сообщения
объединенного
института
ядерных
исследований
дубна

E5-86-662

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A NEW CLASS OF REALISATIONS OF THE LIE ALGEBRA so(q, 2n-q)

1986

1. Introduction

1.1 Canonical (boson) realizations of Lie algebras are used for studying physical systems with symmetries in the framework of the canonical formalism [1]. They are especially useful in connection with the method of collective variables, e.g. in nuclear physics [2]. Moreover, they play a role in purely mathematical investigations (e.g., in connection with the models for su(3) in terms of so(n,2) and so (2n) algebras [3]).

1.2 In our recent paper [4], the method of constructing realizations for an arbitrary real semisimple algebra g was presented. It was shown that any induced representation can be rewritten as the so-called boson representation. The construction starts from a decomposition $g = n_+^b \oplus g_0^b \oplus n_-^b$ of g, which is a simple generalization of the triangle decomposition [5]; it employs substantially induced representations [6] of g with respect to a suitable representation σ of the subalgebra $g_0^b \oplus n_-^b$. It was proved in Ref. 4 that the method gives realizations which posses two properties permitting their application in the representation theory. They are skew-Hermitean and Shurean.

1.3 In the papers [7-8] we have applied this method for the Lie algebras gl(n+1,R) and sp(n,R). In the case of the algebras gl(n+1,R) we have constructed recurrent formulae which give realizations of gl(n+1,R) in terms of r(n+1-r). Canonical pairs and generators of the subalgebra gl(r,R)$\oplus$gl(n+1-r,R) for r=1,2,...,n. For the sp(n,R) we have obtained recurrent formulae in terms of $r(2n - \frac{3}{2}r + \frac{1}{2})$ canonical pairs and generators of the subalgebra gl(r,R)$\oplus$sp(n-r,R).

1.4 In the present paper, we apply the method of Ref. 4 to the case of algebras so(q,2n-q) which are the real forms of the complex algebras so(2n,$\mathbb{C}$). For the construction we use the explicit forms of the triangle decompositions of this real algebras which we have constructed in the paper /9/. We obtain recurrent formulae which give realizations of so(q,2n-q) in terms of 2n-1 canonical pairs and generators of the subalgebra $gl(1,\mathbb{R}) \oplus so(q-1,2n-q-1)$. The resulting realizations are Schurean and skew-Hermitian. The calculation can be easily adapt for the algebras so(q,2n+1-q) too.

1.5 The paper is organised as follows. All necessary prerequisites are listed in Sect. 2. The Sect. 3 contains the main results. Here the new wide families of realizations are derived. In the last section the results are discussed, and in particular, a detailed comparison with the realizations which were derived in papers /10,11/.

2. Preliminaries

2.1 The Weyl algebra W_{2N} is the associative algebra over with identity generated by 2N elements p_i, q_i where i=1,2,...,N, which satisfy the relations

$$[p_i, q_j] = \delta_{i,j} \quad , \quad [p_i, p_j] = [q_i, q_j] = 0 \qquad (1)$$

for any i,j = 1,2,...,N.

2.2 Let g, g_o are real Lie algebras. By $\tilde{g}$, $\tilde{g}_o$ we denote their complexifications, furthermore, $U(\tilde{g})$, $U(\tilde{g}_o)$ are the enveloping algebras of these complexifications.

Definition: A realization of a Lie algebra g is a homomorphism τ

$$\tau : g \rightarrow W_{2N} \otimes U(\tilde{g}_o) \ . \qquad (2)$$

2.3 The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2N} \otimes U(g_o)$.

Definition: Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realization τ is called Schurean or Schur-realization if all central elements $C \in Z(g)$ are realised by $1 \otimes C_o$ where the C_o's are central elements of the enveloping algebra $U(\tilde{g}_o)$.

2.4 In view of possible applications to the representation theory we introduce the involution "+" in W_{2N} by means of the following relations

$$q_i^+ = -q_i$$
$$p_i^+ = p_i \qquad \text{for } i=1,2,...,N. \qquad (3a)$$

Similarly, the involution "+" on $U(\tilde{g}_o)$ is defined by

$$Y^+ = -Y \qquad \text{for } Y \in g_o \ . \qquad (3b)$$

These involutions define naturally an involution on W_{2N} $U(g_o)$:

$$\left(\sum_j \alpha_j \Pi_j \otimes g_j \right)^+ = \sum_j \bar{\alpha}_j \Pi_j^+ \otimes g_j^+ , \qquad (3c)$$

where $\Pi_j \in W_{2N}$ and $g_j \in U(\tilde{g}_o)$.

Definition: Let g be a real Lie algebra and let "+" be the involution on $W_{2N} \otimes U(\tilde{g}_o)$ described above. A realization τ of g on $W_{2N} \otimes U(\tilde{g}_o)$ is called skew-Hermitean, if for all elements $X \in g$ the following relations hold

$$(\tau(X))^+ = -\tau(X). \qquad (4)$$

2.5 The algebra so(2n,$\mathbb{C}$) is the n(2n-1) - dimensional complex Lie algebra with the standard basis L_{ij}; i,j= ±1, ±2,..., ±n the elements of which obey:

$$L_{ij} = -L_{-j,-i} \qquad (5)$$

and the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{il} L_{kj} - \delta_{j,-l} L_{i,-k} + \delta_{i,-k} L_{-l,j} \ . \qquad (6)$$

2.6 In our paper /9/ we have specified an explicit form of the automorphisms which give the real forms of this algebra. Using these automorphisms we obtain for the algebras so(q,2n-q) the following bases:

$$\begin{aligned} &L_{st} \\ &X_{\alpha\beta} = (L_{\alpha\beta} - L_{\beta\alpha}) \\ &Y_{\alpha\beta} = i(L_{\alpha\beta} + L_{\beta\alpha}) \\ &Y_{s\alpha} = (L_{s\alpha} + L_{s,-\alpha}) \\ &Y_{s\alpha} = i(L_{s\alpha} - L_{s,-\alpha}), \end{aligned} \qquad (7)$$

where s,t = ±1, ±2,..., ±q and α,β = ±(q+1), ±(q+2),..., ±n.

The commutation relations in this basis are introduced in Appendix A.

2.7 For $b=L_{11}$ we define a decomposition of algebra so(q,2n-q) in this way:

$$
\begin{aligned}
g &= n_+^b \oplus g_0^b \oplus n_-^b \\
n_+^b &= \mathbb{R}\left\{ X \in g;\ [b,X] = \alpha_X X \quad \text{where } \alpha_X > 0 \right\} \\
g_0^b &= \mathbb{R}\left\{ X \in g;\ [b,X] = 0 \right\} \qquad\qquad (8) \\
n_-^b &= \mathbb{R}\left\{ X \in g,\ [b,X] = -\alpha_X X \quad \text{where } \alpha_X > 0 \right\}.
\end{aligned}
$$

This decompositions we use as a starting point for our construction (see also Ref. 4 Sec. 4).

3. Construction of realizations

3.1 Using the commutation relations (see Appendix A) we can bring the decomposition (8) into the form:

$$
\begin{aligned}
n_+^b &= \mathbb{R}\left\{ L_{1i},\ X_{1\alpha},\ Y_{1\alpha} \right\} \\
g_0^b &= \mathbb{R}\left\{ L_{11}, L_{ij}, X_{\alpha\beta}, Y_{\alpha\beta},\ X_{i\alpha},\ Y_{i\alpha} \right\} \qquad\qquad (9) \\
n_-^b &= \mathbb{R}\left\{ L_{i1},\ X_{\alpha 1},\ Y_{\alpha 1} \right\},
\end{aligned}
$$

where again $i,j = \pm 2, \pm 3, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \pm(q+2), \ldots, \pm n$. The relation (5) implies that the basis in $\tilde{n}_+^b$ forms the following (2n-2)-elements: .

$$
\left|
\begin{array}{llll}
L_{12}, & L_{13}, & \ldots, & L_{1q} \\
L_{1,-2}, & L_{1,-3}, & \ldots, & L_{1,-q} \\
X_{1,q+1}, & X_{1,q+2}, & \ldots, & X_{1,n} \\
Y_{1,q+1}, & Y_{1,q+2}, & \ldots, & Y_{1,n}
\end{array}
\right| . \qquad\qquad (10)
$$

We introduce an ordering in the above basis in which its elements are ordered lexicographically. The monomials of $U(\tilde{n}_+^b)$ can be then written as the matrices

$$
\left|
\begin{array}{l}
n_2^L,\ n_3^L,\ \ldots, n_q^L \\
n_{-2}^L,\ n_{-3}^L,\ \ldots, n_{-q}^L \\
n_{q+1}^X, n_{q+2}^X, \ldots, n_n^X \\
n_{q+1}^Y, n_{q+2}^Y, \ldots, n_n^Y
\end{array}
\right|
= \left(L_{12}^{n_2^L}, \ldots, L_{1q}^{n_q^L}\right) \times \left(L_{1,-2}^{n_{-2}^L}, \ldots, L_{1,-q}^{n_{-q}^L}\right)
\left(X_{1,q+1}^{n_{q+1}^X}, \ldots, X_{1,n}^{n_n^X}\right)\left(Y_{1,q+1}^{n_{q+1}^Y}, \ldots, Y_{1n}^{n_n^Y}\right), \qquad (11)
$$

where of course n_i^L, n_α^X, n_α^Y belongs to N_0, the set of all non-negative integers.

3.2 Now we are able to apply the general construction described in Ref. 4. Let σ be an auxiliary representation of the algebra $g_0^b \oplus n_-^b$ on a vector space V such that

$$
\begin{aligned}
&\sigma(n_-^b) = 0 \\
&\sigma(g_0^b) \text{ is faithfull.} \qquad\qquad (12)
\end{aligned}
$$

We denote by W the carrier space of the induced representation $\rho = \text{ind}\,(g,\sigma)$. If $v_1, \ldots, v_d$ is a basis in the space V, then the vectors

$$
\left|
\begin{array}{l}
n_2^L,\ \ldots,\ n_q^L \\
n_{-2}^L,\ \ldots,\ n_{-q}^L \\
n_{q+1}^X, \ldots,\ n_n^X \\
n_{q+1}^Y, \ldots,\ n_n^Y
\end{array}
\right| \otimes v_i \ . \qquad\qquad (13)
$$

3.3 We define the creation and annihilation operators $\bar{a}_\alpha^X$, a_α^X on the space W in the following way:

$$
\bar{a}_\alpha^X
\left|
\begin{array}{lll}
n_2^L, & \ldots, & n_q^L \\
n_{-2}^L, & \ldots, & n_{-q}^L \\
n_{q+1}^X, \ldots, n_\alpha^X & , \ldots, & n_n^X \\
n_{q+1}^Y, & \ldots, & n_n^Y
\end{array}
\right| \otimes v_i =
\left|
\begin{array}{lll}
n_2^L, & \ldots, & n_q^L \\
n_{-2}^L, & \ldots, & n_{-q}^L \\
n_{q+1}^X, \ldots, & n_\alpha^X + 1, \ldots, & n_n^X \\
n_{q+1}^V, & \ldots, & n_n^Y
\end{array}
\right| \otimes v_i , \qquad (14a)
$$

$$a^X_\alpha \left| \begin{matrix} n^L_2, & \dots, & n^L_{-q} \\ n^L_{-2}, & \dots, & n^L_{-q} \\ n^X_{q+1},\dots, & n^X_\alpha, & \dots,n^X_n \\ n^Y_{q+1}, & \dots, & n^Y_n \end{matrix} \right| \otimes v_i = n^X_\alpha \left| \begin{matrix} n^L_2, & \dots, & n^L_q \\ n^L_{-2}, & \dots, & n^L_{-q} \\ n^X_{q+1},\dots, & n^X_\alpha -1, & \dots,n^X_n \\ n^Y_{q+1}, & \dots, & n^Y_n \end{matrix} \right| \otimes v_i \qquad (14a)$$

and similarly we define the operators $\bar a^Y_\alpha$, a^Y_α, $\bar a^L_i$, a^L_i and $\bar a^L_{-i}$, a^L_{-i} for any $\alpha = (q+1),\dots,n$; $i=2,3,\dots,q$. For any $= (q+1),(q+2),\dots,$ we put

$$\bar a^X_{-\alpha} = \bar a^X_\alpha,\quad \bar a^Y_{-\alpha} = -\bar a^Y_\alpha ,\quad a^X_{-\alpha} = -a^X_\alpha \quad \text{and} \quad a^Y_{-\alpha} = -a^Y_\alpha .$$

Furthermore we define the operators $\tilde X$ for any $X \in g^b_0$ by the relation

$$\tilde X = 1 \otimes \sigma(X) . \qquad (14b)$$

3.4 According to theorem 3.6 of Ref. 4 the induced representation $\rho = \mathrm{ind}(g,\sigma)$ can be rewritten using the above defined operators (14a-b). We get the formulae

$$\rho(L_{11}) = \sum_{k=2}^{q} (\bar a^L_k a^L_k + \bar a^L_{-k} a^L_{-k}) + \sum_{\alpha=q+1}^{n} (\bar a^X_\alpha a^X_\alpha + a^Y_\alpha a^Y_\alpha) + L_{11}$$

$$\rho(L_{ij}) = -\bar a^L_j a^L_i + \bar a^L_{-i} a^L_{-j} + L_{ij}$$

$$\rho(X_{\alpha\beta}) = -\bar a^X_\beta a^X_\alpha + \bar a^X_\alpha a^X_\beta - \bar a^Y_\beta a^Y_\alpha + \bar a^Y_\alpha a^Y_\beta + X_{\alpha\beta} \qquad (15)$$

$$\rho(Y_{\alpha\beta}) = -\bar a^Y_\beta a^X_\alpha - \bar a^Y_\alpha a^X_\beta + \bar a^X_\beta a^Y_\alpha + \bar a^X_\alpha a^Y_\beta + Y_{\alpha\beta}$$

$$\rho(X_{i\alpha}) = -\bar a^X_\alpha a^L_i + 2\bar a^L_{-i} a^X_\alpha + X_{i\alpha}$$

$$\rho(Y_{i\alpha}) = -\bar a^Y_\alpha a^L_i + 2a^L_{-i} a^Y_\alpha + Y_{i\alpha}$$

where $i,j = \pm 2, \pm 3,\dots, \pm q$ and $\alpha,\beta = \pm(q+1),\dots, \pm n$ and further

$$\rho(L_{12}) = a^L_2$$

$$\rho(L_{21}) = -\rho(L_{11})a^L_2 + \sum_{k=2}^{q} \bar a^L_{-2} a^L_{-k} a^L_k + \sigma(L_{2k})a^L_k + \sigma(L_{2,-k})a^L_{-k} + \sum_{\alpha=q+1}^{n} \sigma(X_{2\alpha}) a^X_\alpha + \sigma(Y_{2\alpha}) a^Y_\alpha - \bar a^L_{-2}(a^X_\alpha a^X_\alpha + a^Y_\alpha a^Y_\alpha)$$

The representation of the remaining generators we obtain using the commutation rules (see Appendix A).

3.5 Now the skew-Hermitean realizations sought are obtained easily by replacing the operators in the above expressions by suitable algebraic objects.

$$\bar a \longrightarrow q \qquad a \longrightarrow p \qquad \sigma(X) \longrightarrow X \qquad (16)$$

For details, see Ref. 4 (Sect. 3.7-3.9). They are given by the formulae

$$\tau(L_{11}) = \sum_{k=2}^{q} (q^L_k p^L_k + q^L_{-k} p^L_{-k}) + \sum_{\alpha=q+1}^{n} q^X_\alpha p^X_\alpha + q^Y_\alpha p^Y_\alpha + L_{11} + (n-1)$$

$$\tau(L_{ij}) = -q^L_j p^L_i + q^L_{-i} p^L_{-j} + L_{ij}$$

$$\tau(X_{\alpha\beta}) = -q^X_\beta p^X_\alpha + q^X_\alpha p^X_\beta - q^Y_\beta p^Y_\alpha + q^Y_\alpha p^Y_\beta + X_{\alpha\beta} \qquad (17)$$

$$\tau(Y_{\alpha\beta}) = -q^Y_\beta p^X_\alpha - q^Y_\alpha p^X_\beta + q^X_\beta p^Y_\alpha + q^X_\alpha p^Y_\beta + Y_{\alpha\beta}$$

$$\tau(X_{i\alpha}) = -q^X_\alpha p^L_i + 2q^L_{-i} p^X_\alpha + X_{i\alpha}$$

$$\tau(Y_{i\alpha}) = -q^Y_\alpha p^L_i + 2q^L_{-i} p^Y_\alpha + Y_{i\alpha} ,$$

where $i,j = \pm 2, \pm 3,\dots, \pm q$; $\alpha,\beta = \pm(q+1),\pm(q+2),\dots, \pm n$ and further

$$\tau(L_{12}) = q^L_2$$

$$\tau(L_{21}) = -\tau(L_{11})p^L_2 + \sum_{k=2}^{q} q^L_{-2} p^L_{-k} p^L_k + p^L_k L_{2k} + p^L_{-k} L_{2,-k} + \sum_{\alpha=q+1}^{n} (X_{2\alpha}) p^X_\alpha + (Y_{2\alpha}) p^Y_\alpha - q^L_{-2}(p^X_\alpha p^X_\alpha + p^Y_\alpha p^Y_\alpha) .$$

The element $b=L_{11}$ has the same meaning as the element b from Ref.4. Therefore, we can apply theorem 4.3 of that paper to the realizations (17) thus obtaining the following proposition.

Proposition. τ are Schur-realizations of $so(q,2n-q)$ in the $W_{2N} \otimes U(gl(1,R) \oplus so(q-1,2n-q-1))$.

4. Discussion

Explicit forms of realizations for so(n,2) have been constructed by Le Blanc and Rowe [3] using the method of coherent state representation. These realizations are defined by means of n canonical pairs and generators of a subalgebra so(n)⊕so(2). Another class of realizations has been described by Havlíček and Exner [10]; in their paper, realizations of so(m,n) in terms of (m+n-2) - canonical pairs and generators of so(m-1,n-1)⊕gl(1,R). Also realization given in presented paper are similar to those given explicitly in the papers mentioned, but we cannot give explicitly the relation which transfer these realizations one another.

Appendix A.

Using the relations (6) we can compute commutation relations in the basis (7). In this appendix we give their explicit form:

$$[L_{ij},L_{kl}]=\delta_{jk}L_{il}-\delta_{il}L_{kj}-\delta_{j,-l}L_{i,-k}+\delta_{i,-k}L_{-l,j}$$

$$[L_{ij},X_{k\alpha}]=\delta_{jk}X_{i\alpha}-\delta_{i,-k}X_{-j,\alpha},\quad [L_{ij},X_{\alpha\beta}]=0$$

$$[L_{ij},Y_{k\alpha}]=\delta_{jk}Y_{i}+\delta_{i,-k}Y_{-j},\quad [L_{ij},Y_{\alpha\beta}]=0$$

$$[X_{i\alpha},X_{j\beta}]=-\delta_{i,-j}X_{\alpha\beta}+\delta_{i,-j}X_{\alpha,-\beta}-2(\delta_{\alpha\beta}+\delta_{\alpha,-\beta})L_{i,-j}$$

$$[X_{i\alpha},Y_{j\beta}]=-\delta_{i,-j}Y_{\alpha\beta}+\delta_{i,-j}Y_{\alpha,-\beta}$$

$$[Y_{i\alpha},Y_{j\beta}]=-\delta_{i,-j}X_{\alpha\beta}+\delta_{i,-j}X_{\alpha,-\beta}-2(\delta_{\alpha\beta}-\delta_{\alpha,-\beta})L_{i,-j}$$

$$[X_{i\alpha},X_{\beta\gamma}]=(\delta_{\alpha\beta}+\delta_{\alpha,-\beta})X_{i\gamma}-(\delta_{\alpha\gamma}+\delta_{\alpha,-\gamma})X_{i\beta}$$

$$[X_{i\alpha},Y_{\beta\gamma}]=(\delta_{\alpha\beta}+\delta_{\alpha,-\beta})Y_{i\gamma}+(\delta_{\alpha\gamma}+\delta_{-\alpha,\gamma})Y_{i\beta}$$

$$[Y_{i\alpha},Y_{\beta\gamma}]=-(\delta_{\alpha\beta}-\delta_{\alpha,-\beta})X_{i\gamma}-(\delta_{\alpha\gamma}-\delta_{\alpha,-\gamma})X_{i\beta}$$

$$[Y_{i\alpha},X_{\beta\gamma}]=(\delta_{\alpha\beta}-\delta_{\alpha,-\beta})Y_{i\gamma}-(\delta_{\alpha\gamma}-\delta_{\alpha,-\gamma})Y_{i\beta}$$

$$[X_{\alpha\beta},X_{\gamma\delta}]=\delta_{\beta\gamma}X_{\alpha\delta}-\delta_{\alpha\delta}X_{\gamma\beta}-\delta_{\beta,-\delta}X_{\alpha,-\gamma}+\delta_{\alpha,-\gamma}X_{-}\quad ,$$

$$[X_{\alpha\beta},Y_{\gamma\delta}]=\delta_{\beta\gamma}Y_{\alpha\delta}-\delta_{\alpha\delta}Y_{\gamma\beta}-\delta_{\beta,-\delta}Y_{\alpha,-\gamma}+\delta_{\alpha,-\gamma}X_{-\delta,\beta}$$

$$[Y_{\alpha\beta},Y_{\gamma\delta}]=-\delta_{\beta\gamma}X_{\alpha\delta}+\delta_{\alpha\delta}X_{\gamma\beta}+\delta_{\beta,-\delta}X_{\alpha,-\gamma}-\delta_{\alpha,-\gamma}X_{-\delta,\beta}\,,$$

where $i,j,k,l=\pm 1,\pm 2,\dots,\pm q$ and $\alpha,\beta,\gamma,\delta=\pm(q+1),\pm(q+2),\dots,\pm n$.

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Received by Publishing Department
on October 4, 1986.

Бурдик Ч. E5-86-662

Новый класс реализаций алгебр Ли so(q,2n-q)

В данной работе применяется метод [1] построения бозонных реализаций алгебр Ли so(q,2n-q). Эти реализации описываются рекуррентными формулами, содержащими (2n-2) бозонных пар и генераторов подалгебры gl(1,R)⊕so(q-1, 2n-q-1). Они антиэрмитовы и шуровские.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

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A New Class of Realizations of the Lie Algebra so(q,2n-q)

The method of Ref. [1] is applied to the construction of boson realizations for Lie algebtas so(q,2n-q), q=2,3,.. ,n. There realizations are expressed by means of certain recurrent formulae in terms of (2n-2) - boson pairs and generators of the subalgebra gl(1,R)⊕so(q-1, 2n-q-1). They are skew-Hermitean and Schurean.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986