, K(1,t,\varsigma,\nu)\, dt$$

tends to zero if n goes to infinity.

Proof:

Replace t by $1/t$ in Lemma 3.

Lemma 5:

If $\varphi(t)$ is a function continuous on R_+ and there exist real numbers λ_1 and λ_2 such that for some a, b

$$\int_0^a t^{\lambda_1} \varphi(t)\, dt \quad \text{and} \quad \int_b^\infty t^{\lambda_2} \varphi(t)\, dt \qquad \text{converge,}$$

then

$$\int_0^\infty \varphi(t)\, \bar{L}_{n,\varsigma,\nu}\, K(1,t,\varsigma,\nu)\, dt$$

converges to $\varphi(1)$ if n goes to infinity.

Proof:

By (6.8) we have

$$\int_0^\infty \varphi(t)\, \bar{L}_{n,\varsigma,\nu}\, k(1,t;\varsigma,\nu)\, dt - \varphi(1) =$$

$$= \int_0^\infty (\varphi(t) - \varphi(1))\, \bar{L}_{n,\varsigma,\nu}\, k(1,t;\varsigma,\nu)\, dt \;.$$

Fix $\varepsilon > 0$ such that $|\varphi(t) - \varphi(1)| < \delta$ if only $|t-1| < \varepsilon$. Now decompose the integration into the integration over $0 \leqslant t \leqslant 1-\varepsilon$, $1-\varepsilon < t < 1+\varepsilon$, $1+\varepsilon \leqslant t < \infty$ and denote the corresponding integrals by I_1, I_2 and I_3 respectively. In view of Lemma 3 and Lemma 4 I_1 and I_3 tend to zero and for I_2 a simple computation leads to

$$|I_2| \leqslant \int_{1-\varepsilon}^{1+\varepsilon} |\varphi(t) - \varphi(1)|\, \bar{L}_{n,\varsigma,\nu}\, k(1,t;\varsigma,\nu)\, dt \leqslant$$

$$\leqslant \delta \int_0^\infty \bar{L}_{n,\varsigma,\nu}\, k(1,t;\varsigma,\nu)\, dt \leqslant \delta .$$

Thus, the lemma is proved.

Now we can prove the inversion theorem for the transform (1.3).

Theorem 3:

Suppose $\alpha > \max(0, \nu-1)$, $\Lambda < \nu + \min(0, \varsigma-1)$ and let $\phi \in \mathcal{M}_{\alpha,\Lambda}$. Then the sequence $\{L_{n,\varsigma,\nu}\, \mathcal{G}[\phi;\varsigma,\nu]\}$ converges in $\mathcal{M}_{\alpha,\Lambda}$ to ϕ .

Proof:

We have to prove that for $k \in Z_+$

$$\mu_{\alpha,\Lambda,k}\left(L_{n,\varsigma,\nu}\, \mathcal{G}[\phi,\varsigma,\nu] - \phi \right) \longrightarrow 0$$

as $n \to \infty$. Note that the sequence $\{L_{n,\varsigma,\nu}\, \mathcal{G}[\phi;\varsigma,\nu]\}$ tends to ϕ if and only if $\{\bar{L}_{n,\varsigma,\nu}\, \mathcal{G}[\phi;\varsigma,\nu]\}$ tends to ϕ so that it is sufficient to consider the sequence $\{\bar{L}_{n,\varsigma,\nu}\, \mathcal{G}[\phi;\varsigma,\nu]\}$. From (6.6) and (6.8) we have

$$L_{n,\varrho,\nu,x}\ \mathcal{S}[\phi;\varrho,\nu](x) - \phi(x) =$$

$$= \gamma_{n,\varrho,\nu} \int_0^\infty (\phi(t)-\phi(x)) \int_0^\infty \frac{x^{n+\varrho+\nu-2}\ y^{2n}\ t^n}{(x+y)^{2n+\varrho}\ (y+t)^{2n+\nu}}\, dy\, dt$$

or with $t = x\cdot u$

(6.11) $$\gamma_{n,\varrho,\nu} \int_0^\infty (\phi(u\cdot x)-\phi(x)) \int_0^\infty \frac{y^{2n}\ u^n}{(1+y)^{2n+\varrho}(y+u)^{2n+\nu}}\, dy\, du\ .$$

Put

$$R_u(x) = x\int_1^u \phi'(x\cdot q)\, dq = \phi(ux)-\phi(x)\ .$$

Then

$$R_u^{(k)}(x) = x\int_1^u \phi^{(k+1)}(x\cdot q)\, q^k\, dq + k\int_1^u \phi^{(k)}(q\cdot x)\, q^{k-1}\, dq$$

and by (2.1)

$$|R_u^{(k)}(x)| \le x^{\alpha-k-1}\left(\mu_{\alpha,\beta,k+1}(\phi) + k\mu_{\alpha,\beta,k}(\phi)\right)\left|\int_1^u q^{\alpha-2}(1+q\cdot x)^{\beta-\alpha}\, dq\right|.$$

Furthermore, from (3.6)

$$(1+qx)^{\beta-\alpha} \le (1+x)^{\beta-\alpha}\ \max(1, q^{\beta-\alpha})$$

for $0\le x, q<\infty$ and so we get

$$\mu_{\alpha,\beta,k}(R_u) \le \left(\mu_{\alpha,\beta,k+1}(\phi)+k\mu_{\alpha,\beta,k}(\phi)\right)\ \left|\int_1^u \max(q^\alpha, q^\beta)\, q^{-2} dq\right|.$$

Taking into account the last inequality and (6.11) we have

$$\mu_{\alpha,\beta,k}(L_{n,\varrho,\nu}\ \mathcal{S}[\phi;\varrho,\nu]-\phi) \le$$

$$\le \left(\mu_{\alpha,\beta,k+1}(\phi)+k\mu_{\alpha,\beta,k}(\phi)\right)\cdot$$

$$\cdot\, \gamma_{n,\varrho,\nu}\int_0^\infty\int_0^\infty \frac{y^{2n}\ u^n}{(1+y)^{2n+\varrho}}\frac{dy}{(y+u)^{2n+\nu}}\cdot\left|\int_1^u \max(q^\alpha,q^\beta)\, q^{-2}\, dq\right|.$$

The function $\psi(u)$ defined by

$$\psi(u) = \left|\int_1^u \max(q^\alpha, q^\beta)\, q^{-2} dq\right|$$

is continuous on R_+ , satisfies all conditions of Lemma 5 and vanishes at $u=1$. Consequentely

$$\gamma_{n,\varrho,\nu}\int_0^\infty \psi(u)\int_0^\infty \frac{y^{2n}\ u^n\ dy}{(1+y)^{2n+\varrho}(y+u)^{2n+\nu}}\, du \longrightarrow \psi(1) = 0$$

as $n\longrightarrow\infty$. Thus

$$\mu_{\alpha,\beta,k}\left(L_{n,\varrho,\nu}\ \mathcal{S}[\phi;\varrho,\nu]-\phi\right) \longrightarrow 0$$

as $n\longrightarrow\infty$ and the theorem is proved.

<u>Lemma 6:</u>
Suppose

(6.12) $\gamma > 1-\varrho+\max(0, 1-\nu)$, $\delta < 1+\min(0, 1-\varrho)$

and let $\phi\in\mathcal{M}_{\gamma,\delta}$. Then the following commutation relation holds for the operator $L_{n,\varrho,\nu}$

(6.13) $$L_{n,\varrho,\nu,x}\ \mathcal{S}[t^{\varrho+\nu-2}\phi(t);\varrho,\nu](x) = x^{\varrho+\nu-2}\,\mathcal{S}[L_{n,\varrho,\nu,t}\phi(t);\varrho,\nu](x)$$

<u>Proof:</u>
If γ and δ satisfy (6.12), then $\alpha=\gamma+\varrho+\nu-2$ and $\beta=\delta+\varrho+\nu-2$ satisfy (5.1) so that the S_2-transform of $t^{\varrho+\nu-2}\phi(t)$ and of $L_{n,\varrho,\nu,t}\phi(t)$ are well defined.
To prove equality (6.13), we consider the right-hand side of (6.13) and substitute t by $x\cdot u$. Using the homogeneity of the kernel $k(x,t;\varrho,\nu)$ we have

$$x^{\varrho+\nu-2}\int_0^\infty K(x,t,\varrho,\nu)\ (L_{n,\varrho,\nu,t}\ \phi(t))\ dt$$

$$= \int_0^\infty K(1,u;\varrho,\nu)\ (L_{n,\varrho,\nu,x\cdot u}\ \phi(x\cdot u))\ du$$

$$= \int_0^\infty K(1,u;\varrho,\nu)\ u^{\varrho+\nu-2}\ (L_{n,\varrho,\nu,x}\ \phi(x\cdot u))\ du$$

$$= L_{n,\varrho,\nu,x}\int_0^\infty K(1,u;\varrho,\nu)\ u^{\varrho+\nu-2}\ \phi(x\cdot u)\, du$$

Replace ux by t, the last integral gets to

$$L_{n,\rho,\nu,x}\int_0^\infty K(x,t;\rho,\nu)\,t^{\rho+\nu-2}\,dt$$

and equality (6.13) is proved.

Let γ,δ satisfy condition (6.12). Then from Theorem 3 and Lemma 6 we see that the sequence $\{\mathcal{S}[L_{n,\rho,\nu}\phi;\rho,\nu]\}$ converges in $\mathcal{M}_{\gamma,\delta}$ to ϕ.

Now the results on the inversion of the generalized S_2-transform (in the sense of (1.7)) can be summarized as follows:

Theorem 4:
Suppose $\alpha > \max(0,\nu-1)$, $\alpha \geq a$, $\beta < \nu+\min(0,\rho-1)$, $\beta \leq b$ and let $\phi \in \mathcal{M}_{\alpha,\beta}$ and $f \in \mathcal{M}'_{a,b}$. Then

i) $\langle \mathcal{S}[L_{n,\nu,\rho} f;\nu,\rho],\phi\rangle \longrightarrow \langle f,\phi\rangle$ as $n\to\infty$

if

$a < \nu+\min(0,\rho-1)$ and $a < \nu$ if $\rho = 1$

$b \geq \max(0,\nu-1)$ and $b > 0$ if $\nu = 1$

ii) $\langle L_{n,\nu,\rho}\,\mathcal{S}[f;\nu,\rho],\phi\rangle \longrightarrow \langle f,\phi\rangle$ as $n\to\infty$

if

- $a \leq 1+\min(0,1-\rho)$ and $a < 1$ if $\rho = 1$
- $b \geq 1-\rho+\max(0,1-\nu)$ and $b > 1-\rho$ if $\nu = 1$

Proof:
We prove only the first part. Under the conditions mentioned above we have $L_{n,\nu,\rho} f \in \mathcal{M}'_{a+\rho+\nu-2,\,b+\rho+\nu-2}$ and $\mathcal{S}[L_{n,\nu,\rho} f;\nu,\rho] \in \mathcal{M}'_{\alpha,\beta}$ so that

$$\langle \mathcal{S}[L_{n,\nu,\rho} f,\nu,\rho],\phi\rangle = \langle L_{n,\nu,\rho} f,\ \mathcal{S}[\phi;\rho,\nu]\rangle$$

$$= \langle f,\ L_{n,\rho,\nu}\mathcal{S}[\phi,\rho,\nu]\rangle .$$

Finally, using Theorem 3 we obtain the first part of Theorem 4.

Remark:
An inversion theorem for the S_2-transform of distributions defined by (1.5) can be proved by following the lines of /4, Theorem 3/, where as sequence of differential operators a sequence similar to (6.2) must be elected.

7. Examples

i) Let $\rho=\nu=1$. Then

$$K(x,t;\rho,\nu) = \begin{cases} \dfrac{\ln x - \ln t}{x-t} & t \neq x \\[2ex] \dfrac{1}{x} & t = x \end{cases}.$$

We get the classical S_2-transform considered by Boas and Widder /2/.

(7.1) $$\mathcal{S}[\phi;1,1](z) = \int_0^\infty \frac{\ln z-\ln t}{z-t}\,\phi(t)\,dt.$$

If $z = x > 0$, then with $t = x\cdot y$ (7.1) gets to

(7.2) $$\mathcal{S}[\phi;1,1](x) = \int_0^\infty \frac{\ln y}{1-y}\,\phi(x\cdot y)\,dy.$$

ii) Let $\rho=\nu=2$, $0<\rho<1$. Then by formula 2.2.5.19 from /8/ we have

$$K(x,t;\rho,2-\rho) = \begin{cases} \dfrac{1}{1-\rho}\,\dfrac{x^{1-\rho}-t^{1-\rho}}{(x-t)\,t^{1-\rho}} & t \neq x \\[2ex] \dfrac{1}{x} & t = x \end{cases}$$

so that

(7.3) $$\mathcal{S}[\phi;\rho,2-\rho](z) = \frac{1}{1-\rho}\int_0^\infty \frac{z^{1-\rho}-t^{1-\rho}}{(z-t)\,t^{1-\rho}}\,\phi(t)\,dt$$

and if $z = x > 0$, then with $t = x\cdot y$ (7.3) gets to

(7.4) $$\mathcal{S}[\phi;\rho,2-\rho](z) = \frac{1}{1-\rho}\int_0^\infty \frac{1-y^{1-\rho}}{(1-y)\,y^{1-\rho}}\,\phi(xy)\,dy.$$

Acknowledgements
The author would like to thank Professors Ju.N.Droshinov and B.I. Zavialov for fruitful discussions.

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Received by Publishing Department
on December 23, 1986.

Трегер Г. E5-86-834
Об итерированном преобразовании Стильтеса для обобщенных функций

Описывается итерированное преобразование Стильтеса для обобщенных функций с помощью техники дуального преобразования. Показывается, что обратный оператор есть линейный дифференциальный оператор бесконечного порядка.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Tröger G. E5-86-834
On the Iterated Stieltjes Transform of Generalized Functions

The iterated Stieltjes transform is discussed using the method of adjoint mapping. It is proved that the inversion operator is a linear differential operator of infinite order.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1986