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REMARK ON JOINT DISTRIBUTION IN QUANTUM LOGICS.
Noncompatible Observables

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This paper is a continuation of the first part under the same title, hereafter refered to as [I]. Sections, theorems and formulae are numbered consecutively, starting with Section 3. References [1-19] are listed at the end of [I].

3. JOINT DISTRIBUTION OF NONCOMPATIBLE OBSERVABLES

In the present section we concentrate ourselves to the main aim of this paper. We shall study the problem of existence of noncompatible σ-observables. We give the new results which are valid also for measures on $\mathcal{L}$ with infinite values, and which generalize the known results for states. We note the methods developed in [6-11, 19, 20] are not applicable for our case.

The existence of a joint distribution closely depends on the concept of a commutator. Let $\mathcal{L}$ be an OML. For a finite subset $F=\{a_1,\ldots, a_n\}$ of $\mathcal{L}$ let us put, following to L. Beran [21]

$$\mathrm{com}\, F := \bigvee_{j_1\ldots j_n=0}^{1} \bigwedge_{i=1}^{n} a_i^{j_i}, \tag{3.1}$$

where $a^{\circ} := a^{\perp}$, $a^1 := a$. The element com F is called the commutator of a finite set $F\subset\mathcal{L}$. For two-element set $F=\{a,b\}$ a com F has been defined by Marsden [22]. We recall that independently of [21] the commutator of F has been used in [7] to show that the so-called question observables $q_{a_1},\ldots, q_{a_n}$ have a joint distribution in a state m iff $m(\mathrm{com}\,F)=1$ (here q_a is a $\mathcal{B}(R^1)$ - σ-observable with $q_a(\{0\})=a^{\perp}$, $q_a(\{1\})=a$). The elements $\mathrm{com}^{\perp}F$ and com F are called in [21] as the upper and the lower commutator, respectively, of F. It is clear that $a_1,\ldots, a_n\in\mathcal{L}$ are mutually compatible iff $\mathrm{com}\,F=1$, where $F=\{a_1,\ldots,a_n\}$.

Now, for any M, $M\subset\mathcal{L}$, put

$$\mathrm{com}\, M = \wedge\{\mathrm{com}\, F : F \text{ is a finite subset of } M\}, \tag{3.2}$$

if the element on the right-hand side of (3.2) exists in $\mathcal{L}$. Definitorically we put $\mathrm{com}\,\emptyset=1$. The commutator of M, com M, has been firstly introduced in [6] for the study of joint distributions.

The following notion has been defined in [9]. We say that a subset M of $\mathcal{L}$ is partially compatible with respect to a, $a \in \mathcal{L}$, if (i) $a \leftrightarrow b$ for any $b \in M$; (ii) $\{b \wedge a: b \in M\}$ is a set of mutually compatible elements of $\mathcal{L}$. It is known [11] that if $a = \mathrm{com}\, M$ exists, then M is partially compatible with respect to a.

Let $\{a_s : s \in S\}$ be an indexed set of elements of $\mathcal{L}$. The element $a \in \mathcal{L}$ is said to be countably obtainable over $\{a_s : s \in S\}$ [10,8], if $a = \bigwedge_{s \in S} a_s$, and if there is a countable subset $S_0 \subset S$ with $a = \bigwedge_{s \in S_0} a_s$. From [[10] Prop.2.3] it follows that if there is at most countable subset $N \subset M$ of a quantum logic $\mathcal{L}$ which generates the minimal sublogic $\mathcal{L}_0$ of $\mathcal{L}$ containing M, then $\mathrm{com}\, M$ exists and is countably obtainable over $\{\mathrm{com}\, F : F$ is a finite subset of $M\}$.

The following result, due to W.Puguntké [23], is of particular interest for the present study: There is an OML $\mathcal{L}$ and $M \subset \mathcal{L}$ for which the commutator of M does not exists in $\mathcal{L}$.

Let x_i be an $\mathcal{A}_i$-observable of $\mathcal{L}$, $i = 1,\dots,n$. Define

$$a(E_1,\dots,E_n) = \bigvee_{j_1 \dots j_n = 0}^{1} \bigwedge_{i=1}^{n} x_i(E_i^{j_i}), \tag{3.3}$$

where $E_i \in \mathcal{A}_i$, $i = 1,\dots,n$. Then, due to [10], for $M = \bigcup_{i=1}^{n} \mathcal{R}(x_i)$, where x_i is a $\mathcal{B}(R^1)$-σ-observable, $i = 1,\dots,n \le \infty$. The commutator $\mathrm{com}\, M$ exists in a quantum logic $\mathcal{L}$ and it is countably obtainable over $\{\mathrm{com}\, F: F$ finite subset of $M\}$, and, moreover, $\mathrm{com}\, M$ is countably obtainable over $\{a(E_1,\dots,E_n) : E_i \in \mathcal{B}(R^1), i = 1,\dots,n\}$. If there is $a_0 = \mathrm{com}(\bigcup_{t \in T} \mathcal{R}(x_t))$, we call a_0 as the commutator of σ-observables $\{x_t : t \in T\}$.

The following result has been proved in [12].

Theorem 3.1. Let $x_1,\dots,x_n$ be $\mathcal{B}(R^1)$-σ-observables of a quantum logic $\mathcal{L}$ and let m be a measure on $\mathcal{L}$. Let us denote $a_0 = \mathrm{com}(\bigcup_{i=1}^{n} \mathcal{R}(x_i))$ If

$$m(a_0^{\perp}) = 0, \tag{3.4}$$

then there is a joint distribution in m. If at least one x_i is σ-finite with respect to m, then the joint distribution is uniqne.

If $x_1,\dots,x_n$ have a joint distribution in m and at least one x_i is σ-finite with respect to m, then (3.4) holds.

Moreover, maps $x_{i0}: E \to x_i(E) \wedge a_0$, $E \in \mathcal{B}(R^1)$, are mutually compatible σ-observables of a quantum logic $\mathcal{L}(0, a_0) :=$

$= \{b \in \mathcal{L} : b < a\}$ (here an orthocomplementation "$'$" is defined as $b' = b^{\perp} \wedge a_0$, $b < a_0$).

It is known [7,8] that $\mathcal{B}(R^1)$-σ-observables $x_1,\dots,x_n$ have a joint distribution in a state m (finite measure, too) iff

$$m(a^{\perp}(E_1,\dots,E_n)) = 0, \tag{3.5}$$

for any $E_1,\dots,E_n \in \mathcal{B}(R^1)$. This is equivalent to the next condition [7,19,20]

$$m(\bigwedge_{j=1}^{n} x_j(E_{j1} \cup E_{j2})) = \sum_{k_1 \dots k_n = 1}^{2} m(\bigwedge_{j=1}^{n} x_j(E_{jk_j})), \tag{3.6}$$

for any $E_{j1} \cap E_{j2} = \emptyset$, $E_{j1}, E_{j2} \in \mathcal{B}(R^1)$, $j = 1,\dots,n$.

For measures with infinite values this equivalence has been proved only in particular cases [12]: (3.5) for measures with carriers, and (3.6) only on a σ-continuous logic. In the below the equivalence of (3.5) with (3.6) and with the existence of a joint distribution in measures attaining even infinite values will be proved.

In the following we shall suppose that $\mathcal{A}_1,\dots,\mathcal{A}_n$ are independent Boolean sub-σ-algebras of a Boolean algebra $\mathcal{A}$ and x_i is an $\mathcal{A}_i$-σ-observable of a logic $\mathcal{L}$, $i = 1,\dots,n$. A decomposition of 1 in a logic $\mathcal{L}$ is a system $\{a_i\} \subset \mathcal{L}$ such that $a_i \perp a_j$ whenever $i \neq j$, $\bigvee_i a_i = 1$.

Lemma 3.2. Let $h_i : \mathcal{B}_i \to \mathcal{A}_i$ be a σ-homomorphism of a Boolean sub-σ-algebra $\mathcal{B}_i$ of $\mathcal{A}_i$, $i = 1,\dots,n$. Then $x_1,\dots,x_n$ have a joint distribution in a measure m iff $x_1 \circ h_1,\dots,x_n \circ h_n$ have a joint distribution for any σ-homomorphism h_i and any Boolean sub-σ-algebra $\mathcal{B}_i$ of $\mathcal{A}_i$, $i = 1,\dots,n$.

Proof. It is evident. Q.E.D.

Lemma 3.3. Let $x_1,\dots,x_n$ have a joint distribution in a measure m. Then

$$m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge \bigwedge_{k=1}^{K} a(E_1^k,\dots,E_n^k)) = m(\bigwedge_{i=1}^{n} x_i(E_i)), \tag{3.7}$$

for any $E_i, E_i^k \in \mathcal{A}_i$, $i = 1,\dots,n$, $k = 1,\dots,K$, where K may be an integer or $+\infty$.

Proof. It is the same as that of Lemma 2.2 in [12] Q.E.D.

Theorem 3.4. Let at least one of $x_1,\dots,x_n$ be σ-finite with respect to m. If the commutator, a_0, of $x_1,\dots,x_n$ exists and if it is countably obtainable, then $x_1,\dots,x_n$ have a joint distribution in m iff

$$m(a_0^{\perp}) = 0 . \tag{3.8}$$

Proof. From [6,10] it follows that if a_0 is countably obtainable over {com F: F finite subset of $\bigcup_{i=1}^{n} \mathcal{R}(x_i)$}, then a_0 is countably obtainable over $\{a(E_1,\dots,E_n): E_i \in \mathcal{A}_i, i = 1,\dots, n\}$ and vice versa.

Let $x_1,\dots, x_n$ have a joint distribution in m, and let x_1 be σ-finite with respect to m. Using (3.7) we may establish that if $m(x_1(E)) < \infty$ for some $E \in \mathcal{A}_1$, then $m(x_1(E) \wedge a_0^{\perp}) = 0$. Therefore, if $\{E_k\}_{k=1}^{\infty} \subset \mathcal{A}_1$ is a countable decomposition of 1 with $m(x_1(E_k)) < \infty$, $k \geq 1$, then

$$m(a_0^{\perp}) = m(a_0^{\perp} \wedge x_1(\bigvee_{k=1}^{\infty} E_k)) = \sum_{k=1}^{\infty} m(a_0^{\perp} \wedge x_1(E_k)) = 0 ,$$

when we use the property of the partial compatibility of a_0.

Let now (3.8) hold. Then putting $\bar{m} := m \mid \mathcal{L}(0,a_0)$

$$m(\bigwedge_{i=1}^{n} x_i(E_i)) = m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_0) = \bar{m}(\bigwedge_{i=1}^{n} x_{i0}(E_i)) ,$$

where $x_{i0}(E) := x_i(E) \wedge a_0$ is an $\mathcal{A}_i - \sigma$-observable of $\mathcal{L}(0, a_0)$, $i = 1,\dots, n$, which are mutually compatible. Appealing to Theorem 2.2 of [1] we see that $x_1,\dots, x_n$ have a joint distribution in m. Q.E.D.

We note that if x_t is an $\mathcal{A}_t - \sigma$-observable of quantum logic $\mathcal{L}$, $t \in T$, where $\{\mathcal{A}_t : t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$, and if $\{x_t : t \in T\}$ have a countable obtainable commutator a_0, then Theorem 3.4 is valid for $\{x_t : t \in T\}$, too.

Lemma 3.5. Let $x_1,\dots, x_n$ have a joint distribution in m and let x_1, say, be σ-finite with respect to m. Then

$$\text{(i)} \quad m(a^{\perp}(E_1,\dots, E_n)) = 0 , \tag{3.9}$$

for any $E_i \in \mathcal{A}_i$, $i = 1,\dots, n$;

$$\text{(ii)} \quad m(\mathrm{com}^{\perp} F) = 0 , \tag{3.10}$$

for any finite subset $F \subset \bigcup_{i=1}^{n} \mathcal{R}(x_i)$.

Proof. Any finite $\emptyset \neq F \subset \bigcup_{i=1}^{n} \mathcal{R}(x_i)$ generates a finite decomposition, $\mathcal{K}_i$, of 1 in each $\mathcal{A}_i$ in the following manner. If in F there is no elements of $\mathcal{A}_i$, then we put $\mathcal{K}_i = \{0,1\}$. If $\mathcal{A}_i \cap F = \{E_1,\dots, E_k\}$, then we put $\mathcal{K}_i = \{E_1^{j_1} \wedge \dots \wedge E_k^{j_k} : j_s \in \{0,1\}, s = 1,\dots, k\}$. Let $\{F_j\}_{j=1}^{\infty}$ be a countable decomposition of 1 in $\mathcal{A}_1$ with $m(x_1(F_j)) < \infty$, $j \geq 1$. Denote by $\mathcal{B}_1$ the minimal sub-σ-algebra of $\mathcal{A}_1$ containing $\{F_j \cap A : A \in \mathcal{K}_1, j \geq 1\}$, and, for $2 \leq i \leq n$, we put $\mathcal{B}_i = \sigma(\mathcal{K}_i)$. Then $\bar{x}_i := x_i \mid \mathcal{B}_i$ is a $\mathcal{B}_i - \sigma$-observable of $\mathcal{L}$ and, due to Lemma 3.2, they have a joint distribution in m. Since the subset $\mathcal{M} = \{\mathrm{com}\, G: G$ is a finite subset of $\bigcup_{i=1}^{n} \mathcal{R}(\bar{x}_i)\}$ of $\mathcal{L}$ has at most countably many elements, there is a commutator a_0 of $\bar{x}_1,\dots,\bar{x}_n$, and, moreover, a_0 is countably obtainable. Because $\bar{x}_1$ is σ-finite with respect to m, appealing Theorem 3.4 we have $m(a_0^{\perp}) = 0$. Since $F \in \mathcal{M}$ and $a_0 < \mathrm{com}\, F$, we obtain (3.10).

To prove (3.9) it is sufficient to put $F = \{x_1(E_1),\dots, x_n(E_n)\}$. Q.E.D.

The next technical lemmas will be useful in the following. If $F = \{c_1,\dots, c_i\} \subset \mathcal{L}$, $\mathcal{L}$ is an OML, then we put $\mathrm{com}(c_1,\dots,c_i) := \mathrm{com}\, F$.

Lemma 3.6. Let $\mathcal{L}$ be an OML. If $a_1,\dots,a_k$ are mutually orthogonal elements of $\mathcal{L}$, then, for any $b_1,\dots,b_n$

$$(\bigvee_{i=1}^{k} a_i) \wedge \mathrm{com}^{\perp}(a_1,\dots, a_k, b_1,\dots, b_n) = \bigvee_{i=1}^{k} (a_i \wedge \mathrm{com}^{\perp}(a_i, b_1,\dots, b_n)). \tag{3.11}$$

Proof. Lemma 2.1 of [8] implies that

$$\mathrm{com}(a_1,\dots, a_k, b_1,\dots, b_n) =$$

$$= \bigvee_{d \in D^n} (a_1 \wedge b^d \vee \dots a_k \wedge b^d \vee (a_1 \vee \dots \vee a_k)^{\perp} \wedge b^d) ,$$

where $D = \{0,1\}$, $b^d := b_1^{d_1} \wedge \dots \wedge b_n^{d_n}$, $d = (d_1,\dots, d_n) \in D^n$.

Calculate

$$a := (\bigvee_{i=1}^{k} a_i) \wedge \mathrm{com}^{\perp}(a_1,\dots, a_k, b_1,\dots, b_n) =$$

$$= (\bigvee_{i=1}^{k} a_i) \wedge \bigwedge_{d \in D^n} ((a_1^{\perp} \vee b^{d\perp}) \wedge \dots \wedge (a_k^{\perp} \vee b^{d\perp}) \wedge (a_1 \vee \dots \vee a_k \vee b^{d\perp}) .$$

Since, for all $i, j = 1,\dots, n$ and each $d \in D^n$, $a_i \leftrightarrow a_j^{\perp} \vee b^{d\perp}$, $a_i^{\perp} \leftrightarrow (a_1^{\perp} \vee \dots \vee a_k \vee b^{d\perp})$, then, according to [1, Lemma 6.10], we may apply the distributive law. Hence,

$$a = \bigvee_{i=1}^{k} (a_i \wedge \bigwedge_{d \in D^n} (a_1^{\perp} \vee b^{d\perp}) \wedge \dots \wedge (a_k^{\perp} \vee b^{d\perp})) =$$

$$= \bigvee_{i=1}^{k} (a_i \wedge \bigwedge_{d \in D^n} (a_i^{\perp} \vee b^{d\perp})) = \bigvee_{i=1}^{k} (a_i \wedge \mathrm{com}^{\perp}(a_i, b_1,\dots, b_n)) .$$

Q.E.D.

Lemma 3.7. Let $\mathcal{L}$ be an OML. If for $F=\{a, b_1,\dots,b_n\}$ we have $m(\mathrm{com}^\perp F)=0$ then

$$m(a) = \sum_{d\in D^n} m(a\wedge b^d) = m(a\wedge \mathrm{com}\, F). \tag{3.12}$$

Proof. $m(a) = m(a\wedge \mathrm{com}\,F) + m(a\wedge \mathrm{com}^\perp F) = m(a\wedge \mathrm{com}\,F) = m(\bigvee_{d\in D^n} a\wedge b)$, when we use the notations from Lemma 3.6. Q.E.D.

Lemma 3.8. Let $\mathcal{L}$ be an OML. Let $F=\{a_1,\dots,a_k, b_1,\dots,b_n\}\subset \mathcal{L}$, where $a_1,\dots,a_k$ are mutually orthogonal elements. If $m(\mathrm{com}^\perp F)=0$, then

$$m((\bigvee_{i=1}^k a_i)\wedge \bigwedge_{j=1}^n b_j) = \sum_{i=1}^k m(a_i\wedge \bigwedge_{j=1}^n b_j). \tag{3.13}$$

Proof. Using [/8/, Lemma 2.1] and the distributive law we can obtain

$$m((\bigvee_{i=1}^k a_i)\wedge \bigwedge_{j=1}^n b_j) = m((\bigvee_{i=1}^k \wedge a_i)\wedge \bigwedge_{j=1}^n b_j\wedge \mathrm{com}\,F) =$$

$$= m((\bigvee_{i=1}^k a_i)\wedge \bigwedge_{j=1}^n b_j \wedge \bigvee_{d\in D^n}((a_1\wedge b^d)\vee\dots\vee(a_k\wedge b^d)\vee($$

$$\vee\, (a_1\vee\dots\vee a_k)^\perp \wedge b^d)) = m(\bigvee_{i=1}^k (a_i\wedge \bigwedge_{j=1}^n b_j)).$$

Q.E.D.

The following notions are needed for the main result of this section. Let $\mathcal{L}$ be an OML. A non-empty subset $\mathcal{J}\subset\mathcal{L}$ is said to be a p-ideal [/22,3/] if (i) if $a,b\in\mathcal{J}$, then $a\vee b\in\mathcal{J}$; (ii) if $b\in\mathcal{J}$, $a<b$, then $a\in\mathcal{J}$; (iii) $b\in\mathcal{J}$ implies $(b\vee a^\perp)\wedge a\in\mathcal{J}$ for all $a\in\mathcal{L}$. If instead of (i) there holds (i) if $\{a_n\}_{n=1}^\infty\subset\mathcal{J}$, then $\bigvee_{n=1}^\infty a_n\in\mathcal{J}$, then $\mathcal{J}$ is called a σ-p-ideal of $\mathcal{L}$. If h is a σ-homomorphism of a quantum logic $\mathcal{L}$ into a quantum logic $\mathcal{L}_1$, then Ker h is a σ-p-ideal of $\mathcal{L}$. We say that $a\sim b$ iff $(a\vee b)\wedge(a\wedge b)^\perp\in\mathcal{J}$, $a, b\in\mathcal{L}$. Then a relation "$\sim$" is the relation of an equivalence on $\mathcal{L}$, and it holds (i) if $a\sim b$, then $a^\perp\sim b^\perp$; (ii) $a_1\sim b_1$, $a_2\sim b_2$ imply $a_1\vee b_1\sim a_2\vee b_2$. Denote by $\mathcal{L}/\mathcal{J}$ the factor OML defined via $\mathcal{L}/\mathcal{J}=\{[a]_\mathcal{J},\ a\in\mathcal{L}\}$, where $[a]_\mathcal{J} := \{b\in\mathcal{L}: b\sim a\}$, and $[a]_\mathcal{J}^\perp := [a^\perp]_\mathcal{J}$, $[a]_\mathcal{J}\vee[b]_\mathcal{J} := [a\vee b]_\mathcal{J}$. The map $h:\mathcal{L}\to\mathcal{L}/\mathcal{J}$ which assigns $[a]_\mathcal{J}$ to any $a\in\mathcal{L}$, is a homomorphism of $\mathcal{L}$ onto $\mathcal{L}/\mathcal{J}$.

Finally, we present a theorem which generalizes all known conditions concerning the existence of a joint distribution in a measure into two main aspects; (i) measures may attain the infinite values, too (ii) the conditions do not depend on the existence of the commutator of a given system $\{x_t: t\in T\}$ of σ-observables.

Theorem 3.9. Let $x_t:\mathcal{A}_t\to\mathcal{L}$ be a σ-observable of a quantum logic $\mathcal{L}$, $t\in T$, where $\{\mathcal{A}_t: t\in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$. Let at least one σ-observable, x_{t_0} say, be σ-finite with respect to a measure m. Then the following conditions are equivalent

(i) $\{x_t: t\in T\}$ have a joint distribution in m:

$$\text{(ii)}\quad m(\mathrm{com}^\perp(\{x_t(A_t): t\in\alpha\}))=0, \tag{3.14}$$

for any $A_t\in\mathcal{A}_t$, $t\in\alpha$, and any finite $\emptyset\neq\alpha\subset T$;

$$\text{(iii)}\quad m(\mathrm{com}^\perp F)=0, \tag{3.15}$$

for any finite subset F of $\bigcup\{\mathcal{R}(x_t): t\in T\}$;

$$\text{(iv)}\quad m(\bigwedge_{t\in\alpha} x_t(A_{1t}\vee A_{2t})) = \sum_{\substack{i_t=1\\ t\in\alpha}}^{2} m(\bigwedge_{t\in\alpha} x_t(A_{i_t t})), \tag{3.16}$$

for any $A_{1t}\wedge A_{2t}=0$, $A_{1t}, A_{2t}\in\mathcal{A}_t$, $t\in\alpha$, and any finite $\emptyset\neq\alpha\subset T$;

$$\text{(v)}\quad m(\bigwedge_{t\in\alpha} x_t(\bigvee_{k=1}^\infty A_{kt})) = \sum_{\substack{k_t=1\\ t\in\alpha}}^{\infty} m(\bigwedge_{t\in\alpha} x_t(A_{k_t t})), \tag{3.17}$$

for any $\{A_{kt}\}_{k=1}^\infty\subset\mathcal{A}_t$, $A_{it}\wedge A_{jt}=0$, $i\neq j$, $t\in\alpha$, and any finite $\emptyset\neq\alpha\subset T$;

(vi) There exists a Boolean σ-algebra $\mathcal{B}$, $\mathcal{B}\neq\{0\}$, and a σ-homomorphism h of the minimal sublogic $\mathcal{L}_0$ of $\mathcal{L}$ containing all $\mathcal{R}(x_t)$ onto $\mathcal{B}$ such that $m(a)=0$ for all $a\in\mathrm{Ker}\,h$;

(vii) There is a quantum logic $\mathcal{L}_1\neq\{0\}$ and a σ-homomorphism h of $\mathcal{L}_0$ onto $\mathcal{L}_1$ such that $\{h\circ x_t: t\in T\}$ are mutually compatible σ-observables of $\mathcal{L}_1$ and $m(a)=0$ for all $a\in\mathrm{Ker}\,h$;

(viii) There is a (unique) measure μ on $\prod_{t\in T}\mathcal{A}_t$ such that

$$\mu(\bigwedge_{t\in\alpha} A_t) = m(\bigwedge_{t\in\alpha} x_t(A_t)), \tag{3.18}$$

for any $A_t\in\mathcal{A}_t$, $t\in\alpha$ and any finite $\emptyset\neq\alpha\subset T$.

Proof. We shall prove the following implications: (i) => (ii) => (iii) => (iv) => (v) => (i), and (i) => (vi) => (vii) => => (viii) => (i).

(i)=>(ii). Let a non-empty finite subset $\alpha\subset T$ be given. If $t_0\in\alpha$, then we appeal Lemma 3.5 and (3.9). If $t_0\notin\alpha$, we change α to $\alpha\cup\{t_0\}$ and use a monotonicity of m.

(ii)=>(iii). Let a finite $F\subset\bigcup\{\mathcal{R}(x_t): t\in T\}$ be given. Then there is a finite subset $\alpha\subset T$ with diveres indices $t_1,\dots,t_n$. Analogically as in the proof of Lemma 3.5, for each $i=1,\dots,n$, F generates a finite decomposition $\mathcal{K}_i=\{A_1^i,\dots,A_{k_i}^i\}\subset\mathcal{A}_{t_i}$

of 1. Theorem 2.2 of [8] yields $\mathrm{com}\, F = \mathrm{com}(\bigcup_{i=1}^{n} \mathcal{K}_i)$ and a repeated application of (3.11) gives us

$$0 \le m(\mathrm{com}^{\perp} F) = m(1 \wedge \mathrm{com}^{\perp}(\bigcup_{i=1}^{n} \mathcal{K}_i)) = m((\bigvee_{i_1=1}^{k_1} x_{t_1}(A^1_{i_1})) \wedge \mathrm{com}^{\perp}(\bigcup_{i=1}^{n} \mathcal{K}_i)) =$$

$$= \sum_{i_1=1}^{k_1} m(x_{t_1}(A^1_{i_1}) \wedge \mathrm{com}^{\perp}(\{x_{t_1}(A^1_{i_1})\} \cup \bigcup_{i=2}^{n} \mathcal{K}_i)) \le$$

$$\le \sum_{i_1=1}^{k_1} m(\mathrm{com}^{\perp}(\{x_{t_1}(A^1_{i_1})\} \cup \bigcup_{n=2}^{n} \mathcal{K}_i)) \le \dots \le$$

$$\le \sum_{i_1=1}^{k_1} \dots \sum_{i_n=1}^{k_n} m(\mathrm{com}^{\perp}(x_{t_1}(A^1_{i_1}), \dots, x_{t_n}(A^n_{i_n}))) = 0 .$$

(iii)=>(iv). Let now a non-empty set $\alpha = \{t_1, \dots, t_n\} \subset T$ be given. Let $A_{1t_i} \wedge A_{2t_i} = 0$, $A_{1t_i}, A_{2t_i} \in \mathcal{A}_{t_i}$, $i = 1, \dots, n$. Define $a_k = x_{t_1}(A_{kt_1})$, $k = 1,2$, $b_j = x_{t_j}(A_{1t_j} \vee A_{2t_j})$, $j = 2, \dots, n$. Repeated appealing Lemma 3.8 we see that (3.16) is true.

(iv)=>(v). To prove (3.17) we limit ourselves to the following case. Let a non-empty subset $\alpha = \{t_1, \dots, t_n\} \subset T$ be given. Let $\{A_{ki}\}_{k=1}^{\infty} \subset \mathcal{A}_{t_i}$, $A_{ki} \wedge A_{kj} = 0$ whenever $k \ne j$, be given for any $i = 1, \dots, n$. There are two possible cases (i) $t_0 \in \alpha$, then we put $t_1 = t_0$; (ii) $t_0 \notin \alpha$, then without loss of generality we can change α to $\alpha \cup \{t_0\}$ and we also put $t_1 = t_0$. There is a countable decomposition $\{E_v\}_{v=1}^{\infty} \subset \mathcal{A}_{t_1}$ of 1 with $m(x_{t_1}(E_v)) < \infty$, $v \ge 1$. Define $\mathcal{B}_1$ as the minimal Boolean sub-σ-algebra of $\mathcal{A}_{t_1}$ containing $\{E_v \wedge A_{k1}, E_v \wedge A_1^{\perp} : v, k \ge 1\}$, where $A_i = \bigvee_{k=1}^{\infty} A_{ki}$, $i = 1, \dots, n$, and let $\mathcal{B}_i$, $i = 2, \dots, n$, be the Boolean sub-σ-algebras of $\mathcal{A}_{t_i}$ generated by $\{A_t^{\perp}, A_{ki} : k \ge 1\}$.

A mapping $\bar{x}_i := x_i | \mathcal{B}_i$ is a $\mathcal{B}_i$-σ-observable of $\mathcal{L}$, $i = 1, \dots, n$, and $\bar{x}_1$ is σ-finite with respect to m. It is clear that $\bar{x}_1, \dots, \bar{x}_n$ have a countably obtainable commutator a_0 over {com F: F finite subset of $\bigcup_{i=1}^{n} \mathcal{R}(\bar{x}_i)$} because the last set has at most countably many elements.

We claim to show $m(a_0^{\perp}) = 0$. Let $a_0 = \bigwedge_{k=1}^{\infty} \mathrm{com}\, F_k$. It is evident that if F and G are two finite subset of $\mathcal{L}$ with $F \subset G$, then $\mathrm{com}\, G < \mathrm{com}\, F$. Therefore, we can choose F_k to be nondecreasing and containing $\bar{x}_1(E_v)$ for any fixed $v \ge 1$. Indeed, put $G_k = \bigcup_{i=1}^{k} F_i \cup \{\bar{x}_1(E_v)\}$, then $a_0 = \bigwedge_{k=1}^{\infty} \mathrm{com}\, F_k > \bigwedge_{k=1}^{\infty} \mathrm{com}\, G_k > a_0$.

Using the continuity of m from above and the properties of the commutators we obtain

$$m(\bar{x}_1(E_v) \wedge a_0) = \lim_k m(\bar{x}_1(E_v) \wedge \mathrm{com}\, F_k) = \lim_k m(\bar{x}_1(E_v)) = m(\bar{x}_1(E_v)).$$

In the previous step we used Lemma 3.7 and (3.12). This implies $m(\bar{x}_1(E_v) \wedge a_0^{\perp}) = 0$ for any $v \ge 1$, and, consequently, $m(a_0^{\perp}) = \sum_{v=1}^{\infty} m(\bar{x}_1(E_v) \wedge a_0^{\perp}) = 0$. Theorem 3.4 entails that $\bar{x}_1, \dots, \bar{x}_n$ have a joint distribution in m, so, in particular, (3.17) holds.

(v)=>(i). Let $\emptyset \ne \alpha \subset T$, α finite, be given. Without loss of generality we may assume that $t_0 \in \alpha$. The property (v) and the Carathéodory method of measure extension applied to Boolean algebras,[17,18] guarantees that there is a (unique) measure μ_α on $\prod_{t \in \alpha} \mathcal{A}_t$ such that

$$\mu_\alpha(\bigwedge_{t \in \alpha} A_t) = m(\bigwedge_{t \in \alpha} x_t(A_t)), \quad \text{for any } A_t \in \mathcal{A}_t,\ t \in \alpha .$$

(i)=>(vi). Let $\mathcal{L}_0$ be the minimal sublogic of a logic $\mathcal{L}$ containing $M = \bigcup_{t \in T} \mathcal{R}(x_t)$. Define

$$\mathcal{J} = \{a \in \mathcal{L}_0 : a < \bigvee_{i=1}^{\infty} \mathrm{com}^{\perp} F_i ,\ F_i \text{ finite subset of } M,\ i \ge 1\}, \tag{3.19}$$

From [11] it follows that $\mathcal{J}$ is a σ-p-ideal of $\mathcal{L}_0$. Theorem 5 of [22] says that the factor logic $\mathcal{L}_0/\mathcal{J}$ is a Boolean σ-algebra. Let us put $\mathcal{B} = \mathcal{L}_0/\mathcal{J}$. A map $h : \mathcal{L}_0 \to \mathcal{B}$ defined by $h(a) = [a]_{\mathcal{J}}$ is a σ-homomorphism of $\mathcal{L}_0$ onto $\mathcal{B}$.

Now we claim to show that $m(a) = 0$ whenever $h(a) = 0$. In other words $m(a) = 0$ whenever $a \in \mathcal{J}$. From the definition of it follows that there is a sequence, $\{F_i\}_{i=1}^{\infty}$, of finite subsets of M such that $a < \bigvee_{i=1}^{\infty} \mathrm{com}^{\perp} F_i$. Let us put $G_i = F_i \cup \{x_{t_0}(E_i)\}$, $i \ge 1$, where $\{E_i\}_{i=1}^{\infty}$ is a decomposition of 1 in $\mathcal{A}_{t_0}$ with $m(x_{t_0}(E_i)) < \infty$. For any G_i, there is a finite subset α_i of T such that for any $b \in G_i$ there is $t \in \alpha$ with $x_t(A) = b$ for some $A \in \mathcal{A}_t$. Put $G = \bigcup_{i=1}^{\infty} G_i$ and $\alpha = \bigcup_{i=1}^{\infty} \alpha_i$. We order the elements of α as follows $\alpha = \{t_1, t_2, \dots\}$.

Let $\mathcal{B}_i$ be the minimal Boolean sub-σ-algebra of $\mathcal{A}_{t_i}$ generated by $\mathcal{A}_{t_i} \cap G$. Then $\bar{x}_i := x_{t_i} | \mathcal{B}_i$ is a $\mathcal{B}_i$-σ-observable of $\mathcal{L}$, and $\{\bar{x}_i\}_{i=1}^{\infty}$ have a countably obtainable commutator a_0. $\{\mathcal{B}_i : i \ge 1\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$. Since at least one of $\{\bar{x}_i\}_{i=1}^{\infty}$ is σ-finite with respect to m, Lemma 3.2 and Theorem 3.4 say $m(a_0^{\perp}) = 0$.

An easy calculation shows

$$a < \bigvee_{i=1}^{\infty} \mathrm{com}^{\perp} F_i < \bigvee_{i=1}^{\infty} \mathrm{com}^{\perp} G_i < a_0^{\perp},$$

so that, $m(a) = 0$.

It rests to show that $\mathcal{B}$ is not a degenerate Boolean σ-algebra. In the opposite case $0 = 1$ in $\mathcal{B}$ and, therefore, $h(0)=h(1)$ so that, $m(1) = 0$ which is a contradiction.

(vi)=>(vii). Let (vi) hold. Defining $\mathcal{L}_1 := \mathcal{B}$ and taking the σ-homomorphism h from (vi) the condition (vii) is proved.

(vii)=>(viii). Let (vii) hold. Define a measure $\bar{m}$ on $\mathcal{L}_1$ as follows: $\bar{m}(h(a)) = m(a)$. We show that $\bar{m}$ is defined well. Let $h(a) = h(b)$. Then $h(a \vee b) = h(a \wedge b)$ and $h((a \vee b) \wedge (a \wedge b)^{\perp}) = 0$. Therefore $m((a \vee b) \wedge (a \wedge b)^{\perp}) = 0$. Using the orthomodular law we have $m(a \vee b) = m(a \wedge b) + m((a \vee b) \wedge (a \wedge b)^{\perp}) = m(a \wedge b)$. So that, $m(a) = m(b)$.

Clearly, $\bar{m}(0) = 0$. Let now $\{h(a_i)\}_{i=1}^{\infty}$ be orthogonal elements in $\mathcal{L}_1$. In $\mathcal{L}_0$ we define $b_1 = a_1$, $b_n = a_n \wedge (\bigvee_{i=1}^{n-1} a_i)^{\perp}$, $n \geq 2$. Then $\{b_n\}_{n=1}^{\infty}$ are orthogonal elements and $h(b_n) = h(a_n)$. An easy check shows

$$\bar{m}\Big(\bigvee_{n=1}^{\infty} h(a_n)\Big) = \bar{m}\Big(\bigvee_{n=1}^{\infty} h(b_n)\Big) = \bar{m}\Big(h\Big(\bigvee_{n=1}^{\infty} b_n\Big)\Big) = m\Big(\bigvee_{n=1}^{\infty} b_n\Big) =$$

$$= \sum_{n=1}^{\infty} m(b_n) = \sum_{n=1}^{\infty} \bar{m}(h(b_n)) = \sum_{n=1}^{\infty} \bar{m}(h(a_n)).$$

Since at least one σ-observable of $\mathcal{L}_1$ from $\{h \circ x_t : t \in T\}$ is σ-finite with respect to m, then Theorem 2.2 of [7] entails that there is a unique measure μ on $\prod_{t \in T} \mathcal{A}_t$ such that

$$\mu\Big(\bigwedge_{t \in \alpha} A_t\Big) = \bar{m}\Big(\bigwedge_{t \in \alpha} h \circ x_t(A_t)\Big),$$

for any $A_t \in \mathcal{A}_t$ and any finite $\emptyset \neq \alpha \subset T$. Using the definition of $\bar{m}$ we prove (3.18).

(viii)=>(i). This implication is evident.

Theorem is completely proved. Q.E.D.

Remark 1. As an example of a particular interest for the present study might serve the implication (vii)=>(i) when in its proof we do not apply Theorem 2.2.

So let (vii) hold. First of all we show that $\mathcal{L}_1$ is a Boolean σ-algebra. For $b \in \mathcal{L}_0$, denote by $K(b)$ the set of all $a \in \mathcal{L}_0$ such that $h(a) \leftrightarrow h(b)$. If $b = x_t(A)$, where $t \in T$ and $A \in \mathcal{A}_t$ are arbitrary, then $K(b)$ is a sublogic of $\mathcal{L}_0$ containing $\bigcup\{\mathcal{R}(x_t) : t \in T\}$. So, $x_t(A) \leftrightarrow a$ for each $a \in \mathcal{L}_0$. Let now $b \in \mathcal{L}_0$, then the same argument shows $K(b) = \mathcal{L}_0$. Hence, $h(a) \leftrightarrow h(b)$ for any $a, b \in \mathcal{L}_0$. In other words, $\mathcal{L}_1$ is a Boolean σ-algebra.

It is known that Ker h is a σ-p-ideal of $\mathcal{L}_0$. The factor logic $\mathcal{L}_0$/Ker h is σ-isomorphic with $\mathcal{L}_0$ [24, p.41], hence, $\mathcal{L}_0$/Ker h is a Boolean σ-algebra. One result of Marsden[22] shows that in this case Ker h contains as a subset the σ-p-

ideal $\mathcal{J}$ from (3.19). Hence, $m(a)=0$ for any $a \in \mathcal{J}$, in particular, $m(\mathrm{com}^{\perp} F)=0$ for any finite subset F of $\bigcup\{\mathcal{R}(x_t) : t \in T\}$. Appealing the condition (ii) of the last theorem, this is equivalent with (i). Q.E.D.

Remark 2. (a) The implication (v)=>(i) has been proved by Gudder[5] for $\mathcal{B}(R^1)$-σ-observables and states.

(b) The implication (iv)=>(v) was proved by Pulmanová[11] for states and σ-observables defined on Borel σ-algebras of topological spaces equipped with a tight topology and using results of compact approximations on these spaces[25].

(c) The implication (ii)=>(iv) has been proved in [7] for states and σ-observables, where the main tool of the proof has been the following simple observation: of $t_i \leq s_i$ $i \in \{1,2,\dots\}$ and $-\infty < \sum t_i = \sum s_i < \infty$, then $t_i = s_i$ for any i. However when at least one of $t_i (s_i)$ is $+\infty$, then this is not true, in general.

(d) A very elementary proof of (ii)=>(i) for $\mathcal{B}(R^1)$-σ-observables and states was present in[19]. It is based on the properties of the distribution function $F(t_1, \dots, t_n) := m(\bigwedge_{i=1}^{n} x_i((-\infty, t_i)))$, $t_i \in R^1$, $i = 1, \dots, n$. This approach is not applicable for general cases.

(e) The equivalence between (i) and (vi) has been established in[11] for a system of $\mathcal{B}(R^1)$-σ-observables and states.

Finally, in the rest of this section we deal with some corollaries of Theorem 3.9.

Proposition 3.10. Let the assumptions of Theorem 3.9 hold. If (i) of Theorem 3.9 is valid, for any $a \in \mathcal{L}_0$, $m_a(b) := m(a \wedge b)$, $b \in \mathcal{L}_0$ is a σ-additive σ-finite measure on $\mathcal{L}_0$.

Proof. If $m(a) = 0$, the proposition is evident. Let $m(a) > 0$, and let $b = \bigvee_{i=1}^{\infty} b_i$, $\{b_i\} \subset \mathcal{L}_0$, $b_i \perp b_j$ if $i \neq j$. Due to (vi), there is a Boolean σ-algebra $\mathcal{B}$ and σ-homomorphism h from $\mathcal{L}_0$ onto $\mathcal{B}$. Therefore $\bar{m}(h(a)) := m(a)$ is a σ-finite measure on $\mathcal{B}$. Then

$$m_a\Big(\bigvee_{i=1}^{\infty} b_i\Big) = m\Big(a \wedge \bigvee_{i=1}^{\infty} b_i\Big) = \bar{m}\Big(h\Big(a \wedge \bigvee_{i=1}^{\infty} b_i\Big)\Big) =$$

$$= \bar{m}\Big(\bigvee_{i=1}^{\infty} (h(a \wedge b_i))\Big) = \sum_{i=1}^{\infty} \bar{m}(h(a \wedge b_i)) = \sum_{i=1}^{\infty} m_a(b_i).$$

Q.E.D.

Remark 3. If $a \in \mathcal{L}_0$ and $0 < m(a) < \infty$, then $m_a(b)/m(a)$, $b \in \mathcal{L}_0$, as a conditional probability on $\mathcal{L}_0$ may be treated.

Proposition 3.11. Let the assumptions of Theorem 3.9 hold. Then $\{x_t : t \in T\}$ have a joint distribution in a measure m iff, for any $a \in \cup\{\mathcal{R}(x_t) : t \in T\}$, the function $m_a(b) := m(a \wedge b)$, $b \in \mathcal{L}_0$, is additive on $\mathcal{L}_0$, that is, $m_a(b_1 \vee b_2) = m_a(b_1) + m_a(b_2)$ whenever $b_1, b_2 \in \mathcal{L}_0$ and $b_1 \perp b_2$. Moreover, m_a is always σ-additive σ-finite measure on $\mathcal{L}_0$, and $m_a(b \vee c) = m((a \wedge b) \vee (b \wedge c))$, $b, c \in \mathcal{L}_0$.

Proof. One part of the proposition follows from Proposition 3.10.

To prove the second part we claim to show that (iv) of Theorem 3.9 holds. First of all let $a = \{t_1, t_2\} \subset T$ and $A_{1i}, A_{2i} \in \mathcal{A}_{t_i}$, $A_{1i} \wedge A_{2i} = 0$, $i = 1,2$, be given. Then

$$m\left(\bigwedge_{i=1}^{2} x_{t_i}(A_{1i} \vee A_{2i})\right) = m_b(x_{t_1}(A_{11} \vee A_{21})) = m_b(x_{t_1}(A_{11})) +$$

$$+ m_b(x_{t_2}(A_{21})) = m_{a_1}(x_{t_2}(A_{12} \vee A_{22})) + m_{a_2}(x_{t_2}(A_{12} \vee A_{22})) =$$

$$= \sum_{j_1, j_2 = 1}^{2} m\left(\bigwedge_{i=1}^{2} x_{t_i}(A_{j_i i})\right),$$

when we use $b = x_{t_2}(A_{12} \vee A_{21})$, $a_j = x_{t_1}(A_{j1})$, $j = 1,2$.

The general case of (iv) is obtainable from the just established fact using the mathematical induction, which proves $\{x_t : t \in T\}$ has a joint distribution in m.

The last assertion of Proposition follows from Proposition 3.10. Q.E.D.

Corollary 3.11.1. Under the hypotheses of Theorem 3.9 we have (i) let $a \in \mathcal{L}$, $m(a) > 0$; if $\{x_t : t \in T\}$ have a joint distribution in m, then $\{x_t : t \in T\}$, as σ-observables of $\mathcal{L}_0$, have a joint distribution in m_a; (ii) $\{x_t : t \in T\}$ have a joint distribution in m iff (3.10) holds for any finite $F \subset \mathcal{L}_0$.

Proof. (i) If $a \in \cup\{\mathcal{R}(x_t) : t \in T\}$, then the assertion follows from Proposition 3.11. In the general case, according to (vi) of Theorem 3.9, there is a Boolean σ-algebra $\mathcal{B}$ and a σ-homomorphism h from $\mathcal{L}_0$ onto $\mathcal{B}$ such that $m(a) = 0$ whenever $h(a) = 0$. Hence, if $F \subset \cup\{\mathcal{R}(x_t) : t \in T\}$ is a finite subset, then $h(a_1), \ldots, h(a_n)$ are compatible in $\mathcal{B}$, where $F = \{a_1, \ldots, a_n\}$. Therefore

$$m_a(\mathrm{com}^{\perp} F) = \bar{m}(h(a) \wedge h(\mathrm{com}^{\perp} F)) = 0,$$

where $\bar{m}(h(a)) := m(a)$, $a \in \mathcal{L}_0$, is a measure on $\mathcal{B}$.

(ii) Let $\{x_t : t \in T\}$ have a joint distribution in m. Analogically as in the first part we may prove that $h(\mathrm{com}^{\perp} F) = 0$ whenever F is a finite subset of $\mathcal{L}_0$. Hence $m(\mathrm{com}^{\perp} F) = 0$. Q.E.D.

We say that a measure m on a logic $\mathcal{L}$ has a Jauch-Piron property if $m(a) = m(b) = 0$ imply $m(a \vee b) = 0$.

Corollary 3.11.2. Let the assumptions of Theorem 3.9 hold. If $\{x_t : t \in T\}$ have a joint distribution in a measure m, then $m(\bigvee_{i=1}^{\infty} a_i) = 0$ whenever $m(a_i) = 0$, $a_i \in \mathcal{L}_0$, $i \geq 1$.

Proof. This is a consequence of Corollary 3.11.2 and the observation that for a measure $\bar{m}$, on $\mathcal{B}$ we have

$$\bar{m}(h(a) \vee h(b)) + \bar{m}(h(a) \wedge h(b)) = \bar{m}(h(a)) + \bar{m}(h(b)), \quad a, b \in \mathcal{L}_0$$

(this is a valuation property of $\bar{m}$ and m, respectively). Q.E.D.

4. JOINT DISTRIBUTIONS AND COMMUTATORS

We have seen that the cornerstone of the theory of a joint distribution of σ-observables in a measure is the commutator of observables. Although it may not exist, in general, for instance see [23] and in Theorem 3.9 it does not exhibit, it appears implicitely in partial steps of Theorem 3.9. In the present section we shall study some relationships between the existence of a joint distribution of observables and the existence of a commutator of observables.

First of all we remark that the following is true. Let x_t be an $\mathcal{A}_t$-σ-observable of a quantum logic $\mathcal{L}$, $t \in T$. Then

$$\wedge\{\mathrm{com}\, F : F \text{ finite subset of } \cup\{\mathcal{R}(x_t) : t \in T\}\} =$$

$$= \wedge\{\mathrm{com}(\{x_t(A_t) : t \in a\}) : (\forall A_t \in \mathcal{A}_t), (\forall t \in a), (\forall a \text{ finite subset of } T)\}. \qquad (4.1)$$

This is understood as follows: if one of the element in (4.1) exists in $\mathcal{L}$, the second one exists, too, and both are equal. This assertion may be proved similarly as Propositions 2.1 and 2.2 from [10].

Let $\emptyset \neq M \subset \mathcal{L}$, by $\mathcal{L}_0(M)$ we denote the minimal sublogic of $\mathcal{L}$ containing M.

Proposition 4.1. Let $\emptyset \neq M \subset \mathcal{L}$ and $\mathcal{J} = \mathcal{J}(M)$ be the σ-p-ideal of $\mathcal{L}_0(M)$ defined by (3.19). Then (i)

$$a_0^{\perp} = \vee\{x : x \in \mathcal{J}(M)\} \ (\text{in } \mathcal{L}). \qquad (4.2)$$

This means that if one of the elements in (4.2) exists in $\mathcal{L}$, then the second one also exists, and both are equal; here a_0 is the commutator of M.

(ii) The commutator of M, a_0, is countably obtainable if, and only if, $a_0^\perp \in \mathcal{J}(M)$.

Proof. (i) and (ii) follows immediately from the definitions of $\mathcal{J}(M)$ and a_0. Q.E.D.

Proposition 4.2. Let there be $a_0 = \text{com}\, M$ and let $a_0 \neq 0$. Let $\mathcal{L}_{a_0}$ be the minimal sublogic of a logic $\mathcal{L}(0, a_0)$ containing $\{a \wedge a_0 : a \in M\}$. Then $h_{a_0}: a \mapsto a \wedge a_0$, $a \in \mathcal{L}_0(M)$, is a σ-homomorphism of $\mathcal{L}_0(M)$ onto $\mathcal{L}_{a_0}$, and

$$\text{Ker}\, h_{a_0} \supset \mathcal{J}(M). \tag{4.3}$$

Proof. Since the set $K = \{a \in \mathcal{L}_0(M): a \leftrightarrow a_0\}$ is a sublogic of $\mathcal{L}_0(M)$, the map h_{a_0} is well defined and is a σ-homomorphism. Now we show that it transforms $\mathcal{L}_0(M)$ onto $\mathcal{L}_{a_0}$. Denote by $\mathcal{B} \doteq \{a \in \mathcal{L}_{a_0}$: there is $c \in \mathcal{L}_0(M)$ with $c \wedge a_0 = a\}$. Then $\mathcal{B}$ is a sublogic of $\mathcal{L}_{a_0}$ containing $\{a \wedge a_0 : a \in M\}$.

Using the result of Marsden [22] we can establish (4.3), because $\mathcal{L}_{a_0}$ is a Boolean σ-algebra. (4.3) follows also from a simple observation: $\text{Ker}\, h_{a_0} = \{b \in \mathcal{L}_0(M): b \perp a_0\}$. Q.E.D.

We say that an element a, $a \in \mathcal{L}$, is a carrier of a measure m if $m(b) = 0$ whenever $b \perp a$. It is clear that if a carrier exists, then it is unique.

Proposition 4.3. Let the assumptions of Theorem 3.9 be fulfilled, and let the commutator, a_0, of $\{x_t : t \in T\}$ exists in $\mathcal{L}$. If a is a carrier of m, then the following conditions are equivalent

(i) $\{x_t : t \in T\}$ have a joint distribution in m;

(ii) $m(a_0^\perp) = 0$;

(iii) $a < a_0$.

Proof. Using the properties of the carrier and the commutator and appealing (ii) or (iii) of Theorem 3.9, the equivalence may be proved. Q.E.D.

This result may be applied to an important case of quantum logics - to a logic of all closed subspaces, $\mathcal{L}(H)$, of a Hilbert space H whose dimension is a non-measurable cardinal. We recall that the set X has a non-measurable cardinal if there is no trivial measure ν on the power set 2^X such that $\nu(\{x\}) = 0$ for all $x \in X$.

Theorem 4.3. Let $\mathcal{L} = \mathcal{L}(H)$ be a quantum logic of a real or complex Hilbert space whose dimension is a non-measurable cardinal $\neq 2$. Let the assumptions of Theorem 3.9 be fulfilled. Then the following conditions are equivalent

(i) $\{x_t : t \in T\}$ have a joint distribution in m;

(ii) $m(a_0^\perp) = 0$;

(iii) $x_{t_{i_1}}(E_{i_1}) \dots x_{t_{i_n}}(E_{i_n}) f = x_{t_1}(E_1) \dots x_{t_n}(E_n) f$,

for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$, $n \geq 1$, any $E_i \in \mathcal{B}_{t_i}$, and finite $\emptyset \neq \alpha = \{t_1, \dots, t_n\} \subset T$ and any vector $f \in a$, where a is the carrier of a measure m.

Moreover, a Boolean σ-algebra in (vi) of Theorem 3.9 may be chosen as a Boolean sub-σ-algebra of a quantum logic of some Hilbert space.

Proof. Since $\mathcal{L}(H)$ is a complete lattice, the commutator a_0 of $\{x_t : t \in T\}$ always exists in $\mathcal{L}(H)$. According to [25], any σ-finite measure m on $\mathcal{L}(H)$ possesses a carrier which is a separable subspace of H. Proposition 4.3 yields the equivalence of (i) and (ii). The equivalence of (i) and (iii) is a simple modification of the results in [12,25].

The last assertion follows from Proposition 4.2 and (4.3). Moreover, we note that $x_{t_0}: E \mapsto x_t(E) \wedge a_0$ is an $\mathcal{B}_t$-σ-observable of $\mathcal{L}(a_0)$ and $\{x_{t_0} : t \in T\}$ are mutually compatible. Q.E.D.

We finish this section with the following remark. If the commutator a_0 of $\{x_t : t \in T\}$ exists and (3.8) holds, then $\{x_t : t \in T\}$ have a joint distribution in m. The converse implication is known only for special cases, for example, if a_0 is countably obtainable or m has a carrier or $a = 1$. Therefore it would be of interest to exhibit the conditions when (i) and (ii) of Proposition 4.3 are equivalent.

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Заметка о совместном распределении в квантовых логиках.
Некомпатибильные наблюдаемые

Предлагаемая работа является продолжением первой части работы с тем же названием. Изучаются совместные распределения в σ-конечных мерах для некомпатибильных наблюдаемых квантовой логики - аксиоматической модели квантовой механики, определенных на произвольной системе σ-независимых булевых σ-подалгебрах булевой σ-алгебры. Предложены некоторые необходимые и достаточные условия для существования совместного распределения. В частности показано, что любая система наблюдаемых имеет совместное распределение тогда и только тогда, когда она может быть внедрена в систему компатибильных наблюдаемых некоторой квантовой логики. Использованные методы отличаются от методов, известных для конечных мер. В конце работы исследуется соотношение между существованием совместного распределения и существованием коммутатора наблюдаемых, а также упоминается квантовая логика несепарабельного гильбертова пространства.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Remark on Joint Distribution in Quantum Logics.
Noncompatible Observables

This paper is a continuation of the first part under the same title. We study a joint distribution in σ-finite measures for noncompatible observables of a quantum logic - an axiomatic model of quantum mechanics - defined on an arbitrary system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra. We present some necessary and sufficient conditions for the existence of a joint distribution. In particular, it is shown that an arbitrary system of observables has a joint distribution in a measure iff it may be embedded into a system of compatible observables of some quantum logic. The used methods are different of those known for finite measures. Finally, we deal with a connection between the existence of a joint distribution and the existence of a commutator of observables, and the quantum logic of a nonseparable Hilbert space is mentioned.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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