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E5-86-649

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ON COMMUTATORS IN ALGEBRAS OF UNBOUNDED OPERATORS

Submitted to "Zeitschrift für Analysis
und Ihre Anwendungen"

1986

1. Introduction

The structure of commutators in algebras of bounded operators (especially in $\mathcal{B}(\mathcal{H})$ and von Neumann algebras) was investigated by many authors. Let us only remember the paper of Brown and Pearcy /?/ which can be regarded as some final step in clarifying the situation for $\mathcal{B}(\mathcal{H})$: an operator $A \in \mathcal{B}(\mathcal{H})$ is a commutator if and only if it is not of the form $A = \lambda I + C$, $\lambda \neq 0$ and C a compact operator ($\mathcal{H}$ - separable, infinite dimensional Hilbert space). In finite dimensional spaces one has the classical result: a quadratic matrix is a commutator if and only if it has trace zero.
Unbounded operators enter if one considers the CCR (cf./13/). To the author's knowledge up to now commutators in algebras of unbounded operators were not investigated.
The present paper should be regarded as a first step toward a systematic study of commutators in the context of topological algebras of unbounded operators. The aim is first of all to stimulate such investigations by presenting some conjectures and problems on the basis of the results obtained so far. We restrict ourselves to maximal Op^*-algebras $\mathcal{L}^+(\mathcal{D})$ defined on domains of the form $\mathcal{D} = \bigcap_{n \geq 0} \mathcal{D}(T^n)$ (cf. sect. 2). The paper is organized as follows. The second section contains the necessary notions, notations and preliminaries. Section 3 concerns diagonal and quasidiagonal operators. Here, the possibility of representing an operator as commutator is related with the structure of the domain $\mathcal{D}$ of the algebra. Several criteria are given which imply that such operators are commutators. In section 4 we consider finite dimensional operators. If $\mathcal{D}$ is not of type (I), any finite dimensional operator is a commutator. If $\mathcal{D}$ is of type (I), then it is proved that "enough" finite dimensional operators are commutators. Combining considerations of sections 3 and 4 one gets as a main result that for the domains under consideration the commutators are $\tau_{\mathcal{D}}$-dense in $\mathcal{L}^+(\mathcal{D})$. This is the analogous result to the bounded case. Section 5 contains some facts about selfcommutators. In section 6 we indicate connections with quasi-$*$-algebras and formulate some conjectures.

2. Preliminaries

For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathcal{H}$ the set $\mathcal{L}^+(\mathfrak{D}) = \{A: A\mathfrak{D} \subset \mathfrak{D}\ ,\ A^*\mathfrak{D} \subset \mathfrak{D}\}$ forms a $*$-algebra with respect to the usual operations and the involution $A \longrightarrow A^+ = A^* \upharpoonright \mathfrak{D}$. An Op^*-algebra $\mathcal{A}(\mathfrak{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$ containing the identity operator I. The graph topology t on $\mathfrak{D}$ induced by $\mathcal{L}^+(\mathfrak{D})$ is given by the family of seminorms

$$\mathfrak{D} \ni \varphi \longrightarrow \|A\varphi\| \qquad \text{for all } A \in \mathcal{L}^+(\mathfrak{D}).$$

Among the many possible topologies on $\mathcal{L}^+(\mathfrak{D})$ we mention only the so-called uniform topology $\tau_{\mathfrak{D}}$ /6/ given by the seminorms

$$A \longrightarrow \|A\|_{\mathfrak{M}} = \sup_{\varphi,\psi\in\mathfrak{M}} |\langle\varphi\ ,A\psi\rangle|,$$

where $\mathfrak{M}$ runs over all t-bounded subsets of $\mathfrak{D}$. The set

$$\mathcal{C}(\mathfrak{D}) = \{C \in \mathcal{L}^+(\mathfrak{D}) : C\mathfrak{M} \text{ is relatively t-compact for all t-bounded } \mathfrak{M} \subset \mathfrak{D}\}$$

is a two-sided $*$-ideal in $\mathcal{L}^+(\mathfrak{D})$. It appears that this set is a very appropriate generalization of the ideal of compact operators in $\mathcal{B}(\mathcal{H})$/5/,/10/. If $\mathfrak{D}[t]$ is an (F)-space then the $\tau_{\mathfrak{D}}$-closure of the set of finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$ coincides with $\mathcal{C}(\mathfrak{D})$. In this paper we consider only (F)-domains of the form

$$\mathfrak{D} = \mathfrak{D}^\infty(T) \equiv \bigcap_{n\geq 0} \mathfrak{D}(T^n),$$

where $T = T^* \geq I$ is a selfadjoint operator which can be supposed to have the following structure:

$$T\varphi_n = t_n\varphi_n \quad , n = 1,2,\dots\ , (\varphi_n) \text{ - orthonormal basis in } \mathcal{H}.$$

If necessary one can suppose $t_n \in \mathbb{N}$. Write $T \sim (t_n)$ or more exactly $T \sim (t_n),(\varphi_n)$. Further we use the following notations. Let $T = \int_1^\infty \lambda\, dE_\lambda$ be the spectral resolution of T. Then the operators $P_\mu = \int_1^\mu dE_\lambda$ belong to $\mathcal{L}^+(\mathfrak{D})$ for all $1 \leq \mu < \infty$ and $\mathcal{H}^{(\mu)} = P_\mu\mathcal{H} \subset \mathfrak{D}$. It will be frequently used that for all $A \in \mathcal{L}^+(\mathfrak{D})$: $A = \tau_{\mathfrak{D}}\text{-}\lim_{\mu\to\infty} P_\mu A P_\mu$.

We will make use of the classification of domains of closed operators and operator algebras given in /8/,/9/ and recall only some facts.

$\mathfrak{D}$ is of type (I) ($\mathfrak{D} \in$ (I)) if there is no infinite dimensional Hilbert space $\mathcal{H}_o \subset \mathfrak{D}$. This is equivalent to $\lim t_n = \infty$.

$\mathfrak{D}$ is of type (II) ($\mathfrak{D} \in$ (II)) if there is a splitting

$$\mathfrak{D} = \mathcal{H}_o \oplus \mathfrak{D}_o\ ,\quad \mathcal{H}_o\text{-infinite dimensional Hilbert space},\ \mathfrak{D}_o \in (I).$$

This is equivalent to a decomposition $(t_n) = (t_n^o) \cup (t_n^1)$, (t_n^o) - bounded infinite sequence, $\lim t_n^1 = \infty$, i.e., $\mathfrak{D} = \mathcal{H}_o \oplus \mathfrak{D}^\infty(T_1)$,

$T_1 \sim (t_n^1),(\psi_n)$ for some orthonormal basis in $\mathcal{H}_o^\perp$.

$\mathfrak{D}$ is of type (III) ($\mathfrak{D} \in$ (III)) if T has infinite many eigenvalues with infinite multiplicity $(t_n^\cdot)$, $\lim t_n^\cdot = \infty$. Therefore

$$\mathfrak{D} = \sum_{(t_n^\cdot)} \oplus \mathcal{H}_n \oplus \mathfrak{D}_o \quad \text{with } \mathfrak{D}_o \in (I) \text{ or } \mathfrak{D}_o = (0),\ T \upharpoonright \mathcal{H}_n = t_n^\cdot I, \text{i.e.}$$

$\mathcal{H}_n$ is the eigenspace of T corresponding to the eigenvalue $t_n^\cdot$. The sum above means:

$$\sum_{(t_n^\cdot)} \oplus \mathcal{H}_n = \{\psi = \sum \oplus \psi_n : \psi_n \in \mathcal{H}_n,\ \sum_n (t_n^\cdot)^{2k} \|\psi_n\|^2 < \infty \ \text{ for all } k \in \mathbb{N}\}$$

Especially, $\mathfrak{D} \in (III_A)$ if T can be chosen in such a way that $\mathfrak{D}_o = (0)$.

In section 3 we will use the fact that all of the $\mathcal{H}_n$, n=1,2,... can be identified (e.g. via some fixed isometric isomorphism) with some Hilbert space $\mathcal{K}$. Thus it makes sense to consider any ψ_k as an element of $\mathcal{H}_j$, too.

In what follows we will fix the orthonormal basis (φ_n) and if it is not indicated otherwise all constructions will be done with respect to this (φ_n). In general it is even necessary to fix also the ordering $\varphi_1, \varphi_2, \dots$ (i.e. (φ_n) and $(\varphi_{\pi(n)}) = (\psi_n)$ for some permutation π of $\mathbb{N}$ are in general not equivalent with respect to the constructions.). The following operators are mostly used:

diagonal operator	D_a :	$D_a\varphi_n = a_n\varphi_n$
right shift	R :	$R\varphi_n = \varphi_{n+1}$
weighted right shift	R_a:	$R_a\varphi_n = a_n\varphi_{n+1}$
left shift	L :	$L\varphi_n = \varphi_{n-1}$, $\varphi_o = 0$
weighted left shift	L_a:	$L_a\varphi_n = a_n\varphi_{n-1}$, $\varphi_o = 0$

As above, we often write $D_a \equiv D \sim (a_n)$. Here $a = (a_n)$ is a sequence of complex numbers.

A first point is to decide whether or not such an operator belongs to $\mathcal{L}^+(\mathfrak{D})$. It is easy to write down some formal conditions, namely:

$D_a \in \mathcal{L}^+(\mathfrak{D})$ if and only if $|a_n| \leq C\, t_n^r$ for all n and some $C,r > 0$ (1)

R_a resp. $L_a \in \mathcal{L}^+(\mathfrak{D})$ if and only if there are $C,r > 0$ so that

$$|a_n|\, t_{n+1} \leq C\, t_n^r \quad \text{resp.} \quad |a_n|\, t_{n-1} \leq C\, t_n^r \quad \text{for all } n. \qquad (2)$$

For reasons which will become clear a little bit later it is useful to introduce some more general notions.

Definition 2.1

i) A sequence of positive numbers (s_n) is said to be admissible for R_a, L_a resp. if there are $C,r > 0$ so that for all n

$$|a_n|\cdot s_{n+1} \le C s_n^r \quad \text{resp.} \quad |a_n| s_{n-1} \le C s_n^r . \qquad (2')$$

If $a_n = 1$ for all n we call (s_n) simply shift-admissible.

ii) Let (a_n), (t_n) be sequences with $a_n \in \mathbb{C}$, $t_n > 0$. (a_n) is said to be (t_n)-addable (or T-addable if $T \sim (t_n)$) if there are $C, r > 0$ with

$$|b_n| \le C t_n^r \quad \text{for all } n \in \mathbb{N} \text{ with } b_n = \sum_{j=1}^{n} a_j . \qquad (3)$$

Since most of the representations of operators as commutators use (explicitly or implicitly) the shift operator, it seems worthwhile to add some remarks. In general the estimations (2) are not very helpful to decide whether or not $R, L \in \mathcal{L}^+(\mathfrak{D})$. The reason is, roughly speaking, that in the sequence (t_n) the eigenvalues with infinite multiplicity can be arranged in a complicated manner (cf. the classification sketched above). So, it may happen that (t_n) is not shift-admissible, i.e. $R, L \notin \mathcal{L}^+(\mathfrak{D})$ but $(t_{\pi(n)})$ is shift admissible for some permutation π. Without proof we state some observations:

1. $\mathfrak{D} \in (I)$: Then $R, L \in \mathcal{L}^+(\mathfrak{D})$ means that (t_n) does not increase too fast. Note that $t_n \sim n^n$ or even $t_n \sim (n!)^n$ is not yet too fast.
2. $\mathfrak{D} \in (II)$: R, L are never in $\mathcal{L}^+(\mathfrak{D})$. More exactly, there is no permutation π of $\mathbb{N}$ so that $(t_{\pi(n)})$ is shift-admissible.
3. $\mathfrak{D} \in (III)$: Here one can state two results.
   i) If $(t_n^{\cdot})$ is shift-admissible, then automatically $\mathfrak{D} \in (III_A)$.
   ii) If $(t_n^{\cdot})$ is shift-admissible, then there is a π so that $(t_{\pi(n)})$ is also shift-admissible.

The next lemma states that the notions defined in Definition 2.1 are independent of the representing operator T if $\mathfrak{D} \in (I)$.

Lemma 2.2

Let $\mathfrak{D} = \mathfrak{D}^\infty(T) = \mathfrak{D}^\infty(S) \in (I)$, $T\varphi_n = t_n\varphi_n$, $S\psi_n = s_n\psi_n$ and let R_a, L_a $(R_a^{\cdot}, L_a^{\cdot})$ be the weighted shift operators corresponding to (φ_n) $((\psi_n))$. Then

i) $R_a, L_a \in \mathcal{L}^+(\mathfrak{D})$ if and only if $R_a^{\cdot}, L_a^{\cdot} \in \mathcal{L}^+(\mathfrak{D})$

ii) (a_n) is T-addable if and only if (a_n) is S-addable.

Proof:

In /11/ there was proved that for some $C, D, r > 0$ and for all n:

$$C\, s_n^{1/r} \le t_n \le D\, s_n^r .$$

Therefore, if an estimation of type (2) or (3) is valid for (t_n) it is also valid for (s_n) and vice versa (of course with other constants).

q.e.d

Most of our further considerations are based on matrix representations of operators. If not stated otherwise we always use the representation with respect to the canonical basis (φ_n) and write $A \sim (A_{mn})$ with $A_{mn} = \langle \varphi_m, A\varphi_n \rangle$.

3. Diagonal and quasidiagonal operators

In this section we demonstrate some typical features for commutators in $\mathcal{L}^+(\mathfrak{D})$. Therefore we don't start with the most general result (in this context) but prefer a more inductive representation of the results. We start with $\mathfrak{D} \in (I)$.

Let $D_a \in \mathcal{L}^+(\mathfrak{D})$, $a = (a_n)$. Then one has the formal relation

$$D_a = L_b R - R L_b \qquad (4)$$

with $b = (b_n)$ and $b_n = \sum_{j=1}^{n} a_j$.

Since we are interested in commutator representations within $\mathcal{L}^+(\mathfrak{D})$, only operators from $\mathcal{L}^+(\mathfrak{D})$ are allowed. If $R, L \in \mathcal{L}^+(\mathfrak{D})$, then $L_b = D_b.L$ and one has $L_b \in \mathcal{L}^+(\mathfrak{D})$ if and only if $D_b \in \mathcal{L}^+(\mathfrak{D})$.

Lemma 3.1

Let $\mathfrak{D} \in (I)$, $R, L \in \mathcal{L}^+(\mathfrak{D})$ and D_a a diagonal operator with T-addable sequence $a = (a_n)$. Then D_a is a commutator, given by (4).

Remark 3.2

If one uses representation (4), one has, so to say, two contrary restrictions. The first one is $R, L \in \mathcal{L}^+(\mathfrak{D})$, i.e. the (t_n) should not increase too wild. On the other hand, if (t_n) increases "slowly", then it will happen, that some (or even many) diagonal operators $D_a \in \mathcal{L}^+(\mathfrak{D})$ will not have a T-addable diagonal sequence a. For example, if $T \sim (\log(n+1))$, then this sequence itself is not T-addable. Thus, it does not wonder, that there is an optimal case (actually, many such cases), i.e. an optimal growth of (t_n).

Lemma 3.3

Each of the following conditions implies that any diagonal operator $D_a \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

i) $\mathfrak{D}$ is isomorphic to the sequence space s (the Schwartz space of rapidly decreasing sequences).

ii) $\mathfrak{D}$ is isomorphic to a sequence space contained in s and $R, L \in \mathcal{L}^+(\mathfrak{D})$.

Proof:

i) The proof is simple. Without loss of generality we may suppose that $\mathfrak{D} = \mathfrak{D}^\infty(T)$ with $t_n = n$. $D_a \in \mathcal{L}^+(\mathfrak{D})$ means $|a_n| \le C n^r$

for some $C, r > 0$. Then (a_n) is T-addable because of the estimation

$$|b_n| = |\sum_{j=1}^{n} a_j| \leq \sum_{j=1}^{n} |a_j| \leq C \sum_{j=1}^{n} j^r \leq C\, n^{r+1} .$$

Clearly, $R, L \in \mathcal{L}^+(\mathfrak{D})$, and so the proof is complete.

ii) Here one combines an analogous estimation as in i) with the estimation $n \leq C\, t_n^s$ (for some $C, s > 0$) , where $T \sim (t_n)$, $\mathfrak{D}^\infty(T) \subseteq s$ (cf. /8/ and /9/). This gives the T-addability of (a_n). So the assertion follows from $R, L \in \mathcal{L}^+(\mathfrak{D})$ and (4).

q.e.d.

Remark that in the proof of Lemma 3.2 it is implicitly used that both $R, L \in \mathcal{L}^+(\mathfrak{D})$ and the T-addability of (a_n) does not depend on the concrete representation $\mathfrak{D} = \mathfrak{D}^\infty(T)$. As mentioned in section 2 this is true only for $\mathfrak{D} \in (I)$.

Clearly, the representation of an operator as commutator is not unique but it seems that - at least for diagonal operators - the growth of the sequence (a_n) determines the degree of unboundedness of A or/and B if $D_a = AB - BA$. Thus, the following conjecture may be true.

Conjecture 1

If $D_a \in \mathcal{L}^+(\mathfrak{D})$ has a representation $D_a = AB - BA$, $A, B \in \mathcal{L}^+(\mathfrak{D})$, then D_a has also the representation (4). More specifically: if $\mathfrak{D} \in (I)$, then T is a commutator if and only if (t_n) is shift-admissible and T-addable.

The following conjecture also seems reasonable.

Conjecture 2

Let $\mathfrak{D} \in (I)$. If the identity I is a commutator, then $\mathfrak{D}[t]$ is a nuclear space.

Now we turn to quasidiagonal operators, and we will again use some ideas from the bounded case. In /1/ there was proved the following:

Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \dots$; $S = (S_{ij})$ an operator on $\mathcal{H}$ with $S_{ij} \in \mathcal{B}(\mathcal{H}_0)$, $\sum_{i,j} \| S_{ij} \| < \infty$. Then : $S = LW - WL$ where

$L = (L_{ij})$, $L_{ij} = \delta_{i+1,j}\, I_{\mathcal{H}_0}$ ($I_{\mathcal{H}_0}$ - identity on $\mathcal{H}_0$):

$W = (W_{ij})$ with

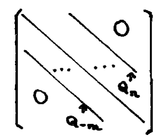
$$W_{1j} = 0 \ , \ j=1,2,\dots \ ; \quad W_{ij} = 0, \ i \leq 0 \ \text{ or } \ j \leq 0 \qquad (5)$$
$$W_{ij} = W_{i-1,j-1} + S_{i-1,j} \ , \ i=2,3,\dots \ ; j=1,2,\dots$$

Now let us return to $\mathcal{L}^+(\mathfrak{D})$ with $\mathfrak{D} \in (I)$. For $S \sim (S_{ij}) \in \mathcal{L}^+(\mathfrak{D})$ we define the matrices $L \sim (L_{ij})$ and $W \sim (W_{ij})$ in the analogous way as above. That means, now all S_{ij}, L_{ij}, W_{ij} are $\in \mathbb{C}$, $I_{\mathcal{H}_0}$ must be replaced simply by 1. Again one has formally

$$S = LW - WL \qquad (6)$$

In case $S = D_a$ (6) is equivalent to (4).

If we again suppose $R, L \in \mathcal{L}^+(\mathfrak{D})$, then the only question to decide is whether or not (W_{ij}) defines an operator $W \in \mathcal{L}^+(\mathfrak{D})$. In general this is a difficult task. Most easily one can handle this for quasidiagonal operators. An operator $Q \in \mathcal{L}^+(\mathfrak{D})$ is said to be <u>quasidiagonal</u> (with respect to (φ_n) as usual) if its matrix representation (Q_{ij}) has only finite many lower and upper subdiagonals different from zero, i.e. schematically:

$Q \sim$  .

Denote these subdiagonals (from left to right) by $Q_{-m}, \dots, Q_0, \dots, Q_n$ and write $Q = [Q_{-m}, \dots, Q_n]$ .Moreover denote the sequences of matrix-elements corresponding to Q_j by $(q_k^{(j)}) = q^{(j)}$, $k=1,2,\dots$; $-m \leq k \leq n$, i.e.

$$q_k^{(j)} = \begin{cases} Q_{k,k+j} \ , \ k=1,2,\dots \ ; \ 0 \leq j \leq n \\ Q_{k-j,k} \ , \ k=1,2,\dots \ ; \ -m \leq j < 0 \end{cases}$$

Proposition 3.4

Let $\mathfrak{D} \in (I)$ and $R, L \in \mathcal{L}^+(\mathfrak{D})$. A quasidiagonal operator $Q = [Q_{-m}, \dots Q_n] \in \mathcal{L}^+(\mathfrak{D})$ is a commutator if all sequences $q^{(j)}$, $-m \leq j \leq n$, are T-addable. In this case one possible representation is given by

$$Q = LW - WL ,$$

where L is the left shift and W is given by (5).

Proof:

That (W_{ij}) defines an operator $W \in \mathcal{L}^+(\mathfrak{D})$ can be seen from the following considerations. The definition of W implies that this is a quasidiagonal operator of the same type as Q:

$$W = [W_{-m}, \dots, W_n] = [W_{-m}, 0, \dots, 0] + \dots + [0, \dots, 0, W_n] . \qquad (7)$$

The corresponding sequences $w^{(j)}$ are formed by the partial sums

$$w_n^{(j)} = \sum_{k=1}^{n} q_k^{(j)} .$$

Observe that any term in the sum (7) is obtained from a diagonal operator $D_w(j)$ via application of some power of R or L. Therefore the assumption implies that any $D_w(j) \in \mathcal{L}^+(\mathfrak{D})$ by (1) and any term of (7) also defines an operator belonging to $\mathcal{L}^+(\mathfrak{D})$. Thus $W \in \mathcal{L}^+(\mathfrak{D})$ and the proof is complete.

q.e.d.

Corollary 3.5

Let $\mathfrak{D} \in (I)$, $R, L \in \mathcal{L}^+(\mathfrak{D})$. Then R_a, L_a are commutators if $a = (a_n)$ is T-addable.

Remark that for R_a, L_a the T-addability of $a = (a_n)$ automatically implies $R_a, L_a \in \mathcal{L}^+(\mathfrak{D})$.

Now we consider the case $\mathfrak{D} \notin (I)$.

For $\mathfrak{D} \in (II)$ we remark only the following. Since in this case

$$\mathfrak{D} = \mathcal{H}_0 \oplus \mathfrak{D}^\infty(T_1), \quad \mathfrak{D}^\infty(T_1) \in (I) \qquad \text{(cf. section 2)},$$

the results for the bounded case and for type (I) can be combined to identify a lot of operators of the form

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad B \in \mathcal{B}(\mathcal{H}_0),\ C \in \mathcal{L}^+(\mathfrak{D}^\infty(T_1))$$

as commutators.

Moreover, operators of the form $A = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ and hence also $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$ are commutators as can be seen trivially from e.g.

$$\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Next let $\mathfrak{D} \in (III)$. In view of the considerations of section 2 we consider first the case $\mathfrak{D} \in (III_A)$, i.e.

$$\mathfrak{D} = \sum_{(t_n')} \oplus \mathcal{H}_n. \tag{8}$$

If $A = \sum \oplus A_n$, $A_n \in \mathcal{B}(\mathcal{H}_n)$ and $\|A_n\| \le C\,(t_n')^s$ for all n and appropriate $C, s > 0$, then $A \in \mathcal{L}^+(\mathfrak{D})$. In the case that the sum representing A contains only finite many terms different from zero, the results of the bounded case can be applied (cf. also Proposition 3.6). In the geneeal case one could proceed as follows. Suppose $A_n = [B_n', C_n] = B_n C_n - C_n B_n$. If $B = \sum \oplus B_n$ and $C = \sum \oplus C_n$ belong to $\mathcal{L}^+(\mathfrak{D})$, then $A = [B, C]$. But this procedure seems to be not

very useful for concrete applications. Let us therefore describe another possibility to construct commutators. To do so, we will introduce a generalized shift operator which does not correspond to the orthonormal basis (φ_n) but to the representation (8). Let $\psi = (\psi_1, \psi_2, \dots) \in \mathfrak{D}$ (cf. sect. 2). Then define

$$\tilde{R}\psi = (0, \psi_1, \psi_2, \dots) \quad ; \quad \tilde{L}\psi = (\psi_2, \psi_3, \dots).$$

Analogously to $\mathfrak{D} \in (I)$ one has: $\tilde{R}, \tilde{L} \in \mathcal{L}^+(\mathfrak{D})$ if and only if (t_n') is shift-admissible.

Now we use the matrix representation of $A \in \mathcal{L}^+(\mathfrak{D})$ with respect to (8), i.e., $A \sim (A_{ij})$, $A_{ij} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j) \cong \mathcal{B}(\mathcal{H})$. It is natural to call an operator of the form

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ \dots & \dots & \ddots & \end{bmatrix}$$

generalized diagonal operator or diagonal operator with respect to the representation (8).

The notion of T-addability is generalized as follows. Let $A = \sum \oplus A_n \in \mathcal{L}^+(\mathfrak{D})$ a generalized diagonal operator and $\|A_n\| = a_n$.

We say that A is T-addable if (a_n) is (t_n')-addable.

Proposition 3.6

Suppose $\mathfrak{D} = \mathfrak{D}^\infty(T) \in (III_A)$, (t_n') - shift-admissible and $A \in \mathcal{L}^+(\mathfrak{D})$ a T-addable generalized diagonal operator. Then A is a commutator.

Proof:

As in the proof of Lemma 3.1 it is seen that formally

$$A = W\tilde{R} - \tilde{R}W \tag{9}$$

with

$$\tilde{R} \sim \begin{bmatrix} 0 & 0 & 0 & \dots \\ I & 0 & 0 & \dots \\ 0 & I & 0 & \dots \\ \ddots & \ddots & \ddots & \end{bmatrix} \in \mathcal{L}^+(\mathfrak{D}) \quad \text{and} \quad W \sim \begin{bmatrix} 0 & A_1 & 0 & 0 \\ 0 & 0 & A_1+A_2 & 0 \\ 0 & 0 & 0 & A_1+A_2+A_3 \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This representation is understood with respect to (8). To see that $W \in \mathcal{L}^+(\mathfrak{D})$ one estimates as follows:

Let $\psi = \sum \oplus \psi_n \in \mathfrak{D}$, then

$$\|T^j W \psi\|^2 = \Big\| T^j \sum_{n=1}^\infty \oplus \Big(\sum_{k=1}^n A_k\Big)\psi_{n+1}\Big\|^2 \le \sum_{n=1}^\infty (t_{n+1}')^{2j} \Big(\sum_{k=1}^n \|A_k\|\Big)^2 \|\psi_{n+1}\|^2 = \sum_{n=1}^\infty (t_{n+1}')^{2j} \Big(\sum_{k=1}^n a_k\Big)^2 = (*).$$

Since (a_n) was (t_n')- addable, i.e. $\sum_{k=1}^n a_k \le C\,(t_n')^r$ for some $C, r > 0$ and all n, the estimation can be continued :

$$(*) \leq C^2 \sum_{n=1}^{\infty} (t'_{n+1})^{2(j+r)} \|\psi_{n+1}\|^2 < \infty .$$

The estimation of $\| T^j W^* \psi \|^2$ is almost the same. Thus $W \in \mathcal{L}^+(\mathfrak{D})$.

q.e.d.

Now one could prove several variants of results analogous to Lemma 3.3 but we mention only one of them.

Corollary 3.7

If $t'_n \sim n^\beta$ for some $\beta > 0$, then any bounded generalized diagonal operator is a commutator.

Now let us give a result which is valid in the general case $\mathfrak{D} \in (III)$. First remark that in a natural manner operators $A \in \mathfrak{B}(\mathcal{H}^{(\mu)})$, $\mu > 1$ (see sect.?) can be viewed as operators in $\mathcal{L}^+(\mathfrak{D})$ (identifying A and $A \oplus 0$, 0 - zero operator on $(\mathcal{H}^{(\mu)})^\perp$).

Proposition 3.8

Let $\mathfrak{D} \in (III)$. Then any operator $A \in \mathfrak{B}(\mathcal{H}^{(\mu)})$, $1 \leq \mu < \infty$ is a commutator.

Proof:

The proof can be reduced to the bounded case. Let $\nu > \mu$ be chosen so that $\mathcal{H}^{(\nu)} \ominus \mathcal{H}^{(\mu)}$ is infinite dimensional. Then A has a large kernel if considered as an operator in $\mathfrak{B}(\mathcal{H}^{(\nu)})$. Hence it is a commutator /4/. Consequently, A is also a commutator in $\mathcal{L}^+(\mathfrak{D})$.

q.e.d.

Remember that for arbitrary $\mathfrak{D}^\infty(T)$ on each $\mathfrak{B}(\mathcal{H}^{(\mu)})$ the topology induced by $\tau_\mathfrak{D}$ coincides with the usual operator norm topology. Moreover, the commutators are norm dense in $\mathfrak{B}(\mathcal{H})$ for any infinite dimensional separable Hilbert space $\mathcal{H}$ /?/. Combining these observations with the fact that $\bigcup_\mu \mathfrak{B}(\mathcal{H}^{(\mu)})$ is $\tau_\mathfrak{D}$-dense in $\mathcal{L}^+(\mathfrak{D})$ (via obvious embeddings) (sect.?), one gets:

Proposition 3.9

Let $\mathfrak{D} = \mathfrak{D}^\infty(T) \in (II)$ or $\in (III)$. Then the commutators are $\tau_\mathfrak{D}$-dense in $\mathcal{L}^+(\mathfrak{D})$.

In section 4 this result will be generalized to include also the case $\mathfrak{D} \in (I)$. Because this will use finite dimensional operators, we did not include it in this section.

4. Finite dimensional operators

The aim of this section is to show that most of the finite dimensional operators are commutators. For $\mathfrak{D} \in (II)$ or $\in (III)$ the problem can be reduced to the bounded case.

Proposition 4.1

Let $\mathfrak{D} \notin (I)$, then any finite dimensional operator $F \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

Proof:

Let $F = \sum_{i=1}^{n} \langle \psi_i, \cdot \rangle \chi_i$. Since $\mathfrak{D} \notin (I)$ there is an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathfrak{D}$. Put $\mathcal{H}_1 = \text{lin}\{\psi_1, \dots, \psi_n, \chi_1, \dots, \chi_n, \mathcal{H}_0\}$. Then $\mathcal{H}_1 \subset \mathfrak{D}$ and $F_1 = F|\mathcal{H}_1 \in \mathfrak{B}(\mathcal{H}_1)$ has a large kernel. Consequently, $F_1 = [A_1, B_1]$, $A_1, B_1 \in \mathfrak{B}(\mathcal{H}_1)$. Thus $F = F_1 \oplus 0 = [A, B]$ with $A = A_1 \oplus 0$, $B = B_1 \oplus 0$. Here 0 denotes the zero operator on $\mathcal{H}_1^\perp$. Clearly, $A, B \in \mathcal{L}^+(\mathfrak{D})$ and we are done.

q.e.d.

Now we consider domains $\mathfrak{D} \in (I)$. The first result says that there are "enough" commutators in $\mathcal{L}^+(\mathfrak{D})$, cf. Proposition 3.9.

Proposition 4.2

Let $\mathfrak{D} \in (I)$, then the commutators are $\tau_\mathfrak{D}$-dense in $\mathcal{L}^+(\mathfrak{D})$.

Proof:

Let $\mathcal{H}_n = \text{lin}\{\varphi_1, \dots, \varphi_n\}$ and Q_n the projection onto $\mathcal{H}_n$, then for all $A \in \mathcal{L}^+(\mathfrak{D})$:

$$A = \tau_\mathfrak{D}\text{-}\lim Q_n A Q_n = \tau_\mathfrak{D}\text{-}\lim A_n . \qquad (10)$$

Then $A_n \in \mathfrak{B}(\mathcal{H}_n)$ has the matrix representation

$$A_n \sim \begin{bmatrix} A_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $a_n = \text{Tr}\, A_n$, then

$$A_n = \tau_\mathfrak{D}\text{-} \lim_j B_j^{(n)} \qquad (11)$$

with $B_j^{(n)} = \begin{bmatrix} A_n & & 0 \\ & \ddots & \\ 0 & & -a_n & \\ & & & \ddots \end{bmatrix}$ where $-a_n$ stands at the diagonal place with number $(n+j+1)$.

Since $\text{Tr}\, B_j^{(n)} = 0$ for all j, n, the $B_j^{(n)}$ are commutators in $\mathcal{L}^+(\mathfrak{D})$.

Relations (10) and (11) together give the desired result.

q.e.d.

Combining Propositions 3.9 and 4.2 we get a main result of the paper.

Theorem 4.3

Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$, then the commutators are $\tau_{\mathfrak{D}}$-dense in $\mathcal{L}^+(\mathfrak{D})$.

This Theorem has an interesting corollary which is worthwhile to be mentioned.

Corollary 4.4

Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$, then on $\mathcal{L}^+(\mathfrak{D})$ there are no nonzero $\tau_{\mathfrak{D}}$-continuous complex homomorphisms (i.e. multiplicative linear functionals).

Our final aim in this section is to show that in the case that R,L belong to $\mathcal{L}^+(\mathfrak{D})$, any finite dimensional $F \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

Proposition 4.5

Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$ and $R,L \in \mathcal{L}^+(\mathfrak{D})$, then any finite dimensional $F \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

To separate the technical details from the main idea of the proof, we start with two Lemmata.

Lemma 4.6

Let $F \in \mathcal{L}^+(\mathfrak{D})$ be finite dimensional, $F = \sum_{j=1}^{k} \langle \chi_j, \cdot \rangle \psi_j$,

(ψ_j) - an orthonormal set. Then there is an operator S so that $S = S^* \geq I$, $S\psi_j = s_j \psi_j$, $(\psi_j) \subset \mathfrak{D}$ an orthonormal basis and $\mathfrak{D} = \mathfrak{D}^\infty(S)$.

Proof:

We give only a sketch of the proof. First, we use a fact which seems to be well-known, but for which we can not give a reference.

Our domain $\mathfrak{D}[t] = \mathfrak{D}^\infty(T)[t]$ is an (F)-space with unconditional basis (φ_n). $\mathfrak{D}_1 = \text{lin}\{\psi_1, \dots, \psi_k\}$ is a topologically complemented subspace. Let P_1 be the orthoprojection from $\mathcal{H}$ onto $\mathfrak{D}_1$. Clearly, $P_1 \in \mathcal{L}^+(\mathfrak{D})$. The above mentioned fact consists in the following:

$\mathfrak{D}_2 = (I-P_1)\mathfrak{D}$ has also an unconditional basis.

Next we apply a result of Mitjagin /12/:

Let $E = \mathfrak{D}^\infty(T)[t]$ and $X \subset E$ a complemented subspace with unconditional basis. Then X is topologically isomorphic to a coordinate subspace of E. Especially, X is isomorphic to some $\mathfrak{D}^\infty(B)$, where $B = B^* \geq I$ is a selfadjoint operator in $\mathcal{H}_2 = \overline{\mathfrak{D}_2}$, $B\varrho_n = b_n \varrho_n$ for some orthonormal basis (ϱ_n) in $\mathcal{H}_2$. Then one can put $S = I_k \oplus B$, I_k the identity on $\mathfrak{D}_1$ and $\psi_{n+k} = \varrho_n$, n=1,2,.... q.e.d.

Let us remark, that in what follows we will aplly Lemma 2.2 several times without explicit mention it. The advantage of the representation of $\mathfrak{D}$ described in Lemma 4.6 is a simple matrix representation for F (with respect to (ψ_n)):

$$F \sim (f_{ij}) = \begin{bmatrix} f_{11} & f_{12} & \cdots \\ \cdots & \cdots & \cdots \\ f_{k1} & f_{k2} & \cdots \\ 0 & 0 & 0 \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where the right-hand matrix has (kxk)-matrices as entries:

$F_1 = (f_{ij})$; $1 \leq i,j \leq k$; ... $F_n = (f_{ij})$, $1 \leq i \leq k$, $(n-1)k+1 \leq j \leq nk$; ...

Moreover, $F = \sum F_n$ in an obvious manner.

This matrix representation suggests a splitting of $\mathcal{H}$:

$\mathcal{H} = \sum \oplus \mathcal{H}_n$, $\dim \mathcal{H}_n = k$; put P_n to be the projection onto $\mathcal{H}_n$.

Without loss of generality we may suppose that $1 \leq s_1 \leq s_2 \leq s_3 \dots$

Put $a_n = s_{nk}$ and form a new operator A by setting $A \restriction \mathcal{H}_n = a_n I_k$.

Using the assumption $R,L \in \mathcal{L}^+(\mathfrak{D})$ we get $\mathfrak{D} = \mathfrak{D}^\infty(A)$, more exactly:

$$\mathfrak{D} = \sum_{(a_n)} \oplus \mathcal{H}_n = \{\varphi = \sum \oplus \bar{\varphi}_n : \bar{\varphi}_n \in \mathcal{H}_n, \sum a_n^{2m} \|\bar{\varphi}_n\|^2 < \infty \quad \forall\, m \in \mathbb{N}\} =$$
$$= \{\varphi : \sum \|P_n \varphi\|^2 a_n^{2m} < \infty \quad \forall\, m \in \mathbb{N}\} \qquad (12).$$

Lemma 4.7

With the notations above one has

$$\sum a_n^{2m} \|F_{n+1}\|^2 < \infty \quad \text{for all } m \in \mathbb{N} . \qquad (13)$$

Proof:

Using $F_n = P_1 F P_n$ one immediately gets $\|F_n\|^2 \leq \sum_{j=1}^{k} \|P_n \chi_j\|^2$.

This together with (12) applied to $\varphi = \chi_1, \dots, \chi_k$ succesively gives

$$\sum_n a_n^{2m} \|F_n\|^2 \leq \sum_n \Big(\sum_{j=1}^{k} \|P_n \chi_j\|^2 \Big) a_n^{2m} < \infty .$$

Since $a_1 \leq a_2 \leq \dots$ (13) follows. q.e.d.

Proof of Proposition 4.5

Referring to Lemma 4.6 and the considerations before Lemma 4.7 we use the representation $\mathfrak{D} = \mathfrak{D}^\infty(A)$ described there. If we put

$$C = \begin{bmatrix} -F_2 & -F_3 & -F_4 & \cdots \\ F_1 & 0 & 0 & \cdots \\ 0 & F_1 & 0 & \cdots \\ 0 & 0 & F_1 & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix} \quad ; \quad D = \begin{bmatrix} 0 & I_k & 0 & 0 & \cdots \\ 0 & 0 & I_k & 0 & \cdots \\ 0 & 0 & 0 & I_k & \cdots \\ \cdots & \cdots & \cdots & & \ddots \end{bmatrix}$$

one has formally $F = DC - CD$. Again $D \in \mathcal{L}^+(\mathfrak{D})$ by the assumptions.

To check $C \in \mathcal{L}^+(\mathfrak{D})$ we first estimate:

$$\|A^m C\varphi\|^2 = a_1^{2m}\Big\|\sum_{n=1}^{\infty} F_{n+1}\bar\varphi_n\Big\|^2 + \sum_{n=2}^{\infty} a_n^{2m}\|F_1\bar\varphi_{n-1}\|^2 \le$$

$$\le C_1\, a_1^{2m}\|\varphi\|^2 + \sum_{n=2}^{\infty} a_n^{2m}\|\bar\varphi_{n-1}\|^2 < \infty .$$

Here (12) and $R, L \in \mathcal{L}^+(\mathfrak{D})$ are used. It remains to estimate $\|A^m C^* \varphi\|^2$

$$\|A^m C^*\varphi\|^2 = \sum_{n=1}^{\infty} a_n^{2m}\|F^*_{n+1}\bar\varphi_1 + F_1^*\bar\varphi_{n+1}\|^2 \le 2\Big\{\sum_{n=1}^{\infty} a_n^{2m}\|F^*_{n+1}\|^2\|\bar\varphi_1\|^2 +$$

$$+ \sum_{n=1}^{\infty} a_n^{2m}\|F_1^*\|^2\|\bar\varphi_{n+1}\|^2\Big\} < \infty .$$

The first sum is finite due to Lemma 4.7 ; the second sum is finite because of (12) and $\ldots a_n \le a_{n+1} \le \ldots$.

q.e.d.

5. Selfcommutators

A special kind of commutators are the so-called selfcommutators. Some information about the situation in the bounded case can be taken from /4/. Concerning the unbounded case we will give only some preliminary results.

Definition 5.1

An operator $S \in \mathcal{L}^+(\mathfrak{D})$ is said to be a selfcommutator (or representable as a selfcommutator) if

$S = [A, A^+] = AA^+ - A^+A$ for some $A \in \mathcal{L}^+(\mathfrak{D})$.

Clearly, selfcommutators are symmetric ($S = S^+$).

The following results are well-known /4/,/13/:

1. If $S \in \mathcal{B}(\mathcal{H})$ is a selfcommutator, the 0 belongs to the spectrum and so S is not invertible in $\mathcal{B}(\mathcal{H})$.
2. If A is a closed operator and on $\mathfrak{D}(A^*A) = \mathfrak{D}(AA^*)$ one has $AA^* - A^*A = I$, then A^*A has eigenvalues 0,1,2,... all with the same multiplicity.

Turning to $\mathcal{L}^+(\mathfrak{D})$ let us remark:

3. We consider here only selfcommutators $S = A, A^+$ for such A that AA^+ and A^+A are essentially selfadjoint operators.
4. $\sigma(AA^+) \cup \{0\} = \sigma(A^+A) \cup \{0\}$. Moreover, if A is a closed operator, then A^*A and AA^* have the same non-zero eigenvalues with the same multiplicity /3/.
5. Property 1. is not valid for $\mathcal{L}^+(\mathfrak{D})$.

In the next lemma we collect some further properties related with selfcommutators.

Lemma 5.2

Let $\mathfrak{D} = \mathfrak{D}^\infty(T) \in (I)$, $0 \le S \sim (s_n), (\varphi_n)$ a diagonal operator in $\mathcal{L}^+(\mathfrak{D})$ with $S \in (I)$, cf./8/. If $S = AA^+ - A^+A$ the following statements are true:

i) All operators $\overline{A^+A}$, $\overline{AA^+}$, $\bar A$, $\overline{A^+}$ are $\in (I)$.

ii) Let (a_n) be the eigenvalues of $\overline{A^+A}$, then $s_n \le a_n$. Moreover, $(a_n) \subset \sigma(\overline{AA^+})$ and if $0 \in \sigma(\overline{AA^+})$, then $0 \notin \sigma_{ess}(\overline{AA^+})$, i.e., 0 can be only an eigenvalue with finite multiplicity.

Proof:

i) $AA^+ = S + A^+A$ leads immediately to

$$\langle S\varphi, \varphi\rangle \le \langle S\varphi, \varphi\rangle + \langle A^+A\varphi, \varphi\rangle = \langle AA^+\varphi, \varphi\rangle = \|A^+\varphi\|^2 \quad (14)$$

Therefore, $\mathfrak{D}(S^{1/2}) \supseteq \mathfrak{D}(\overline{A^+}) \supset \mathfrak{D}(\overline{AA^+})$ and consequently $\mathfrak{D}(\overline{A^+})$ and $\mathfrak{D}(\overline{AA^+})$ belong to (I). Here we used that $\mathfrak{D}(S) \in (I)$ implies $\mathfrak{D}(S^{1/2}) \in (I)$. To see $\mathfrak{D}(\bar A) \in (I)$ suppose that there is an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathfrak{D}(\bar A)$. $\mathcal{H}_0 \cap \mathfrak{D}$ is infinite dimensional and for $\varphi, \psi \in \mathcal{K}_0$ ($\mathcal{K}_0$ - unit ball in $\mathcal{H}_0$) one has

$$\sup_{\varphi,\psi} |\langle\varphi, A^+\psi\rangle| = \sup_{\varphi,\psi} |\langle A\varphi, \psi\rangle| < \infty .$$

Thus, A^+ is bounded on $\mathcal{H}_0 \cap \mathfrak{D}$, i.e., $\overline{\mathcal{H}_0 \cap \mathfrak{D}} \subset \mathfrak{D}(\overline{A^+})$ in contradiction to $\mathfrak{D}(\overline{A^+}) \in (I)$. Hence, $\mathfrak{D}(\bar A) \in (I)$ and so $\mathfrak{D}(\overline{A^+A}) \in (I)$, too.

ii) By (14) and the mini-max-principle one gets $s_n \le a_n$. Property 4. above gives $(a_n) \subset \sigma(\overline{AA^+})$ and i) means especially that 0 cannot be in $\sigma_{ess}(\overline{AA^+})$, because this would imply that $\mathfrak{D}(\overline{AA^+}) \notin (I)$.

q.e.d.

Now we indicate some simple conditions which imply that diagonal operators (with respect to (φ_n)) are selfcommutators. This should be also compared with Lemma 3.1.

Lemma 5.3

Let $\mathfrak{D} = \mathfrak{D}^\infty(T) \in (I)$, $R, L \in \mathcal{L}^+(\mathfrak{D})$, $0 \le D \sim (d_n)$, $\in \mathcal{L}^+(\mathfrak{D})$ so that $d = (d_n)$ is T-addable. Then D is a selfcommutator.

Proof:

We give at once infinite many such representations, namely: Let A_k be the operator which corresponds to the matrix

$$A_k \sim \begin{bmatrix} 0 & \cdots & 0 & a_1 & 0 & \cdots \\ & \ddots & & \ddots & a_2 & \ddots \\ & & \ddots & & \ddots & a_n \\ & & & 0 & & \ddots \end{bmatrix},$$

i.e., in the k^{th} upper subdiagonal there stands the sequence (a_n). With other words $A_k \sim (a_{ij}^{(k)})$, $a_{ij}^{(k)} = \delta_{i,j-k} a_i$. Then the assumptions guarantee that $A_k \in \mathcal{L}^+(\mathfrak{D})$ and it is easy to see that $D = [A_k, A_k^+]$ with appropriately chosen (a_n) (cf. also the next remark).

q.e.d.

Remark 5.4

i) In the classical case $D = I$, of course D and AA^+ commute and this is an essential point in the proof of property 2. above. The representations of D used in the proof of Lemma 5.3 also lead to operators $A_k A_k^+$ (and $A_k^+ A_k$) which commute with D. Moreover, it is easy to determine the spectra of $A_k A_k^+$ (and $(A_k^+ A_k)$:

$$A_k A_k^+ \sim \begin{bmatrix} a_1^2 & & 0 \\ & a_2^2 & \\ 0 & & \ddots \end{bmatrix}, \qquad A_k^+ A_k \sim \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & a_1^2 & \\ & & & & a_2^2 & \\ & & & & & \ddots \end{bmatrix}.$$

Therefore one has:

$a_1^2 = d_1$, $a_2^2 = d_2$, ... , $a_k^2 = d_k$

$a_{k+1}^2 = d_{k+1}+d_1$, $a_{k+2}^2 = d_{k+2}+d_2$, ... , $a_{2k}^2 = d_{2k}+d_k$; ....

$a_{nk+1}^2 = d_{nk+1}+\dots+d_{2k+1}+d_{k+1}+d_1$. $a_{nk+2}^2 = d_{nk+2}+\dots+d_{2k+2}+d_{k+2}+d_2$,

... $a_{(n+1)k}^2 = d_{(n+1)k}+ \dots +d_{2k}+d_k$.

So, we see a clear structure of the spectrum of $A_k A_k^+$ for any k, A little bit more formally it can be described as follows. Consider a decomposition of (d_n) into k-blocks:

$$(d_n) = \bigcup_{j=0}^{\infty} D_j \ , \quad D_j = (d_{jk+1}, \dots, d_{(j+1)k}).$$

For these ordered k-tuples define D_i+D_j as an elementwise addition and for $i<j$ put $(D_i,D_j) = (d_{ik+1},\dots,d_{(i+1)k},d_{jk+1},\dots,d_{(j+1)k})$.

Then $(a_n^2) = (D_0, D_0+D_1, D_0+D_1+D_2, \dots)$ is the sequence of eigenvalues of AA^+. The sequence of eigenvalues of A^+A is then given by $(0,\dots,0,a_1^2, a_2^2, \dots)$; the sequence starts with k zeros. Furthermore, if each eigenvalue of D has equal multiplicity $d \geq 1$, then by appropriate choice of k ($1 \leq k \leq d$) one can generate a spectrum of A^+A with (homogeneous) multiplicity k, $1 \leq k \leq d$.

Thus one sees also in which manner the eigenvalues (0,1,2,...) with equal multiplicity arise in the case $D = I$ (cf.2. above).

ii) Let us remark that the case $\mathfrak{D} \in (III_A)$ can also be handled, if one modifies the notions of diagonal and shift operators in the way described in the second part of section 3.

6. Commutators and quasi-*-algebras. Concluding remarks

We conclude with a small section which contains the relationship between commutators and quasi-*-algebras and some conjectures as well as problems.

First let us recall the notion of a topological quasi-*-algebra as it was introduced by G.Lassner (cf./7/ and the references therein).

Definition 6.1

A topological quasi-*-algebra ($\mathcal{A}[\xi]$, $\mathcal{A}_0$) is a locally convex space $\mathcal{A}[\xi]$ with a distinguished dense linear subspace $\mathcal{A}_0$ satisfying the following properties:

i) Partial multiplications $A \longrightarrow BA$ and $A \longrightarrow AB$ are defined as continuous linear transformations on $\mathcal{A}$ for every $B \in \mathcal{A}_0$. With respect to these multiplications $\mathcal{A}_0$ becomes an algebra and $\mathcal{A}$ is an $\mathcal{A}_0$-modul.

ii) A continuous linear involution $A \longrightarrow A^+$ is defined on $\mathcal{A}$ which leaves $\mathcal{A}_0$ invariant and fulfills $(AB)^+ = B^+A^+$ for all $A \in \mathcal{A}$, $B \in \mathcal{A}_0$.

We use this notion only in a very special context which we describe now. Thereby, we leave aside topological considerations.

Let again $\mathfrak{D} = \mathfrak{D}^{\infty}(T)$ be given. With (φ_n) there is in a natural way associated a domain $\mathfrak{D}_0 \subset \mathfrak{D}$ consisting of all finite linear combinations of the (φ_n), i.e., $\mathfrak{D}_0$ is isomorphic to the sequence space $\mathfrak{d}$ (all finite sequences). Then, $\mathfrak{D}_0 \subset \mathfrak{D} \subset \mathcal{H} \subset \mathfrak{D}_0'$, where $\mathfrak{D}_0'$ is isomorphic to $\mathfrak{d}'$, the space of all sequences. According to Definition 6.1 ($\mathcal{L}(\mathfrak{D}_0, \mathfrak{D}_0')$, $\mathcal{L}^+(\mathfrak{D}_0)$) is a quasi-*-algebra. Clearly, any infinite matrix (a_{ij}) represents an operator $A \in \mathcal{L}(\mathfrak{D}_0, \mathfrak{D}_0')$.

Proposition 6.2

Any operator $A \in \mathcal{L}(\mathfrak{D}_0, \mathfrak{D}_0')$ is a commutator in the quasi-*-algebra ($\mathcal{L}(\mathfrak{D}_0, \mathfrak{D}_0')$, $\mathcal{L}^+(\mathfrak{D}_0)$).

Proof:

We will not pursue a strong definition, what commutator means and so on. The meaning of the Proposition will be clear from the proof. First remark, that R,L belong to $\mathcal{L}^+(\mathfrak{D}_o)$, $\mathcal{L}(\mathfrak{D}_o, \mathfrak{D}_o')$ and $\mathcal{L}(\mathfrak{D}_o')$. Analogously to section 3 we define the matrix (W_{ij}) corresponding to $A \sim (A_{ij})$. Then, (W_{ij}) defines an operator $W \in \mathcal{L}(\mathfrak{D}_o, \mathfrak{D}_o')$ (because any matrix defines an operator belonging to this set). Consequently

$$A = WR - RW$$

and the right-hand side is well-defined as an element from $\mathcal{L}(\mathfrak{D}_o, \mathfrak{D}_o')$. Here, in the product WR we consider R as an element of $\mathcal{L}^+(\mathfrak{D}_o)$, while in RW the operator R is considered as an element of $\mathcal{L}(\mathfrak{D}_o')$.

q.e.d.

To formulate a simple corollary remark that $\mathcal{L}^+(\mathfrak{D}) \subset \mathcal{L}(\mathfrak{D}_o, \mathfrak{D}_o')$.

Corollary 6.3

Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$ be given. With respect to the quasi-$*$-algebra $(\mathcal{L}(\mathfrak{D}_o, \mathfrak{D}_o'), \mathcal{L}^+(\mathfrak{D}_o))$ any $A \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

The facts just mentioned can be interpreted as follows. If one leaves the domain $\mathfrak{D}$ and the Hilbert space $\mathcal{H}$ and uses more general structures, the problem of the representation of operators as commutators becomes trivial.

At the end of the section we list some conjectures and problems for further study. May be, some of them are not very significant or appear to be trivial.

Conjectures 6.4

i) If $\mathfrak{D} \in$ (II) the identity I is not a commutator.

ii) If $\mathfrak{D} \in$ (I), $R,L \in \mathcal{L}^+(\mathfrak{D})$ and (t_n) is T-addable, then any $A \in \mathcal{L}^+(\mathfrak{D})$ is a commutator.

iii) If $\mathfrak{D} \in$ (II) ($\in (III_A)$ resp.) and (t_n^1) ((t_n') resp.) are shift-admissible and (t_n^1) - $((t_n')$-)-addable, then any $A \in \mathcal{C}(\mathfrak{D})$ is a commutator.

In case that $A \in \mathcal{C}(\mathfrak{D})$ is a commutator, $A = BC - CB$, is it possible to take B,C from $\mathcal{C}(\mathfrak{D})$, too?

Problems 6.5

i) Extend the results in an appropriate way to general Op^*-algebras and to the case where $\mathfrak{D}[t]$ is a general (F)-space.

ii) Under which general conditions on D,A one can prove that $\sigma(AA^+)$ has a structure similar to that described in Remark 5.4 i)? Especially: Let $D \sim (d_n)$, $D = AA^+ - A^+A$, D and AA^+ commute. Describe $\sigma(AA^+)$!

Acknowledgement

The author is indebted to A.Ya.Helemskij, M.I.Kadec and K.-D.Kürsten for hints concerning the result about unconditional bases mentioned in the proof of Lemma 4.6

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Received by Publishing Department
on September 30, 1986.

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Тиммерманн В. E5-86-649

О коммутаторах в алгебрах неограниченных операторов

Работа является первым шагом к систематическому исследованию структуры коммутаторов в топологических алгебрах неограниченных операторов. Для диагональных, квазидиагональных и конечномерных операторов даются разные критерии, из которых следует представление этих операторов в виде коммутаторов. Показано, что коммутаторы плотны относительно равномерной топологии в максимальной Op*-алгебре. Работа содержит также некоторые простые факты о самокоммутаторах, гипотезы и проблемы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Timmermann W. E5-86-649

On Commutators in Algebras of Unbounded Operators

The paper is a first step towards a systematic investigation of the structure of commutators in topological algebras of unbounded operators. For diagonal, quasidiagonal and finite dimensional operators there are given several criteria which imply their representability as commutators. It is proved that the commutators are dense in the maximal Op*-algebra with respect to the uniform topology. Simple facts about selfcommutators, some conjectures and problems are also contained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1986