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AN INITIAL VALUE PROBLEM AND THE STABILITY OF SOLITARY-WAVE SOLUTIONS FOR NONLINEAR EQUATIONS IN MATHEMATICAL PHYSICS

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I. INTRODUCTION

Basic ideas of the theory of nonlinear waves were developed with the study of related problems in fluid mechanics,however,almost any part of physics is associated in some way with the wave motion.

In the process of development of the theory of nonlinear waves there were discovered certain "representative" nonlinear wave equations that were, in a sense, universal. Thus, for example, the Korteveg - de Vries equation (KdV),originally introduced in connection with the study of waves on water, combines weak nonlinearity and dispersion and, as such, represents a general physical process. Almost all of these representative equations have solutions of the type of solitary waves. Their interaction gives rise to pure nonlinear effects.

After the remarkable paper of Gardner, Greene, Kruskal and Miura /1/, who were the first to describe the method of solving the KdV equation using the inverse scattering theory, the interest to solitons increased.

A soliton is usually defined to be a solitary traveling wave with the additional property of persisting through an interaction with another soliton. After they pass through one another, they emerge in the same shape and velocity, having suffered no more than perhaps a phase shift.

Zakharov and Shabat have shown that there are also other similar equations and employed /2,3/ the method of the inverse problem for the Dirac operator to the nonlinear Schrödinger equation (NLS) with cubic nonlinearity. Nonlinear evolution equations integrable by an inverse transformation form infinite-dimensional Hamiltonian systems. Complete integrability in the language of the Hamiltonian systems has the following interpretation: The transformation of the initial values to scattering values being the basis of the inverse scattering method appears to be a nonlinear canonical transformation to the action-angle variables. This significant interpretation was initially proposed by Zakharov and Faddeev /4/ for the KdV equation. The most interesting application of this approach is linked to the problem of quantization of nonlinear equations.

Afterwards there has been done much research, in fact there appeared a new direction in the theory of nonlinear differential equations using diverse mathematical techniques.

In the present paper we focus our attention mainly onto the questions of stability of solitary waves.

Studies of stability at an intuitive level were carried out in many physical works. Some of them contain interesting ideas. The stability, as it is understood in physics, of a solitary wave $\varphi_c(x, t)$ is investigated via the quest for a solution to the initial equation in the form $u(x, t) = \varphi_c(x, t) + e^{i\omega t}\psi(x)$. If, in the linear approximation with respect to ψ, the values of ω are real or if they are from the upper half-plane, then $\varphi_c(x, t)$ is said to be stable. Thus the Liapunov theory of stability in the linear approximation is in a naive way extended to infinite-dimensional spaces.

In the mathematical literature there has been done a lot of research on the strict analog of the Liapunov theory in Banach spaces /see eg. 5/. These works contain various sufficient conditions that, upon being fulfilled, allow us to decide the question of stability (instability) of a nonlinear problem through the stability (instability) of its linear approximation. Besides, studying the stability in the linear approximation, it is necessary to consider the contribution of the continuous spectrum of the linearized differential operator. However, for many important equations of mathematical physics the above-mentioned sufficient conditions are not fulfilled. For instance, the solution of the linearized KdV equation is unstable /6/. Nevertheless, as has been shown by Benjamin, who was the first to suggest exact mathematical approach in the study of the stability of solitary waves /7,8/, the solution of the nonlinear problem is stable. The second example is the following. Chen and Kaup /9/, using the linear approximation theory, obtained the instability result of a single-soliton solution of the Benjamin-Ono equation. Anyway, Benet et al. /10/, applying Benjamin's method, have rigorously proved the stability of the shape of a single-soliton solution of the Benjamin-Ono equation. The paper contains an analysis of Chen and Kaup's error. In both cases the question of stability requires exact mathematical methods and to all the investigation at an intuitive level it is inevitable to give an heuristic explanation.

Therefore, we think it to be natural and essential, not only for pure mathematics but also for applications, to perform rigorous research in the nonlinear differential equations of theoretical physics in exact mathematical manner. In other words, conforming to the classical Hadamard concept /54/, it is necessary to investigate the questions of existence of solutions of the initial value problem, uniqueness of solutions and continuous dependence of solutions upon the initial data. An exact investigation of the stability of solitary waves is always connected with the specific metric.

We present now some special results of these investigations.

II. The KdV AND NEJAMIN-BONA-MANONY (BBM) EQUATIONS

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x, 0) = g(x), \; x \in R, \; t \geq 0 \qquad (1)$$

(KdV)

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad u(x, 0) = g(x), \; x \in R, \; t \geq 0 \qquad (2)$$

(BBM)

describe weakly dispersive processes in nonlinear media.

In /11/ Bona and Smith have proved the existence and uniqueness of a global solution of initial value problem and the continuous dependence on the initial conditions in the Sobolev space H^s with the usual norm

$$\| f \|_s^2 = \sum_{j=0}^{s} \int_{-\infty}^{+\infty} |f^{(j)}(x)|^2 dx,$$

where $f \in L_2$, $f^{(j)} = d^j f/dx^j$ is the generalized derivative, $1 \leq j \leq s$. They used the method of pseudoparabolic regularization adding $-\varepsilon u_{xxt}$ to Eq. (1). After a substitution of variables the regularized Equation (1) reduces to Equation (2). Theorems corresponding to (2) can be easily deduced from the fixed point principle. The regularity theory for Equation (1) is obtained by taking the limit $\varepsilon \longrightarrow 0$. In such a way, comparatively easy, not resorting to any complicated mathematical methods, we obtained quite general result for the KdV equation.

The first complete proof of the existence theorem for the KdV equation with periodic initial conditions was given by Temam /12/. He used the method of parabolic regularization, i.e. he added εu_{xxxx} to Eq. (1) and then took the limit for $\varepsilon \to 0$. This method was further developed in /13-15/. Actually, there is proven the existence of a unique local (global) solution of the KdV in the case when the initial data belonged to H^s, $3/2 < s < 2$ ($2 \leqq s$), in them. The same result follows from Kato's theory of quasilinear abstract evolution equations /16,17/, besides, there also follows for $s > 3/2$ the continuous dependence on the initial data in H^s from there. We mention the work /18/ for the non-integer values of s, and /19/ for the proof of the existence of a global solution for certain classes of discontinuous initial conditions. Benjamin gave a dexterous proof of the stability "of the shape" of the solitary waves

$$\Phi(z) = 3c.\mathrm{sech}^2 \left(\tfrac{1}{2} c^{1/2} z\right), \tag{3}$$

$$\Psi(z) = 3c.\mathrm{sech}^2 \left(c^{1/2} z/2(1+c)^{1/2}\right), \tag{4}$$

where $c > 0$, $z = x - (1+c)t$. $\Phi(z)$ and $\Psi(z)$ are the solutions of Eqs. (1) and (2) respectively.

The stability of the shape is defined as the stability with respect to the given on the space H^1 pseudometric

$$d(f, g) = \inf_{y \in R} \| f(\cdot + y) - g(\cdot) \|_1 . \tag{5}$$

The question of stability in the metric (5) for Eqs. (1) and (2) is natural as it reflects the translational invariance of the solutions of Eqs. (1) and (2).

III. THE NLS EQUATION

with the power-type nonlinearity

$$iu_x + u_{xx} - u(a - |u|^{2p}) = 0, \quad a \in R, \ x \in R, \ t \geqq 0 \tag{6}$$

appears in various problems, models many phenomena such as the behaviour of non-ideal Boze-gas with a weak particle interaction, the spreading of the heat impulse in solids, the Langmuir waves in plasma, etc. /see eg. 1,2,20-31/.

The interest in NLS has risen after it became clear that it was universal and fundamental equation in the same sense as the KdV equation and that in the case of a cubic nonlinearity it could be integrated by the method of the inverse problem /2/. Equation (6) has the two-parameter set of solutions

$$\varphi(x, t) = r(x - 2\omega t)e^{i\omega x + i(\alpha - \omega^2)t}, \tag{7}$$

where

$$r(y) = ((p+1)(a + \alpha))^{1/2p} (\cosh py\sqrt{a + \alpha})^{-1/p}, \tag{8}$$

$a + \alpha > 0$ and $\alpha, \omega \in R$. Particularly, for $\omega = 0$ we have the standing wave solution and for $p = 1$, $a = 0$, $\alpha = \xi^2$ we get Zakharov-Shabat's single-soliton solution. Casenave and Lions /32/ call the stability of a standing wave in the pseudometric

$$d(u, \varphi) = \inf_{(\eta, \xi) \in R^2} \| u(x, t) - e^{i\eta} \varphi(x - \xi, t) \|_1 \tag{9}$$

the orbital stability, what reflects the translational and rotational invariance of the solutions of Equation (6) and, particularly, in the real-valued case, the stability of the shape. It is easy to show the instability of $\varphi(x, t)$ in a more complex metric eg

$$d_1(u, \varphi) = \| \varphi - u \|_1 .$$

In the paper /32/, via the use of the concentration-compactness principle introduced by Lions in /33/, the authors prove the orbital stability of a stationary solution (standing wave) for multi-dimensional NLS equation with a more general nonlinearity. The boundary dividing stable and unstable solutions in one-dimensional case /34/ is the nonlinearity $U(|u|).u$, where the nonlinear function $U(|u|)$ has the power-type growth $2p = 4$. Let us mention that the first who pointed out this fact was Makhankov /25,26, 35/.

It is well-known /36-38/ that the initial value problem for Eq. (6) under the assumptions $0 < 2p < 4$, $u(x, 0) = g(x) \in H^1(R)$ has a unique global solution (in a generalized-functions space) $u(x, t)$, with $u(\cdot, t) \in C([0, \infty), H^1(R))$.

In /39,40/, developing the ideas propounded by Benjamin /7,8/, we have proved the following

THEOREM. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u(x, t)$ is a solution of Eq. (6) with $0 < 2p \leqq 3$, $u(x, 0) = g(x) \in H^1(R)$ and $d(u, \varphi)|_{t=0} < \delta$, then $d(u, \varphi) < \varepsilon$ for all $t \in [0, \infty)$ /$d(u, \varphi)$ and φ are given by the formulas (9) and (7) respectively/.

A fundamental rôle in the proof have played the three functionals

$$Q = i\int_{-\infty}^{+\infty} \bar{u}_x u\,dx, \qquad P = \int_{-\infty}^{+\infty} |u|^2 dx,$$

$$E = \int_{-\infty}^{+\infty} \left(|u_x|^2 + a|u|^2 - |u|^{2(p+1)}/(p+1) \right) dx,$$

which are time-invariant whenever the solutions of Eq. (6) belong to $C([0,\infty), H^1(R))$. The difficult part of the proof is concerned with estimating an effective lower bound for the second variation of the functional $M = E + (\alpha + \omega^2)P - 2\omega Q$, what we achieved via suitable choice of the spectral problems. We remark, that though in the spectral decomposition of the KdV and BBM equations it was sufficient to consider only the contribution of the discrete spectrum, in this case we have had to take into account the positive contribution of the continuous spectrum, too. It seems likely that a more subtle evaluation of the contribution of the continuous spectrum in the spectral analysis would give the possibility of the second proof (using spectral theory) of Cazenave and Lions's result in one dimension.

The stability of the solitary wave (7) has been rigorously proved by Zhidkov /41/ for the case p = 1.

Beside the NLS equation, in description of waves in media with a weak dispersion governed by cubic nonlinearity one can encounter also the modified Korteveg-de Vries equation (mKdV) /see also /42/ /

$$u_t + \delta |u|^2 u_x + u_{xxx} = 0, \quad u(x,0) = g(x), \quad x \in R, \quad t \geqq 0 \qquad (10)$$

which, as it is noted by Zakharov in /27/, has a universal nature, too.

Complex-valued Equation (10) was studied by Zakharov /27/, Fornberg and Whitham /43/, Makhankov /25,26,35/ and others. Let us mention that for the real-valued $u(x,t)$ Wadati /44/ have solved Eq. (10) by the inverse scattering method. Repeating the arguments given in /21, § 8-10/ the complex-valued analog of Eq. (10) can also be solved by the inverse-problem method.

In /45/ the existence of unique global solution $u(x,t) \in C([0,\infty), H^s)$ of the initial value problem for Eq. (10) with initial data $u(x,0) = g(x) \in H^s(R)$, $s \geqq 0$, using the theory of quasilinear evolution equations /16, 17/, is established. The same work /i.e. /45/ contains the proof of the orbital stability of a solitary wave using the spectral theory. Let us note again that in this case, in order to obtain the needed estimate, one has to take into account the positive contribution of the continuous spectrum.

Numerical solution of the BBM equation with step-type initial conditions was investigated in /46/ and the existence theorem is given in /47/.

The KdV equation with the initial data having different limits at $\pm\infty$ was originally studied in /48, 49/, where, upon assuming the existence of a global solution, the authors, making use of Whitham's approximation method, have found its asymptote at $t \longrightarrow +\infty$. With the same assumption, i.e. the existence of a solution of the initial value problem for the KdV and mKdV equations with the step-type initial data, the authors of /50,51/, using the inverse scattering method, have found an asymptotical solution at $t \longrightarrow +\infty$.

We use X^s, $s \geqq 1$, to denote the set of functions $f(x)$, defined on the real axis R, with the following property: For every function $f(x)$ there is a constant c_f such that

$$|||f|||_s^2 = \int_{-\infty}^{+\infty} (f - c_f \operatorname{sgn}(x))^2 dx + \sum_{k=1}^{s} \int_{-\infty}^{+\infty} |f^{(k)}(x)|^2 dx < \infty \qquad (11)$$

Linear set X^s, $s \geqq 1$, normed by (11), forms the Banach space of absolutely continuous, on each finite interval, functions $f(x)$ such that

$$\lim_{x \to +\infty} f(x) = -\lim_{x \to -\infty} f(x) = c_f.$$

Moreover, there holds the estimate

$$|c_f| \leqq \sup_{x \in R} |f(x)| \leqq \text{const.}\ |||f|||_s, \qquad (12)$$

that implies

$$f^2 - c_f^2 \in H^s(R).$$

On the assumption that the initial data belong to X^s, $s \geqq 3$, one can, with the help of the pseudoparabolic regularization method /52/, show the existence, uniqueness and continuous dependence on the initial data of a global solution of the real-valued mKdV equation

$$u_t - u^2 u_x + u_{xxx} = 0, \quad u(x,0) = g(x), \quad x \in R, \quad t \geqq 0. \qquad (13)$$

Further, one can prove the stability of the shape of the solitary-wave solution of Eq. (13) that belongs to X^s for all $t \geq 0$. Here we understand the stability of the shape/pertaining to the inequality (12) /as the stability with respect to the pseudometric

$$d(u, \varphi) = \inf_{\xi \in R} |||u(x, t) - \varphi(x - \xi, t)|||_1.$$

The KdV equation in higher-dimensional real spaces or in more general higher-dimensional complex spaces, as well as the multi-dimensional analog of NLS equation, BBM equation, and other universal equations of mathematical physics, such as the Klein-Gordon equation, etc./see eg. 30,35, 53/, are beyond the scope of this survey. Let us just cite, very briefly, some works for completeness.

The existence and stability of the multi-dimensional NLS equation is studied for example in /23, 28, 36-38,55/, the same problem for the Klein-Gordon equation is studied in/55-58/, either of them and others can be found in /30/ and still other problems in /59/.

IV. CONCLUSIONS

Summing up, one could say that the problem of finding the soliton solutions and the problem of analyzing their stability have, in principle, been solved. Yet, the development of the theory is at its very beginnings.

REFERENCES

1. Gardner C.S., Greene J.M., Kruskal M.D., Miura R.M. Phys. Rev. Lett. 1967, 19, p.1095.
2. Захаров В.Е., Шабат А.Б. ЖЭТФ, I97I, 6I, с. II8.
3. Захаров В.Е., Шабат А.Б. ЖЭТФ, I972, 62, с. 34; I973, 64, с.I627; I973, 65, с. 2I9.
4. Захаров В.Е., Фаддеев Л.А. Функц. анализ и прил., I97I, 5, с. I8
5. Далецкий Ю.Л., Крейн М.Г. Устойчивость решений дифференциальных уравнений в банаховом пространстве, М., Наука, I970.
6. Scharf G., Wreszinski W.F. Ann. Phys. 1981, 134, p.56.
7. Benjamin T.B. Proc. R. Soc. Lond. A. 1972, 328, p.153.
8. Bona J. Proc. R. Soc. Lond. A. 1975, 344, p.363.
9. Chen H.H., Kaup D.Y. Phys. fluids 1980, 23, p.235.
10. Bennet R.W. et al. Math. Proc. Cambridge Phil. Soc. 1983, 94, p.351.

11. Bona J.L., Smith R. Phil. Trans. Roy. Soc. Lond. 1975, 278A, p. 555.
12. Temam R.J. Math. Pures et Appl. 1969, 48, p.159.
13. Duslane T.E. Proc. Symp. Pure Math. Am. Math. Soc. 1979, 23, p.303.
14. Saut J.C., Temam R.J. Israel J. Math. 1976, 24, p.78.
15. Saut J.C. Math. Pures et Appl. 1979, 58, p.21.
16. Kato T. Lecture Notes in Math. 1975, 448, p.25.
17. Kato T. Manuscripta Math. 1975, 28, p.89.
18. Bona J.L., Scott R. Duke Math. J. 1976, 43, p.87.
19. Cohen Murray A. Duke Math. J. 1978, 45, p.149.
20. Уизем Дж. Линейные и нелинейные волны. М., Мир, I977.
21. Теория солитонов. Метод обратной задачи. /под ред. С.П. Новикова/. М., Наука, I980.
22. Солитоны в действии. М., Мир, I98I.
23. Солитоны. Сборник статей. /под ред. Р. Буллафа, Ф. Кодри/. М., Мир, I983.
24. Scott A.C., Chu F.Y.F., McLaughlin D.W. Proc. I.E.E.E. 1973, 61, p.1443.
25. Makhankov V.G. Phys. Reports. 1978, 35, p. I.
26. Makhankov V.G. Comp. Phys. Comm. 1980, 21, p. I.
27. Захаров В.Е. В кн.: Кунин И.А. Теория упругих сред с микроструктурой. М., Наука, I975, с. 82, с. 226.
28. Фаддеев Л.Д. Современные проблемы математики. I974, т. 3, с. 93.
29. Ablowitz M.J., Kaup D.J., Newell A.C., Segur H. Stud. Appl. Math. 1974, 53, p.249.
30. Петвиашвили В.И., Яньков В.В. В сб.: Вопросы теории плазмы. Вып. I4./под ред. В.В. Кадомцева/. М., Энергоатомиздат, I983, с. 3.
31. Филлипс О. В кн.: Современная гидродинамика. Успехи и проблемы. М., Мир, I984, с. 297.
32. Cazenave T., Lions P.L. Comm. Math. Phys. 1982, 85, p.549.
33. Lions P.L. C. R. Paris 1982, 294, p.261.
34. Berestiсki H., Cazenave T. Comp. Rend. Acad. Sci. 1981, 293, p.489.
35. Маханьков В.Г. ЭЧАЯ, I983, I4, вып. I, с. I23.
36. Ginibre J., Velo G. Ann. Inst. H. Poincare 1978, 28, p.287.
37. Ginibre J., Velo G. Journ. Funct. Anal. 1979, 32, p. I.
38. Strauss W.A. In:"Proc. Inter. Symp. Inst. Math. Univ. Fed. Rio de Janeiro 1977", North Holland 1978, p.452.

39. Жидков Е.П., Илиев И.Д., Кирчев К.П. ОИЯИ, Р5-83-771, Дубна, 1983.
40. Жидков Е.П., Кирчев К.П. ЭЧАЯ, 1985, 16, вып. 3, с. 547.
41. Жидков П.Е. ЭЧАЯ, 1985, 16, с. 602.
42. Calogero F., Degasperis A. Inverse Problems 1985, I, p. 57.
43. Fornberg B., Whitham G.B. Phil. Trans. Roy. Soc. 1978, A289, p. 373.
44. Wadati M. Journ. Phys. Soc. Japan 1972, 32, p. 1781.
45. Жидков Е.П., Илиев И.Д., Кирчев К.П. ОИЯИ, Р5-83-812, Дубна, 1983; Сибир. Мат. Ж. 1985, 26, №6, с. 39.
46. Peregrine D. H. Journ. Fluid. Mech. 1966, 25, p.321.
47. Benjamin T.B., Bona J.L., Mahony J.L. Phil. Trans. R. Soc. 1972, A272, p.47.
48. Гуревич А.В., Питаевский А.П. ЖЭТФ, 1973, 65, вып. 2, с. 8.
49. Гуревич А.В., Питаевский А.П. Письма в ЖЭТФ, 1973, 17, вып. 5, с. 268.
50. Хруслов Е.Я. Матем. Сб. 1976, 99 /141/, с. 261.
51. Ohmiya M.I. J. Mt. Tokushima Univ. 1978, 12, p.9.
52. Жидков Е.П., Кирчев К.П. ОИЯИ, Р5-82-183, Дубна, 1982; Сибир. Мат. Ж. 1984, 25, с. 30.
53. Janenko N.N., Larkin N.A., Novikov V.A. In:"The Proc. Fifth Czecoslovak Conf. Diff. Eq. Appl. EQUADIFF 5, Bratislava 1981". /Ed. by Greguš M./. p. 159.
54. Hadamard J. Princeton Univ., Bull. 1902, 13, p.49.
55. Makhankov V.G., Shvachka A.B. In:"The IIIrd Inter. Symp. Sel. Topics Stat. Mech.", JINR, D17-84-850, Dubna 1984, vol. II, p. 73.
56. Боев Т.В., Жидков Е.П. ОИЯИ, Р5-85-605, Дубна, 1985 г.
57. Strauss W.A. Contemporary Math. 1983, 17, p.429.
58. Shatah J. Comm. Math. Phys. 1983, 91, p.313.
59. Косевич А.П., Иванов Е.А., Ковалев А.С. Нелинейные волны намагниченности. Динамические и топологические солитоны. Киев, Наукова думка, 1983.

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Задача Коши и устойчивость решений вида уединенных волн некоторых нелинейных уравнений математической физики

Данный обзор посвящен исследованию устойчивости уединенных волн некоторых нелинейных уравнений математической физики. Рассмотрены вопросы существования и единственности решения задачи Коши в некоторых функциональных пространствах.

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An Initial Value Problem and the Stability of Solitary-Wave Solutions for Nonlinear Equations in Mathematical Physics

A review of selected results on the stability of soliton solutions of several one-dimensional universal nonlinear equations of mathematical physics is given. The questions of existence and uniqueness of solutions of initial value problem in some functional spaces are discussed, too.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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