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REMARK ON JOINT DISTRIBUTION IN QUANTUM LOGICS.
Compatible Observables

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This and subsequent papers are devoted to the notion of a joint distribution of observables in a σ-finite measure on a quantum logic for a given system of observables defined on an arbitrary collection of σ-independent Boolean sub-σ-algebras.

In this paper we study the problem of existence of a joint distribution for mutually compatible observables in a measure. It is shown that in this case the joint distribution in a measure always exists, however a joint observable may not exist.

We postpone a detailed study of the existence of a joint distribution in a measure for noncompatible observables to a subsequent paper.

1. PRELIMINARIES

Assume that the set, $\mathcal{L}$, of all experimentally verifiable propositions of a physical system forms a quantum logic. So, we suppose, according to /1/, that $\mathcal{L}$ is a σ-lattice with the first and the last elements 0 and 1, respectively, with an orthocomplementation $\perp: a \to a^{\perp}, a, a^{\perp} \in \mathcal{L}$, which satisfies: (i) $(a^{\perp})^{\perp} = a$ for any $a \in \mathcal{L}$; (ii) if $a < b$, then $b^{\perp} < a^{\perp}$; (iii) $a \vee a^{\perp} = 1$ for any $a \in \mathcal{L}$; (iv) if $a < b$, then $b = a \vee (b \wedge a^{\perp})$ (the orthomodular law).

In particular, it is of interest also the notion of an orthomodular lattice (OML in abbreviation), this is, a lattice $\mathcal{L}$ with (i)-(iv) above.

Two elements $a, b \in \mathcal{L}$ are (i) orthogonal, and we write $a \perp b$, if $a < b^{\perp}$; (ii) compatible, and we write $a \leftrightarrow b$ if there are three mutually orthogonal elements $a_1, b_1, c \in \mathcal{L}$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. It is known that $a \leftrightarrow b$ iff $a = (a \wedge b) \vee (a \wedge b^{\perp})$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics. A map $h: \mathcal{L}_1 \to \mathcal{L}_2$ is called a σ-homomorphism of $\mathcal{L}_1$ into $\mathcal{L}_2$ if (i) $h(1) = 1$; (ii) $h(a) \perp h(b)$ whenever $a \perp b$, $a, b \in \mathcal{L}_1$; (iii) $h(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} h(a_i)$ for any $\{a_i\}_{i=1}^{\infty} \subset \mathcal{L}_1$, $a_i \perp a_j$, $i \neq j$. A kernel of a σ-homomorphism is the set $\mathrm{Ker}\, h := \{a \in \mathcal{L}_1 : h(a) = 0\}$.

An OML $\mathcal{L}$ (logic $\mathcal{L}$) is called a Boolean algebra (Boolean σ-algebra) if the distributive law holds on $\mathcal{L}$, that is, for all $a, b, c \in \mathcal{L}$ $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. Due to [1, Corollary 6.15] the notion of a Boolean algebra (σ-algebra) coincides with that in [2]. The notions of sub OML, sublogic, subalgebra and sub-σ-algebra of σ are defined in a straightforward way, see [1, 2, 3], for instance.

Physical quantities of physical systems are identified with the observables of a quantum logic. Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{L}$ be an OML. We say that a map $x: \mathfrak{A} \to \mathfrak{L}$ is an $\mathfrak{A}$-observable of $\mathfrak{L}$ if (i) $x(1) = 1$; (ii) $x(E) \perp x(F)$ whenever $E \wedge F = 0$, $E, F \in \mathfrak{A}$ (iii) $x(E \vee F) = x(E) \vee x(F)$ if $E \wedge F = 0$, $E, F \in \mathfrak{A}$. If $\mathfrak{A}$ is a Boolean σ-algebra and $\mathfrak{L}$ is a quantum logic, then an $\mathfrak{A}$-observable x of $\mathfrak{L}$ is called an $\mathfrak{A}$-σ-observable of $\mathfrak{L}$ if $x(\bigvee_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ for any $\{E_i\}_{i=1}^{\infty} \subset \mathfrak{A}$, $E_i \wedge E_j = 0$, $i \neq j$.
(Shortly observable, σ-observable, respectively, if $\mathfrak{A}$ is specified.)

For the quantum mechanics is of a great importance the case when $\mathfrak{A}$ is a Boolean (σ-) algebra of subsets of a set X, in particular, when $X = R^1$ and $\mathfrak{A} = \mathfrak{B}(R^1)$ is a Borel σ-algebra of subsets of the real line R^1.

The range of an $\mathfrak{A}$-(σ-) observable x, $\mathfrak{R}(x) := \{x(E) : E \in \mathfrak{A}\}$, is a Boolean sub-($\sigma$-) algebra of $\mathfrak{L}$. A Boolean σ-algebra $\mathfrak{B}$ is separable if it is generated by countably many elements. A $\mathfrak{B}$ is separable sub-σ-algebra of $\mathfrak{L}$ iff there is a $\mathfrak{B}(R^1)$ - σ -observable x such that $\mathfrak{B} = \mathfrak{R}(x)$ [[1], Lemma 6.16].

An $\mathfrak{A}$-observable x and a $\mathfrak{B}$-observable y are compatible if $x(E) \leftrightarrow y(F)$ for any $E \in \mathfrak{A}$, $F \in \mathfrak{B}$. It is known [[1], Lemma 6.14, Corollary 6.15] that if x_t is an $\mathfrak{A}_t$ -(σ-)observable of $\mathfrak{L}$ and $\{x_t : t \in T\}$ are mutually compatible observables, then there is a Boolean sub-(σ-) algebra of $\mathfrak{L}$ containing all ranges $\mathfrak{R}(x_t)$, $t \in T$.

Physical states we shall identify with measures. A map $m: \mathfrak{L} \to [0, \infty]$ is a measure if (i) $m(0) = 0$; (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$, $i \neq j$. A measure m is (i) finite if $m(1) < \infty$; (ii) a state if $m(1) = 1$; (iii) σ-finite if there is a sequence of mutually orthogonal elements of $\mathfrak{L}$, $\{a_i\}_{i=1}^{\infty}$, such that $\bigvee_{i=1}^{\infty} a_i = 1$ and $m(a_i) < \infty$, for any $i \geq 1$. An observable x is σ-finite with respect to m if there is a sequence $\{E_i\}_{i=1}^{\infty} \subset \mathfrak{A}$ such that $E_i \wedge E_j = 0$ whenever $i \neq j$, $\bigvee_{i=1}^{\infty} E_i = 1$ and $m(x(E_i)) < \infty$, $i \geq 1$.

We say that a system, $\{\mathfrak{A}_t : t \in T\}$, of Boolean sub-($\sigma$-)algebras of a Boolean (σ-)algebra $\mathfrak{A}$ is independent (σ-independent) if for any finite (countable) subset $\alpha \subset T$

$$\bigwedge_{t \in \alpha} A_t \neq 0, \tag{1.1}$$

for any $0 \neq A_t \in \mathfrak{A}_t$, and any $t \in \alpha$.

For example, let $(X_t, \mathcal{S}_t)$, $t \in T$, be a measure space, that is, $\mathcal{S}_t$ is a (σ-) algebra of subsets of a set $X_t \neq \emptyset$. Denote by X the Cartesian product of all spaces X_t, i.e., the set of all $\omega = \{\omega_t : t \in T\}$, $\omega_t \in X_t$ for $t \in T$. Let π_t be the t-th projection function of X onto X_t, that is $\pi_t \omega = \omega_t$, $\omega \in X$. Let $\mathcal{S}_t^* = \{\pi_t^{-1}(A) : A \in \mathcal{S}_t\}$, $t \in T$. Then $\mathcal{S}_t$ is (σ-) isomorphic to $\mathcal{S}_t^*$. The minimal sub-(σ-) algebra of X generated by all $\mathcal{S}_t^*$ is denoted by $\mathcal{S} = \prod_{t \in T} \mathcal{S}_t$, and the system $\{\mathcal{S}_t^* : t \in T\}$ of Boolean sub-(σ-) algebras of $\mathcal{S}$ is (σ-) independent [2].

Let $\{\mathfrak{A}_t : t \in T\}$ be a system of (σ-) independent Boolean sub-(σ-) algebras of a Boolean (σ-) algebra $\mathfrak{A}$. Denote by $\mathfrak{I}$ the system of all Boolean rectangles $\bigwedge_{t \in \alpha} A_t$ defined for any $A_t \in \mathfrak{A}_t$, $t \in \alpha$, and each finite $\alpha \subset T$. As in the Cartesian product of (σ-) algebras of subsets of X_t, one may verify the minimal subalgebra, $\mathfrak{R}$, of $\mathfrak{A}$, generated by all $\mathfrak{A}_t$, $t \in T$, consists of all finite joins of orthogonal elements from $\mathfrak{I}$. The minimal sub-σ-algebra of $\mathfrak{A}$ generated by all sub-σ-algebras $\{\mathfrak{A}_t : t \in T\}$ is denoted by $\prod_{t \in T} \mathfrak{A}_t$.

2. JOINT DISTRIBUTION OF COMPATIBLE OBSERVABLES

One of important problems of the quantum logic theory is a determination of a joint distribution for noncompatible observables, as it is indicated in [[4], Problem VII]. Following to Gudder [5] we give the next generalization of the notion of the joint distribution.

Definition. Let m be a measure on a quantum logic $\mathfrak{L}$. We are said that (i) a finite system $x_1, \ldots, x_n$, where x_i is an $\mathfrak{A}_i$ -σ-observable of $\mathfrak{L}$, $i = 1, \ldots, n$, and $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathfrak{A}$, has a joint distribution in m if there is a measure μ on the minimal Boolean sub-σ-algebra $\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ of $\mathfrak{A}$ generated by $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ such that

$$\mu(\bigwedge_{i=1}^{n} A_i) = m(\bigwedge_{i=1}^{n} x_i(A_i)), \tag{2.0}$$

for any $A_i \in \mathfrak{A}_i$, $i = 1, \ldots, n$;

(ii) an infinite system $\{x_t : t \in T\}$, where x_t is an $\mathfrak{A}_t$ -σ-observable, $t \in T$, and $\{\mathfrak{A}_t : t \in T\}$ are σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathfrak{A}$, has a joint distribution in an m if $\{x_t : t \in \alpha\}$ has a joint distribution in m for any finite $\alpha \subset T$.

S.P. Gudder introduced the notion of a joint distribution only for $\mathcal{B}(R^1)$-σ-observable and states. The necessary and sufficient conditions for the existence of a joint distribution for $\mathcal{B}(R^1)$-σ-observables in a given state may be found in [5-11]. The case of σ-finite measure, including also a logic $\mathcal{L} = \mathcal{L}(H)$ of a separable Hilbert space H, is investigated in [12].

It is known [6,7] that the existence of a joint distribution in a measure closely depends on mutually compatible σ-observables of some quantum logic. Therefore in this section we concentrate ourselves to the study of a joint distribution of mutually compatible observables.

Lemma 2.1. If x_i, $i \geq 1$, are mutually compatible $\mathcal{B}(X)$-σ-observables of a quantum logic $\mathcal{L}$, where X is a separable Banach space and $\mathcal{B}(X)$ is the Borel σ-algebra of subsets of X, then there is a unique $\prod_{i=1}^{\infty} \mathcal{B}(X)$-$\sigma$-observable x of $\mathcal{L}$, with

$$x\Big(\bigcap_{i\in\alpha} \pi_i^{-1}(E_i)\Big) = \bigwedge_{i\in\alpha} x_i(E_i)$$

for any $E_i \in \mathcal{B}(X)$, $i \in \alpha$, and any finite subset α of $\{1,2,\dots\}$. Here π_i denotes the i-th projection function from $\prod_{i=1}^{\infty} X$ onto X.

Proof. According to P.Pták [13] there is a $\mathcal{B}(X)$-σ-observable z of $\mathcal{L}$ and the Borel measurable functions $f_n : X \to X$ such that $x_n(A) = z(f_n^{-1}(A))$ for any $A \in \mathcal{B}(X)$. Define $f(t) = (f_1(t), f_2(t), \dots) : X \to \prod_{i=1}^{\infty} X$. Then $z: B \to x(f^{-1}(B))$, $B \in \prod_{i=1}^{\infty} \mathcal{B}(X)$ is the σ-observable in question. Q.E.D.

Theorem 2.2. Let $\{\mathcal{A}_t : t \in T\}$ be a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$. Let x_t be an $\mathcal{A}_t$-σ-observable of a logic $\mathcal{L}$, $t \in T$. If $\{x_t : t \in T\}$ are mutually compatible observables and at least one of them is σ-finite with respect to m, then $\{x_t : t \in T\}$ have a joint distribution in m. Moreover, there is a unique σ-finite measure μ on $\prod_{t\in T} \mathcal{A}_t$ with

$$\mu\Big(\bigwedge_{t\in\alpha} A_t\Big) = m\Big(\bigwedge_{t\in\alpha} x_t(A_t)\Big), \qquad (2.1)$$

for any $A_t \in \mathcal{A}_t$ and any finite subset $\emptyset \neq \alpha \subset T$.

Proof. (i) First of all we show that if x_t is an $\mathcal{A}_t$-observable of $\mathcal{L}$, $t \in T$, where $\{\mathcal{A}_t : t \in T\}$ is a system of independent Boolean subalgebra of a Boolean algebra $\mathcal{A}$, then there is a unique $\mathcal{R}$-observable of $\mathcal{L}$, x, such that

$$x\Big(\bigwedge_{t\in\alpha} A_t\Big) = \bigwedge_{t\in\alpha} x_t(A_t), \qquad (2.2)$$

for any $A_t \in \mathcal{A}_t$ and any finite subset $\alpha \subset T$. Here $\mathcal{R}$ denotes the minimal Boolean subalgebra of $\mathcal{A}$ containing all $\mathcal{A}_t$, $t \in T$.

Taking into account the simple observation that any two Boolean rectangles $\bigwedge_{t\in\alpha} A_t$ and $\bigwedge_{s\in\beta} B_s$ can be assumed on the same finite index subset $\alpha \cup \beta$. Indeed, if we put $A_t^* = A_t$ if $t \in \alpha$, $A_t^* = 1$ if $t \in \beta - \alpha$, and $B_t^* = B_t$, if $t \in \beta - \alpha$, $B_t^* = 1$, $t \in \alpha$, then $\bigwedge_{t\in\alpha} A_t = \{A_t^* : t \in \alpha \cup \beta\}$, $\bigwedge_{s\in\beta} B_s = \wedge\{B_t^* : t \in \alpha \cup \beta\}$. Therefore (i) $\bigwedge_{t\in\alpha} A_t = 0$, $A_t \in \mathcal{A}_t$, $t \in \alpha$, iff at least one $A_t = 0$; (ii) $0 \neq \bigwedge_{t\in\alpha} A_t < \bigwedge_{t\in\alpha} B_t$ iff $A_t < B_t$ for any $t \in \alpha$; (iii) $0 \neq \bigwedge_{t\in\alpha} A_t = \bigwedge_{t\in\alpha} B_t$ iff $A_t = B_t$ for any $t \in \alpha$.

Hence, the map x defined via (2.2) is well defined on the set $\mathcal{D}$ of all Boolean rectangles. Using the remark on the form of the minimal subalgebra, $\mathcal{R}$, containing all $\mathcal{A}_t$, $t \in T$, and the fact that there is a Boolean subalgebra of $\mathcal{L}$ containing all ranges $\mathcal{R}(x_t)$ [1], x may be uniquely extended to an $\mathcal{R}$-observable of $\mathcal{L}$. The uniqueness of x follows from (2.2).

(ii) Now we show that if x_t is an $\mathcal{A}_t$-σ-observable of a logic $\mathcal{L}$ and $\{\mathcal{A}_t : t \in T\}$ are σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$, then, for an $\mathcal{R}$-observable x of $\mathcal{L}$ we have: if $A_n \in \mathcal{R}$, $n \geq 1$, and $A = \bigvee_{n=1}^{\infty} A_n \in \mathcal{R}$, then $x(A) = \bigvee_{n=1}^{\infty} x(A_n)$.

To prove this it is necessary only to show that if $A \in \mathcal{D}$ and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{D}$, $A_n \wedge A_m = 0$ whenever $n \neq m$, $A = \bigvee_{n=1}^{\infty} A_n$, then

$$x(A) = \bigvee_{n=1}^{\infty} x(A_n). \qquad (2.3)$$

Let us put $A_0 = A$. For any $n \geq 0$, there is a finite subset α_n of T such that $A_n = \wedge\{A_t^n : t \in \alpha_n\}$, where $A_t^n \in \mathcal{A}_t$. Without loss of generality we can assume that $\alpha_0 \subset \alpha_1 \subset \dots$, so that, there is a set of nondecreasing indices $\{k_i\}_{i=0}^{\infty}$ with $1 \leq k_0 \leq k_1 \leq \leq \dots$ such that $\alpha_n = \{t_1, t_2, \dots, t_{k_n}\}$, $n = 0, 1, 2, \dots$. Denote by $\mathcal{A}_k$, $k \geq 1$, the minimal Boolean sub-σ-algebra of $\mathcal{A}_{t_k}$ generated by $\{A^0_{t_k}, A^1_{t_k}, \dots\}$. Clearly, $\{\bar{\mathcal{A}}_k\}_{k=1}^{\infty}$ are σ-independent Boolean sub-σ-algebras of $\mathcal{A}$. Due to the Loomis-Sikorski theorem [14,15] for any $k = 1, 2, \dots$, there is a measurable space $(X_k, \mathcal{S}_k)$ and a σ-homomorphism h_k of $\mathcal{S}_k$ onto $\mathcal{A}_k$. Let $\{B^0_k, B^1_k, \dots\}$ be a countable system of subsets of $\mathcal{S}_k$ such that $h_k(B^n_k) = A^n_k$, $n \geq 0$. Denote by $\mathcal{S}_{k0}$ the minimal sub-σ-algebra of $\mathcal{S}_k$ generated by $\{B_k^0, B_k^1, \ldots\}$. The function $c_k: x \to (I_{B_k^0}(x), I_{B_k^1}(x), \ldots)$, $x \in X_k$, (where I_A is the indicator function of a set A) is $\mathcal{S}_{ko}$-measurable from X_k into the compact metric space $Y = \prod_{i=1}^{\infty}\{0,1\}$. It is clear that $\mathcal{S}_{ko} = \{c_k^{-1}(F): F$ Borel subset in $Y\}$. Due to Kuratowski /16/, there is a Borel isomorphism d of Y onto R^1 such that $d_k: x \to d(c_k(x))$, $x \in X_k$, is an $\mathcal{S}_{ko}$-measurable real-valued function, and $\mathcal{S}_{ko} = \{d_k^{-1}(E): E \in \mathcal{B}(R^1)\}$.

Define $\bar{x}_k := x_{t_k} \mid \mathcal{A}_k$. The maps $\hat{x}_k: E \to \bar{x}_k(h_k(d_k^{-1}(E)))$, $E \in \mathcal{B}(R^1)$, $k \geq 1$, are compatible $\mathcal{B}(R^1)$-σ-observables of $\mathcal{L}$. Therefore from Lemma 2.1 we have that there is a unique $\mathcal{B}(R^\infty)$-σ-observable $\hat{x}$ of $\mathcal{L}$ such that

$$\hat{x}\Big(\bigcap_{i\in\alpha}\pi_i^{-1}(E_i)\Big) = \bigwedge_{i\in\alpha}\hat{x}_i(E_i), \tag{2.4}$$

for any $E_i \in \mathcal{B}(R^1)$, $i \in \alpha$, and any finite $\alpha \subset \{1,2,\ldots\}$, where $\pi_i: R^\infty \to R^1$ is the i-the projection function.

Analogically, there is a unique $\mathcal{B}(R^\infty)$-σ-observable of $\mathcal{A}$, h, where $\mathcal{B}(R^\infty) = \prod_{i=1}^{\infty}\mathcal{B}(R^1)$, such that

$$h\Big(\bigcap_{i\in\alpha}\pi_i^{-1}(E_i)\Big) = \bigwedge_{i\in\alpha} h_i(d_i^{-1}(E_i)), \tag{2.5}$$

for any $E_i \in \mathcal{B}(R^1)$, $i \in \alpha$, and any finite subset α of $\{1,2,\ldots\}$.

Now we claim to show that $(\bar{x}\circ h)(E) = (x\circ h)(E) = \hat{x}(E)$ for all cylindrical subsets E of $\mathcal{B}(R^\infty)$, where $\bar{x}$ is a unique $\mathcal{A}_0$-observable of $\mathcal{L}$ determined by (2.2), and $\mathcal{A}_0$ is the minimal Boolean sub-σ-algebra of $\mathcal{A}$ containing all $\mathcal{A}_{t_k}$. Indeed, let α be a finite subset of $\{1,2,\ldots\}$. Then

$$\bar{x}\Big(h\Big(\bigcap_{i\in\alpha}\pi_i^{-1}(E_i)\Big)\Big) = \bar{x}\Big(\bigwedge_{i\in\alpha} h(\pi_i^{-1}(E_i))\Big) = \bar{x}\Big(\bigwedge_{i\in\alpha} h_i(d_i^{-1}(E_i))\Big) =$$

$$= \bigwedge_{i\in\alpha}\bar{x}_i(h_i(d_i^{-1}(E_i))) = \bigwedge_{i\in\alpha}\hat{x}_i(E_i) = \hat{x}\Big(\bigcap_{i\in\alpha}\pi_i^{-1}(E_i)\Big).$$

Moreover, we see that the $\mathcal{B}(R^\infty)$-σ-observable $\hat{x}$ of $\mathcal{L}$ is a unique extension of an $\mathcal{R}(R^\infty)$-observable $\bar{x}\circ h$ of $\mathcal{L}$ to a $\mathcal{B}(R^\infty)$-σ-observable, where $\mathcal{R}(R^\infty)$ is the minimal subalgebra of subsets of R^∞ containing all measurable rectangles.

To prove (2.3) we choose the Borel subsets E_k^n of $\mathcal{B}(R^1)$ such that $A_k^n = h_k(d_k^{-1}(E_k^n))$, $n \geq 0$, $k \geq 1$. Then

$$x\Big(\bigvee_{n=1}^{\infty} A_n\Big) = x\Big(\bigvee_{n=1}^{\infty}\bigwedge_{i=1}^{k_n} A_i^n\Big) = x\Big(\bigvee_{n=1}^{\infty}\bigwedge_{i=1}^{k_n} h_i(d_i^{-1}(E_i^n))\Big) =$$

$$= x\Big(\bigvee_{n=1}^{\infty} h\Big(\bigcap_{i=1}^{k_n}\pi_i^{-1}(E_i^n)\Big)\Big) = x\Big(h\Big(\bigcup_{n=1}^{\infty}\bigcap_{i=1}^{k_n}\pi_i^{-1}(E_i^n)\Big)\Big) =$$

$$= \hat{x}\Big(\bigcup_{n=1}^{\infty}\bigwedge_{i=1}^{k_n}\pi_i^{-1}(E_i^n)\Big) = \bigvee_{n=1}^{\infty}\hat{x}\Big(\bigcap_{i=1}^{k_n}\pi_i^{-1}(E_i^n)\Big) = \bigvee_{n=1}^{\infty}\bigwedge_{i=1}^{k_n}\hat{x}_i(E_i^n) =$$

$$= \bigvee_{n=1}^{\infty}\bigwedge_{i=1}^{k_n}\bar{x}_i(h_i(d_i^{-1}(E_i^n))) = \bigvee_{n=1}^{\infty}\bigwedge_{i=1}^{k_n}\bar{x}_i(A_i^n) = \bigvee_{n=1}^{\infty}\bar{x}(A_n) = \bigvee_{n=1}^{\infty}x(A_n),$$

and, consequently, (2.3) is proved for $A, A_n \in \mathcal{D}$, $n \geq 1$.

Let now $A, A_1, A_2, \ldots \in \mathcal{R}$, $A_n \wedge A_m = 0$ whenever $n \neq m$, and let $A = \bigvee_{n=1}^{\infty} A_n$. The simple usage of the just established property on $\mathcal{D}$ yields $x(A) = \bigvee_{n=1}^{\infty} x(A_n)$. For the general case, let $A, A_1, A_2, \ldots \in \mathcal{R}$ with $A = \bigvee_{n=1}^{\infty} A_n$ be given. Define $B_1 = A_1$, $B_n = A_n \wedge (\bigvee_{i=1}^{n-1} A_i)^\perp$, for $n \geq 2$. Then $B_n \wedge B_m = 0$ if $n \neq m$, and $B_1 \vee \ldots \vee B_n = A_1 \vee \ldots \vee A_n$. Hence, $x(A) = \bigvee_{n=1}^{\infty} x(B_n) = \bigvee_{n=1}^{\infty}\bigvee_{i=1}^{n} x(B_i) = \bigvee_{n=1}^{\infty} x(A_n)$.

We note that for the existence of x with (2.2) and (2.3) we do not need the existence of a measure on $\mathcal{L}$.

(iii) Let x be the $\mathcal{R}$-observable on $\mathcal{L}$ guaranted by the first part of the present proof. Let m be a measure on $\mathcal{L}$ fulfilling the conditions of Theorem. Then, due to (i) and (ii), $\mu(A) := m(x(A))$, $A \in \mathcal{R}$, is a σ-finite σ-additive measure on $\mathcal{R}$. Using the familiar Carathéodory extension method concerning with the extension of a σ-additive σ-finite set function defined on an algebra of subsets to a measure defined on the minimal σ-algebra generated by the algebra, /17/, we may obtain the analoguous result also for Boolean subalgebra $\mathcal{R}$ and $\prod_{t\in T}\mathcal{A}_t$ /18/. It is clear that μ is the joint distribution of $\{x_t: t \in T\}$ in m, and the proof is complete. Q.E.D.

We note that for the $\mathcal{R}$-observable x of $\mathcal{L}$ with (2.2) one may exist no extension of x to a $\prod_{t\in T}\mathcal{A}_t$-$\sigma$-observable of $\mathcal{L}$. To establish this interesting fact, we need the following notions. Let $\mathcal{A}$ be a Boolean σ-algebra. The non-empty subset $\mathcal{J} \subset \mathcal{A}$ is said to be a σ-ideal if (i) $A_n \in \mathcal{J}$, $n \geq 1$, then $\bigvee_{n=1}^{\infty} A_n \in \mathcal{J}$; (ii) if $A < B$ and $B \in \mathcal{J}$ then $A \in \mathcal{J}$. The factor σ-algebra, $\mathcal{A}/\mathcal{J}$ is the system of all $[A]_{\mathcal{J}} := \{B \in \mathcal{A}: B \wedge A^\perp \vee A \wedge B^\perp \in \mathcal{J}\}$, $A \in \mathcal{A}$. The Boolean operations in $\mathcal{A}/\mathcal{J}$ are defined via $[A]_{\mathcal{J}} \vee [B]_{\mathcal{J}} := [A \vee B]_{\mathcal{J}}$, $[A]_{\mathcal{J}}^\perp := [A^\perp]_{\mathcal{J}}$.

The next result is a simple consequence of the last Theorem (see Preliminaries).

Corollary 2.2.1. Let x_t be an $\mathcal{S}_t$-σ-observable of a quantum logic $\mathcal{L}$, $t \in T$, and let $x_t \leftrightarrow x_s$ for any $s, t \in T$, where $\mathcal{S}_t$ is a σ-algebra of subsets of a set X_t. If at least one of x_t's is σ-finite with respect to m, then $\{x_t: t \in T\}$ has a joint distribution in m, and there is a unique σ-finite measure μ on $\prod_{t \in T} \mathcal{S}_t$ with

$$\mu(\bigcap_{t \in \alpha} \pi_t^{-1}(E_t)) = m(\bigwedge_{t \in \alpha} x_t(E_t))$$

for any $E_t \in \mathcal{S}_t$, $t \in \alpha$, and any finite subset $\emptyset \neq \alpha \subset T$.

Example 1. There is a quantum logic $\mathcal{L}$ with a non-empty set of states (even with two-valued states), and with two compatible σ-observables $x_i: \mathcal{S}_i \to \mathcal{L}$, where $\mathcal{S}_i$ is a separable σ-algebra of subsets of a set X_i, $i = 1, 2$, such that there is no $\mathcal{S}_1 \times \mathcal{S}_2$ - σ-observable x of $\mathcal{L}$ with

$$x(E \times F) = x_1(E) \wedge x_2(F), \quad E \in \mathcal{S}_1, \quad F \in \mathcal{S}_2 . \tag{2.6}$$

On the other hand, x_1 and x_2 have a joint distribution in any σ-finite measure m on $\mathcal{L}$.

Proof. Let C be some analytic subset of R^1 which is not a Borel set. Let $X_1 = R^1 - C$, $X_2 = R^1$ and $\mathcal{S}_1: = \mathcal{B}(R^1) \cap (R_1 - C) : = \{B \cap C: B \in \mathcal{B}(R^1)\}$, $\mathcal{S}_2: = \mathcal{B}(R^1)$. It is clear that $\mathcal{S}_1$ and $\mathcal{S}_2$ are separable σ-algebras of subsets, i.e., they contain generators with countably many elements. Denote by $\mathcal{J}'$ the σ-ideal of the Borel σ-algebra $\mathcal{B}(R^2)$ of the real plane R^2 generated by all sets $B \times R^1$, where $B \in \mathcal{B}(R^1)$ and $B \subset C$. Let us put $\mathcal{L} = \mathcal{B}(R^2)/\mathcal{J}'$. formulae

$$x_1(B \cap X_1): = [B \times R^1]_{\mathcal{J}'}, \quad B \in \mathcal{B}(R^1),$$

$$x_2(B): = [R^1 \times B]_{\mathcal{J}'}, \quad B \in \mathcal{B}(R^1),$$

determine two compatible σ-observables $x_i: \mathcal{S}_i \to \mathcal{L}$, $i = 1,2$. Moreover, x_i is a σ-isomorphism of $\mathcal{S}_i$ into $\mathcal{L}$. As it has been shown in /18, p.17; 2, § 37, Example A /, there is no $\mathcal{S}_1 \times \mathcal{S}_2$ - σ-observable of $\mathcal{L}$ with (2.6).

Now we claim to prove the second part of the proposition. Define the σ-ideal of $\mathcal{B}(R^2)$, $\mathcal{J}$, as follows: $\mathcal{J} = \{A \in \mathcal{B}(R^2): A \subset C \times R^1\}$. It is obvious that $\mathcal{J}' \subset \mathcal{J}$. We show that $\mathcal{J}'$ is a proper subset of $\mathcal{J}$. If it was $\mathcal{J}' = \mathcal{J}$, then $\mathcal{B}(R^2)/\mathcal{J}$ would be σ-isomorphic to $\mathcal{S} = \mathcal{B}(R^2) \cap ((R^1 - C) \times R^1): = \{B \cap (R^1 - C) \times R^1: B \in \mathcal{B}(R^2)\}$ (a σ-isomorphism, h, of $\mathcal{B}(R^2)/\mathcal{J}$ onto $\mathcal{S}$ is defined by $h(B \cap ((R^1 - C) \times R^1)) = [B]_{\mathcal{J}}$ for any $B \in \mathcal{B}(R^2)$). Consequently, $\mathcal{L}$ possesses the strong σ-extension property (for definition see below or /2/) and, therefore, there is an x with (2.6) which contradicts the first part of the proof.

Now we define an $\mathcal{L}$-σ-observable, z, of a quantum logic $\mathcal{L}_1: = \mathcal{B}(R_2)/\mathcal{J}$ via $z([A]_{\mathcal{J}'}) = [A]_{\mathcal{J}}$, $A \in \mathcal{B}(R^2)$. The z is defined well, because if $[A_1]_{\mathcal{J}'} = [A_2]_{\mathcal{J}'}$, then $A_1 \wedge A_2^{\perp} \vee A_2 \wedge A_1^{\perp} \in \mathcal{J}' \subset \mathcal{J}$ and $[A_1]_{\mathcal{J}} = [A_2]_{\mathcal{J}}$.

The logic $\mathcal{L}_1$ is σ-isomorphic to the σ-algebra of subsets, $\mathcal{B}(R^2)$ $((R^1 - C) \times R^1)$, hence, $\mathcal{L}_1$ has an order determining system of states (and also an order determining system of two-valued states.) (We recall that a system $\mathcal{M}$ of states on some quantum logic is order determining if $m(a) \leq m(b)$ for any $m \in \mathcal{M}$ iff $a < b$).

Let m be a measure on $\mathcal{L}_1$, then $\bar{m}: a \to m(z(a))$, $a \in \mathcal{L}$, is a measure on $\mathcal{L}$. Let now m be a σ-finite measure on $\mathcal{L}$ and let $\bigvee_{i=1}^{\infty} a_i = 1$, $a_i \perp a_j$ whenever $i \neq j$, $a_i \in \mathcal{L}$, $0 < m(a_i) < \infty$, $i \geq 1$.) Then $m_i(a) = m(a \wedge a_i)$, $a \in \mathcal{L}$, is a finite measure for any $i \geq 1$. Using the result of Corollary 2.2.1 we see that x_1, x_2 have a joint distribution in any m_i, $i \geq 1$, consequently, in $m = \sum_{i=1}^{\infty} m_i$. Q.E.D.

Motivating the above we say that mutually compatible σ-observables $x_t: \mathcal{A}_t \to \mathcal{L}$ of a quantum logic $\mathcal{L}$, $t \in T$, where $\{\mathcal{A}_t: t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$ has a joint σ-observable if there is a $\prod_{t \in T} \mathcal{A}_t$ - σ-observable x of $\mathcal{L}$ with (2.2).

Lemma 2.1 determines an important class of compatible observables which has a joint σ-observable. According to /2/, we say that a Boolean σ-algebra $\mathcal{A}'$ has the strong σ-extension property if, for every Boolean σ-algebra $\mathcal{A}$, every map f (from a set $\mathcal{G}$ σ-generating $\mathcal{A}$) into $\mathcal{A}'$ satisfying the following

$$\text{if} \quad \bigwedge_{i=1}^{\infty} E_i^{\epsilon(i)} = 0, \quad \text{then} \quad \bigwedge_{i=1}^{\infty} f(E_i)^{\epsilon(i)} = 0, \tag{2.7.}$$

for every sequence $\{E_i\}_{i=1}^{\infty} \subset \mathcal{G}$, and for every function $\epsilon(i) \in \{0,1\}$, $i \geq 1$, can be extended to a σ-homomorphism h from $\mathcal{A}$ into $\mathcal{A}'$; here $E^0: = E$, $E^1: = E$.

Theorem 2.3. Let $x_t: \mathcal{A}_t \to \mathcal{L}$, $t \in T$, be compatible σ-observables of a quantum logic $\mathcal{L}$, where $\{\mathcal{A}_t: t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathcal{A}$, and let the minimal sub-σ-algebra of $\mathcal{A}$ generated by all ranges $\mathcal{R}(x_t)$, $t \in T$, have the strong σ-extension property (in particular, it is σ-isomorphic to some σ-algebra of subsets). Then $\{x_t: t \in T\}$ has a joint σ-observable of $\mathcal{L}$.

Proof. It follows immediately from /2, Theorem 37.1 /. Q.E.D.

It may be interesting in the frame of the study of a joint σ-observable of compatible observables, in particular, in a connection with Lemma 2.1, to note that P.Pták [13] found the example of a quantum logic with two compatible $\mathcal{B}(X)$-σ-observables x and y such that the equalities $x = z\circ f^{-1}$, $y = z\circ g^{-1}$ do not simultaneously hold for any two Borel mappings $f, g: X\to X$ and any $\mathcal{B}(X)$-σ-observable z of $\mathcal{L}$. Here X is a Banach space of non-measurable cardinality, $\mathcal{B}(X)$ is its Borel σ-algebra and $\mathcal{L} = \mathcal{B}(X)\times\mathcal{B}(X)'$. However, in this case there is the joint σ-observable of x and y, because x and y are induced by point transformations $T_i: X\times X\to X$ such that $x = T_1^{-1}$, $y = T_2^{-1}$.

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Заметка о совместном распределении в квантовых логиках. Компатибильные наблюдаемые

Изучается совместное распределение в σ-конечных мерах для наблюдаемых квантовой логики - аксиоматической модели квантовой механики, определенных на произвольной системе σ-независимых булевых σ-подалгебр булевой σ-алгебры. В настоящей первой части заметки мы изучаем совместное распределение компатибильных наблюдаемых. Показано, что оно всегда существует, хотя совместная наблюдаемая может и не существовать.

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Remark on Joint Distribution in Quantum Logics. Compatible Observables

The notion of a joint distribution in σ-finite measures of observables of a quantum logic - an axiomatic model of quantum mechanics - defined on an arbitrary system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra is studied. In the present first part of the paper we study a joint distribution of compatible observables. It is shown that it always exists, although a joint observable of compatible observables can fail.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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