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THREE-DIMENSIONAL TOPOLOGICAL SOLITONS IN THE LATTICE MODEL OF A MAGNETIC WITH COMPETING INTERACTIONS

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The question of the existence of three dimensional (D=3) stable localized distributions (solitons) of the N-component unit vector $\vec{s}$, $\sum_{a=1}^{N} s_a^2 = 1$, is important when one investigates He^3 superfluid phases [1], models of magnetics [2], and extended models of elementary particles [3,4]. There are much more possibilities for the existence of stable 3D solitons in models possessing topological charges (e.g., when N=3,4) [5]. But up to now topological solitons have not been obtained in the important case D=3, N=3 [1,6].

We stress that the presence of the topological charge Q_t does not yet guarantee the soliton existence, and in the case of its existence it does not guarantee its stability. For example, in the continuous model with the Hamiltonian density:

$$E = \int \mathcal{H}\, d^3x \,, \qquad \mathcal{H} = \frac{1}{2}\left(\frac{\partial \vec{s}}{\partial x_m}\right)^2, \tag{1}$$

$$\vec{s} = (s_1, s_2, s_3)\,, \qquad \vec{s}^{\,2} = 1\,, \qquad m = 1, 2, 3.$$

localized distributions with an arbitrary Q_t corresponding to the energy E minima cannot exist; it can be easily demonstrated by scaling transformations [7]. Further we note that in the isotropic Heisenberg model with the same density $\mathcal{H}$, the presence of the integral of motion [2]

$$S_3 = \int (1 - s_3)\, d^3x \tag{2}$$

does not guarantee the soliton existence and stability (see [8] and papers cited therein), because some part of the invariant S_3 can be carried away by outgoing waves at the soliton destruction.

Now we consider a spin system with the Hamiltonian density

$$\mathcal{H}=\frac{1}{2}\left[\alpha\left(\frac{\partial\vec{s}}{\partial x_m}\right)^2+\beta\left(\frac{\partial^2\vec{s}}{\partial x_m^2}\right)^2\right],\quad \alpha,\beta>0,\quad m=1,2,3\ . \qquad (3)$$

Making the scaling transformation $\vec{s}(\vec{x}) \to \vec{s}(a\vec{x})$, we see that stable 3D solitons can exist in this model (3)

Now we pass from the continuous model (3) to a discrete system of spins located at cubic lattice sites, we label the sites with the integer numbers n_m for the x_m axes correspondingly. Let the distance a between the neighbour sites be unity. Substituting derivatives by finite differences, we get [8] the spin system with the interaction energy being calculated by summing over all lattice sites $M(\vec{n})$

$$E=\sum_M\Big[(3\alpha+21\beta)-\vec{s}_M\Big(C_n\sum_{m=1}^{3}\vec{s}_{\vec{n}\pm\vec{e}_m}+C_{nn}\sum_{m=1}^{3}\vec{s}_{\vec{n}\pm2\vec{e}_m}+ \qquad (4)$$
$$+C_{nd}\sum_{\substack{k,m=1\\k<m}}^{3}\vec{s}_{\vec{n}\pm\vec{e}_k\pm\vec{e}_m}\Big)\Big],\quad C_n=0.5\alpha+6\beta,\quad C_{nn}=-0.5\beta,\quad C_{nd}=-\beta.$$

Here C_n, C_{nn}, C_{nd} are the coefficients characterizing interactions of $\vec{s}_M$ with the nearest neighbours in axis directions, with the next-to-nearest neighbours in the same directions and with the nearest neighbours in the Mx_mx_k plane diagonal directions correspondingly; $\vec{e}_m$ is the unit vector in the positive direction of the x_m axis. Note that numerical relation of these coefficients corresponds to a fast interaction intensity decrease with the distance between lattice sites and the nearest-neighbour (n) interactions are ferromagnetic ones (FM), but nn and nd interactions are of antiferromagnetic (AFM) type. So one can regard the system (4) as a model of a magnetic with the competing FM and AFM interactions. When all the spins are parallel to each other, the energy E equals zero, it corresponds to the absolute energy minimum in the $Q_t=0$ sector. The existence of solitons with $E>0$ in the same sector cannot be excluded.

For continuous distributions at D=3, N=3 the intrinsic topological charge Q_t is the Hopf index H [1,5]. We have elaborated a method to compute the analogous lattice quantity, i.e., the topological charge H of spin configurations defined at the cubic lattice (see a detailed description in [8]); this method makes it possible to compute the charge H_ℓ even in the case of sharp variations of $\vec{s}$ at distances of the order of an elementary lattice link.

To obtain a topological soliton on the 3D lattice we assign an initial lattice spin configuration with the wanted topological charge H_ℓ (in the paper we investigate only the $H_\ell=1$ sector). Then we change spins at all lattice sites successfully to minimize the interaction energy of the spin under consideration with neighbour site spins (see Eq. (4)). Such a successful updating of all lattice spins is called an iteration. In the course of such a minimization (obviously, monotonic one) of the energy E the lattice configuration charge H_ℓ can change its value (unlike the continuous distributions) but under certain conditions it can be conserved up to the formation of the soliton spin distribution corresponding to the energy E minimum at a given H_ℓ value.

We use the following boundary conditions: $\vec{s}=\vec{e}_z=(0,0,1)$ outside the finite lattice region used in computer experiments. To raise the computer experiment efficiency we made computations using only a quarter of the lattice region containing a localized spin distribution. It is located inside a dihedral angle between the Oxz and Oyz planes containing the symmetry axis Oz. At all sites of the Oz axis we assign $\vec{s}=\vec{e}_z$ ($x=x_1$, $y=x_2$, $z=x_3$). The lattice quarter sizes are as follows: $L_x=14$, $L_y=15$, $L_z=19$. We consider two variants of spin configuration symmetries with respect to the 90° rotation around the Oz axis:

A) $s_x \to s_y$, $\quad s_y \to -s_x$, $\quad s_z \to s_z$,

B) $s_x \to -s_y$, $\quad s_y \to s_x$, $\quad s_z \to s_z$.

At first we chose the symmetry A, and the initial distribution was taken in the form (κ = 0.3):

$$\begin{gathered} S_x = 2(N_1 N_4 + N_2 N_3), \qquad S_y = 2(N_1 N_3 - N_2 N_4), \\ S_z = N_3^2 + N_4^2 - N_1^2 - N_2^2, \qquad N_i = \frac{2x_i}{1+z^2}, \quad i = 1,2,3, \\ N_4 = \frac{1 - z^2}{1 + z^2}, \quad z^2 = \sum_{i=1}^{3} x_i^2, \quad x_i = \kappa n_i . \end{gathered} \tag{5}$$

One can easily estimate the soliton size in the continuous model (3): $R_{sol} \sim \sqrt{\beta \alpha^{-1}}$ Computations have been made at α =2 and two different β values: β = 1,5 and β = 10. Final results of the configuration (5) evolution at these two β differ qualitatively. At β = 1,5 the value H_ℓ = 1 survives during exactly 199 iterations, beginning from the 200-th iteration H_ℓ = 0. The energy E dependence on the iteration number is presented in Fig. 1. Beginning from the 200-th iteration the spin system under investigation rapidly evolves to the uniform distribution (all $\vec{S}_M = \vec{e}_z$). Such a charge H_ℓ non-conservation is the consequence of the fact that at β = 1,5 we have an estimate $R_{sol} \lesssim 1$, and hence this case differs essentially from the continuous one.

At β = 10 the discreteness of the spin system is less essential, because in this case R_{sol} is $\sqrt{20/3} \approx 2{,}6$ times larger than at β = 1,5. The charge H_ℓ = 1 conserves its value during the whole experiment (2650 iterations), as a result the integral of energy tends to a constant value $E = E_1$ = 679 (the relative E variation per one iteration is less than $2{,}5 \cdot 10^{-5}$), and so does a mean value $\langle S_z \rangle$, it becomes $\langle S_z \rangle$ = 0,827. The spins of the soliton distribution obtained are plotted in Figs. 2a, 2b. In Fig. 2a the S_x and S_z components of the spins located at the Oxz plane sites are represented (the crosses correspond to $S_y \geq 0$, the dots mean $S_y < 0$). In Fig. 2b the S_x and S_y components of the spins located at the middle Oxy plane sites are plotted (the crosses mean $S_z \geq 0$, the dots signify $S_z < 0$). This soliton configuration has not any symmetry with respect to the Oxy plane (see Fig. 2a). When $\rho = \sqrt{x^2 + y^2}$ increases, the spin distribution picture tends to the axially symmetric one (Fig. 2b). The comparison of the results at β = 1,5 and β = 10 demonstrates that at β = 0, i.e., in the lattice 3D ferromagnetic Heisenberg model there are no soliton solutions. Solitons arise only if there are interactions with not only nearest neighbours, for example the nn - and nd - interactions, here we have demonstrated this fact for the case of competing FM and AFM spin interactions. when $C_n > 0$, $C_{nn} < 0$, $C_{nd} < 0$.

The soliton solution found above is not unique even at H_ℓ = 1. The second localized solution has been obtained [9] assuming the symmetry B and using the initial data (5), κ = 0,24. This soliton possesses the symmetry with respect to the Oxy plane: when one makes the change $z \to -z$ in the Oxz plane, then $S_x \to S_x$, $S_z \to S_z$, $S_y \to -S_y$. The Oz axis is the second order axis for this soliton. Its energy E_2 = 686 does not differ strongly from E_1.

Making the transformation $S_x \to S_x$, $S_y \to -S_y$, $S_z \to S_z$ of the first soliton quarter and completing this quarter to the full configuration by assuming the symmetry A, we get the third soliton solution with $E_3 = E_2$. This soliton has the fourth order axis Oz. The charge H_ℓ of this soliton equals to -1; so to obtain the third soliton with H_ℓ = 1 it is sufficient to exchange spins in every pair of the lattice sites located symmetrically with respect to the Oxy plane.

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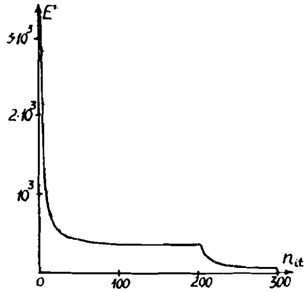

Fig. 1.

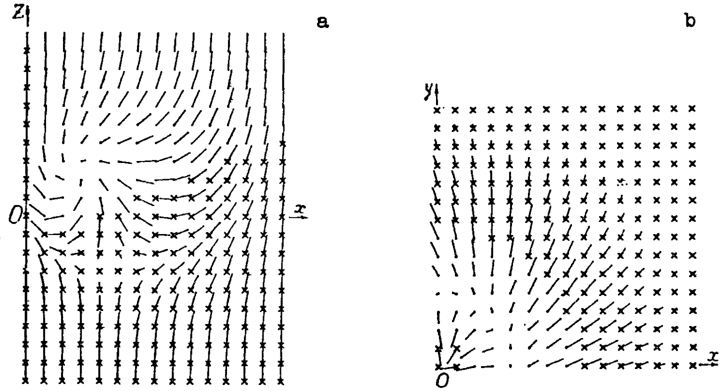

Fig. 2.

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Боголюбский И.Л. E5-86-687

Трехмерные топологические солитоны в решеточной модели магнетика с конкурирующими взаимодействиями

Изучаются топологические солитоны векторного поля $\vec{s}(\vec{x})$, $s_a s_a = 1$, $a = 1,2,3$ на трехмерной кубической решетке. Используется метод последовательной локальной минимизации функционала энергии E. В исследуемой модели имеют место конкурирующие ферромагнитные и антиферромагнитные взаимодействия спинов, расположенных в узлах решетки. Обнаружено, что решеточный топологический заряд (индекс Хопфа H_ℓ) в процессе минимизации E может изменяться, однако в случае его сохранения при $H_\ell \neq 0$ эволюция начальных распределений заканчивается формированием стабильных солитонов. Найдено несколько солитонных распределений.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Bogolubsky I.L. E5-86-687

Three-Dimensional Topological Solitons in the Lattice Model of a Magnetic with Competing Interactions

Topological solitons of the vector field $\vec{s}(\vec{x})$, $s_a s_a = 1$, $a = 1,2,3$ defined on the three-dimensional lattice are investigated. The method of successful local minimization of the energy E functional is used. In the model under investigation there are competing ferromagnetic and antiferromagnetic interactions of spin, localized in the cubic lattice sites. It has been obtained that the lattice topological charge (Hopf index H_ℓ) may change its value during the minimization process,but in the case of its conservation at $H_\ell \neq 0$, the evolution of initial spin distributions results in formation of stable solitons. Several soliton distributions have been found.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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