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A.Dvurečenskij

ON GLEASON'S THEOREM FOR MEASURES WITH INFINITE VALUES

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1. INTRODUCTION

Let H be a Hilbert space over the field, C of real or complex numbers. Denote by $\mathfrak{L}(H)$ the orthocomplemented orthomodular complete lattice of all closed subspaces of H ; $\mathfrak{L}(H)$ is named a quantum logic of a Hilbert space H. By $\underset{t \in T}{\oplus} M_t$ we denote the join of mutually orthogonal subspaces $\{M_t : t \in T\}$. A measure on $\mathfrak{L}(H)$ is a function $m : \mathfrak{L}(H) \to [0,\infty]$ such that (i) $m(0) = 0$; (ii) $m(\overset{\infty}{\underset{i=1}{\oplus}} M_i) = \sum_{i=1}^{\infty} m(M_i)$. Measure m is (a) finite if $m(M) < \infty$ for each $M \in \mathfrak{L}(H)$; (b) σ-finite if there is a sequence $\{M_i\}_{i=1}^{\infty}$ such that $\overset{\infty}{\underset{i=1}{\oplus}} M_i = H$ and $m(M_i) < \infty$, $i \ge 1$.

Measures characterize the states of quantum physical systems (see Varadarajan/1/, for instances).

The crucial result of the theory of quantum logics is the famous Gleason theorem/2/ which asserts that any finite measure m on $\mathfrak{L}(H)$ of a separable Hilbert space H, $\dim H \neq 2$, is induced by a positive Hermitian trace operator T on H via

$$m(P) = \mathrm{tr}(TP), \quad P \in \mathfrak{L}(H), \tag{1.1}$$

(we identify P with the orthoprojector T^P onto P). Sherst nev/3/ and the author/4/ extended (1.1) to all bounded signe measures on $\mathfrak{L}(H)$. We recall that a function $m : \mathfrak{L}(H) \to [-\infty,\infty]$ for which (i) and (ii) of above hold, and from the values $\pm\infty$ it attains only one is called a signed measure. In papers of Eilers and Horst/5/ (for measures) and of Drisch/6/ (for signed measures) it is proved that the assumption of separabilit of H is superfluous. The separability of H is changed to th dimension of H with non-real measurability of a cardinal. Wo recall that the cardinal of a set I is said to be non-real measurable/7/ if there is no measure $\nu \neq 0$ on the power set of I with $\nu(\{a\}) = 0$, for each $a \in I$.

Sherstnev/8/ studied measures on ideals of a separable Hil bert space. The generalization of Gleason's theorem for σ-fi nite measures on $\mathfrak{L}(H)$ of a separable infinite-dimensional Hil bert space is given in/9/. This result has been extended to σ finite signed measures/10/ for a Hilbert space whose dimensio is a non-real measurable cardinal $\neq 2$.

Let m be a measure. An element $P \in \mathfrak{L}(H)$ is said to be a car rier of m if $m(M) = 0$ holds iff $M \perp P$. It is clear that if a ca

rier exists, then it is unique. In/11/ there has been proved that any σ-finite measure on $\mathfrak{L}(H)$ of a separable Hilbert space has a carrier. Moreover, any measure m with $m(H) = \infty$ on $\mathfrak{L}(H)$ with $4 \le \dim H < \infty$, for which there is a three-dimensional subspace Q such that $m(Q) < \infty$, has a carrier.

In the present note we prove that any σ-finite measure on a quantum logic of a Hilbert space whose dimension is a non-real measurable cardinal $\neq 2$ has a separable carrier. This gives the positive answer onto the question posed in/11/ related to the existence of a carrier. Moreover, we prove that this measure is totally additive.

2. σ-FINITE MEASURES

Let H be a Hilbert space over the field C of real or complex numbers with elements x , y ,... and the inner product (.,.). By $\|x\| := (x,x)^{1/2}$ we denote the norm of $x \in H$. If $0 \neq x \in H$, then by P_x we denote the subspace of H generated by x. An orthogonal complement to $P \in \mathfrak{L}(H)$ is the subspace $P^{\perp} := \{x \in H : (x,y) = 0$ for any $y \in P\}$. By Tr(H) we denote the class of all bounded linear operators T in H such that, for every orthonormal basis $\{x_a : a \in I\}$ of H, the series $\sum_{a \in I} (Tx_a, x_a)$ absolutely converges and it is independent of the basis used; the expression $\mathrm{tr}\, T := \sum_{a \in I} (Tx_a, x_a)$ is called the trace of T.

A bilinear form is a function $t : D(t) \times D(t) \to C$, where D(t) is a linear submanifold of H (not necessarily dense or closed in H), named the domain of the definition of t, such that is linear in both arguments, and $t(\alpha x, \beta y) = \alpha \bar{\beta} t(x,y)$, $x, y \in D(t)$, $\alpha, \beta \in C$. If $t(x,y) = \overline{t(y,x)}$ for all $x, y \in D(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x,x) \ge 0$ for all $x \in D(t)$, then t is said to be positive. Let $P \in \mathfrak{L}(H)$ and let $P \subset D(t)$. Then by $t \circ P$ we mean a symmetric bilinear form defined by $t \circ P(x,y) = t(Px, Py)$, $x, y \in H$. If $t \circ P$ is induced by a trace operator T, that is, $t \circ P(x,y) = (Tx,y)$, for all $x, y \in H$, then we say $t \circ P \in \mathrm{Tr}(H)$ and we put $\mathrm{tr}(t \circ P) = \mathrm{tr}\, T$.

An element $P \in \mathfrak{L}(H)$ is a separable if it is a separable subspace of H. We say that a function $m : \mathfrak{L}(H) \to [-\infty,\infty]$ with $m(0) = 0$ is totally additive if

$$m(\underset{a \in I}{\oplus} P_a) = \sum_{a \in I} m(P_a) \tag{2.0}$$

for an arbitrary system $\{P_a : a \in I\}$ of mutually orthogonal subspaces of H. If (2.0) holds only for sequences m is said to be σ-additive.

It is known/7/ that if I is a non-real measurable cardinal, then so is 2^I. Hence, $\aleph_0$ and $C = 2^{\aleph_0}$ are too. Moreover, if $J \le I$, then J is a non-real me[illegible] too.

The following result has been proved in [9]:

Theorem 2.1. Let H be an infinite-dimensional separable Hilbert space. Let m be a σ-finite measure on $\mathcal{L}(H)$ with $m(H)=\infty$. Then there is a unique symmetric bilinear form defined on a dense domain such that

$$m(P) = \begin{cases} tr(t \circ P) & \text{if } t \circ P \in Tr(H), \\ \infty & \text{elsewhere.} \end{cases} \qquad (2.1)$$

This result has been extended in [10] to non-separable case:

Theorem 2.2. Let H be a non-separable Hilbert space whose dimension is a non-real measurable cardinal. Let m be a σ-finite measure on $\mathcal{L}(H)$. Then there is a unique symmetric bilinear form t defined on a dense domain such that

$$m(P) = \begin{cases} tr(t \circ P) & \text{if } m(P) < \infty, \\ \infty & \text{elsewhere.} \end{cases} \qquad (2.2)$$

Moreover, if $m(P)<\infty$, then, for any $\{P_a : a \in A\}$ with $P = \bigoplus_{a\in A} P_a$, we have (2.0).

The basic result for measures with infinite values is the next result.

Lemma 2.3. (Lugovaja - Sherstnev [9]). Let $\dim H = 3$ and let m be a measure on $\mathcal{L}(H)$ with $m(H)=\infty$. If there is a two-dimensional Q, $m(Q)<\infty$, then for any one-dimensional P with $m(P)<\infty$ we have $P<Q$.

This result can be extended as follows.

Lemma 2.4. Let $\dim H \geq 3$ and let m be a signed measure on $\mathcal{L}(H)$. If there are P and Q such that $\dim P^{\perp} = 1 = \dim Q$, $m(P)<\infty$, $m(Q)<\infty$, then $Q<P$.

Proof. It is clear that $Q \not< P$. It may be proved that $\dim(Q^{\perp}\wedge P)\geq 2$. Hence, there is $x_1 \in Q^{\perp}\wedge P$ so that $P_{x_1} \perp Q$, $P_{x_1} \perp P^{\perp}$. Since $\dim(Q \vee P^{\perp}) = 1$, , $\dim(Q \vee P_{x_1} \vee P^{\perp}) = 3$. The conditions of Lemma entail

$$m(Q \vee P_{x_1} \vee P^{\perp}) = \infty, \qquad (2.3)$$

in the opposite case $m(P^{\perp})<\infty$ and $m(H)<\infty$.

Now we show that $\dim((Q \vee P_{x_1} \vee P^{\perp})\wedge P) = 2$. It is evident that $x_1 \in M := (Q \vee P_{x_1} \vee P^{\perp})\wedge P$. Let us suppose that $Q = P_q$, $P^{\perp} = P_{x_0}$

for suitable q, $x_0 \in H$. Then it may be shown that $\alpha q + \beta x_0 \in M$, where $\alpha \in C$ and $\beta \in C$ satisfy $\alpha(q, x_0) + \beta = 0$.

We claim to show $Q<P$. If not, then there is $x_2 \in M$ such that $x_2 \perp x_1$ and $Q \vee (P_{x_1} \oplus P_{x_2}) = Q \vee P_{x_1} \vee P^{\perp}$. Using Lemma 2.3 we have $m(Q \vee (P_{x_1} \oplus P_{x_2}))<\infty$ which is a contradiction with (2.3).

Q.E.D.

Lemma 2.5. Let H have a non-real measurable cardinal ≥ 4 and let m be a measure on $\mathcal{L}(H)$ with $m(H)=\infty$. Let there be a three-dimensional Q with $m(Q)<\infty$. If $m(M) = m(N) = 0$ and $\dim N < \infty$, then $m(M \vee N) = 0$.

Proof. Using Lemma 2.4 we have that $m(Q \vee M \vee N)<\infty$. Hence, using Gleason's theorem [5] for $m^* := m|\mathcal{L}(P)$, where $P = Q \vee M \vee N$, we obtain the desired result.

Q.E.D.

Example 1. The assumption of the existence of a three-dimensional Q with $m(Q)<\infty$ is not superfluous in Lemma 2.5. Indeed, let Q_0 be some two-dimensional subspace of H. Define $m(P)=\infty$ iff $P \not< Q_0$, $m(Q_0)=2$, $m(0)=0$. Choose $x_1, x_2, y_1, y_2 \in Q_0$ such that $x_1 \perp x_2$, $y_1 \perp y_2$, $x_1 \perp y_1 \neq x_1$ and put $m(P_{x_1}) = m(P_{y_1}) = 0$, $m(P_{x_2}) = m(P_{y_2}) = 2$. For all other one-dimensional P, $P<Q_0$, let $m(P)=1$. Then for this m Lemma 2.5 is not valid.

Lemma 2.6. Let $T \in Tr(H)$ and let T be a positive operator. Then m, defined by (1.1), has a separable carrier.

Proof. Due to [12], it follows for T there is an orthonormal basis in H, $\{f_a : a \in A\}$, and non-negative numbers $\{\lambda_a : a \in A\}$ such that

$$T = \sum_{a\in A} \lambda_a f_a \otimes \bar{f}_a , \qquad (2.4)$$

where $f \otimes \bar{f}$ is an operator in H which assigns $(x,f)f$ to any $x \in H$. Since $tr(T) = \sum_{a\in A} \lambda_a$, then the set $\{a \in A : \lambda_a > 0\}$ is countable. An easy calculation shows that $M := \oplus\{P_{f_a} : \lambda_a > 0\}$ is a separable carrier of m.

Q.E.D.

Example 2. Let H be an arbitrary Hilbert space, and let $H = \bigoplus_{i=1}^{\infty} H_i$. Let $T_i \in Tr(H_i)$, $i \geq 1$, be a positive operator; it may be extended to a whole H. Define $m(M) = \sum_i tr(T_i M)$, $M \in \mathcal{L}(H)$. Then m is a σ-finite measure on $\mathcal{L}(H)$. Using (2.4) we see that this measure has a separable carrier.

The following result gives the affirmative answer to the question posed in /11/: Does there exists a carrier for a σ-finite measure?

Theorem 2.7. Any σ-finite measure on $\mathcal{L}(H)$ of a Hilbert space H whose dimension is a non-real measurable cardinal $\neq 2$ has a separable carrier.

Proof. If m is a finite measure, then, due to the generalized Gleason theorem /5/, m has a form (1.1), and Lemma 2.6 implies that m has a separable carrier.

Let now $m(H)=\infty$ and let $H=\bigoplus_{i=1}^{\infty} H_i$, where $m(H_i)<\infty$, $i\ge 1$. Without loss of generality we may assume $\dim H_i \ge 3$, $i \ge 1$. The restriction of m onto $\mathcal{L}(H_i)$, $m_i := m|\mathcal{L}(H_i)$, entails that there is, for any $i\ge 1$, a separable carrier P_i and an N_i with $H_i=P_i \oplus N_i$. It is evident that $m_i(M)=0$ for any $M<N_i$. Let us put $P_\infty = \bigoplus_{i=1}^{\infty} P_i$ and $N_\infty = \bigoplus_{i=1}^{\infty} N_i$. The map $m_\infty := m|\mathcal{L}(P_\infty)$ is a σ-finite measure on $\mathcal{L}(P_\infty)$ of an infinite-dimensional Hilbert space P_∞ with $m_\infty(P_\infty)=\infty$. Hence, due to /11/ Lemma 3.6 m_∞ has a separable carrier P_0. Therefore $m_\infty(P_x)>0$ for any $x \in P_0$.

We claim to show that P_0 is a carrier of m.

Let $\mathfrak{N}=\{x\in H: m(P_x)=0\}\cup\{0\}$ and let $N=N_\infty \oplus (P_\infty \wedge P_0^\perp) = P_0^\perp$.

Then $N_i \subset \mathfrak{N}$, for any $i \ge 1$, and $P_\infty \wedge P_0^\perp \subset \mathfrak{N}$. Using Lemma 2.5 we may show that if $x, y \in \mathfrak{N}$, then $x+y \in \mathfrak{N}$; and if $x \in \mathfrak{N}$ and $\alpha \in C$, then $\alpha x \in \mathfrak{N}$. Now we show that $\mathfrak{N}$ is a closed submanifold in H. Let $\{x_n\} \subset \mathfrak{N}$ and $\|x_n - x\| \to 0$ for some $x \in H$. Define $M_n = P_{x_1} \vee \dots \vee P_{x_n}$.

Due to Lemma 2.5, $m(M_n)=0$. Using the continuity of measure m from below we have $m(\bigvee_{n=1}^{\infty} M_n) = \lim_n m(M_n)=0$. Therefore $x \in \bigvee_{n=1}^{\infty} M_n \subset \mathfrak{N}$.

Now we show $\mathfrak{N}=N$. If not, then there is $x \in \mathfrak{N} \wedge N^\perp$. Hence, $x \in P_0$ and $m(P_x)=0$. On the other hand, using that P_0 is a carrier of m_∞ we have $m(P_x)>0$ which is a contradiction.

Therefore, P_0 is a carrier of m. Q.E.D.

We say that a measure m has a Jauch-Piron property if $m(M)=m(N)=0$ implies $m(M \vee N)=0$.

Corollary 2.7.1. Under the conditions of Theorem 2.7, m has a Jauch-Piron property. Moreover, if $m(M_a)=0$, $a \in A$, then $m(\bigvee_{a\in A} M_a)=0$.

Proof. This is a transparent consequence of the fact that m has a carrier. Q.E.D.

Note 1. The assertion of Corollary 2.7.1 remains valid under the conditions of Lemma 2.6.

We note that Theorem 2.7 is in a sense an analogue of Ulam's result /7/ from which follows that if μ is a finite measure on the power set of I, where the set I has a non-real measurable cardinal, then there is at most countable subset $N \subset I$ such that $\mu(I-N)=0$. It is easy to verify that the same is true for a σ-finite measure μ on 2^I.

Theorem 2.8. Under the hypotheses of Theorem 2.7 m is totally additive on $\mathcal{L}(H)$. Moreover (2.1) holds.

Proof. From Corollary 2.7.1 it follows that if $M=\bigoplus_{a\in A} M_a$ and $m(M_a)=0$ for any $a \in A$, then $m(M)=\sum_{a\in A} m(M_a)$.

Let now $\{P_a : a \in A\}$ be an arbitrary system of mutually orthogonal subspaces of H with the join P. If $m(\bigoplus_{a\in J} P_a)=\infty$ for some countable subset J of A, then $m(P)=\infty=\sum_{a\in A} m(P_a)$. Hence, suppose that $m(\bigoplus_{a\in J} P_a)<\infty$ for any countable subset J of A. Denote, for any $n\ge 1$, $A_n=\{a\in A: m(P_a)\ge 1/n\}$. Our assumption yields that any A_n is a finite subset of A. Put $A_0=\bigcup_{n=1}^{\infty} A_n$. Then, for any $a \in A-A_0$, $m(P_a)=0$. Since $P=\bigoplus_{a\in A_0} P_a \oplus \bigoplus_{a\in A-A_0} P_a$, we see that $m(P)=\sum_{a\in A_0} m(P_a) + \sum_{a\in A-A_0} m(P_a)$, when within the first series we use the σ-additivity of m.

To show (2.1) it is necessary to verify that $m(P)<\infty$ iff $t \circ P \in Tr(H)$. One direction of this equivalence is obvious from (2.2). For the second one, we use the total additivity of m: $\mathrm{tr}\, t \circ P = \sum_i t(x_i, x_i) = \sum_i m(P_{x_i}) = m(P)$, where $\{x_i\}$ is an orthonormal basis in P. Q.E.D.

We recall that in /10/ the problem of a total additivity has been raised. The positive answer for measures is given in Theorem 2.8, but for signed measures this question is still open. Moreover, in /10/ there has been argued that $m(P)<\infty$ iff $t \circ P \in Tr(H)$ although only the implication "if $m(P)<\infty$, then $t\circ P \in Tr(H)$" has been proved. The complete proof of the mentioned equivalence for measures is given in Theorem 2.8.

Note 2. Under the conditions of Lemma 2.6 m is totally additive. The same is true for Example 2, too.

3. n-FINITE MEASURES

In this section we generalize the notions of σ-finiteness and σ-additivity of measures on $\mathcal{L}(H)$ to a more general case, and the results analogous to those in Section 2 will be proved.

Let n and m be two cardinals. We say that a function $m: \mathcal{L}(H) \to [0,\infty]$ with $m(0)=0$ is (i) m-additive if $m(\bigoplus_{a\in A} P_a) = \sum_{a\in A} m(P_a)$ whenever the cardinal of A is m ; (ii) n-finite if there is $\{H_i : i \in B\}$ such that $H = \bigoplus_{i\in B} H_i$, $m(H_i)<\infty$ for any $i \in B$, and the cardinal of B is n .The extension of these notions to signed measures is straightforward: for definiteness we shall suppose $m: \mathcal{L}(H) \to (-\infty,\infty]$.

It is evident that if $m_1 \le m_2$, then any m_2-additive measure is m_1-additive. If $n_1 \le n_2$, then any n_1 finite measure is n_2 -finite. We recall that unbounded finitelly additive σ -finite measures on $\mathcal{L}(H)$ are studied in [13].

Lemma 3.1. Let $\mathcal{L}(H)$ be the quantum logic of a Hilbert space H whose dimension is an infinite cardinal I. Then any I -additive I -finite measure has a carrier of dimension $\le$I.

Proof. Define $\mathcal{N} = \{x : m(P_x) = 0\} \cup \{0\}$. Similarly as in the proof of Theorem 2.7 we may show $\mathcal{N}$ is a closed subspace of H. Hence, if $x \in \mathcal{N}$, then $m(P_x)=0$. Choose an orthogonal basis $\{x_a\}$ in $\mathcal{N}$. Using the I-additivity we have $m(\mathcal{N}) = \sum_a m(P_{x_a}) = 0$. Therefore $P := \mathcal{N}^{\perp}$ is the carrier of m.

Q.E.D.

Theorem 3.2. Let $\mathcal{L}(H)$ be a quantum logic of a Hilbert space H whose dimension, I, is a nonreal measurable cardinal $\ne 2$. Let n and m be two cardinals such that $m \le I$ and $m \ge \aleph_0$ .Then any m-additive n-finite measure m on $\mathcal{L}(H)$ has a carrier whose dimension is $\le \max\{\aleph_0, n\}$.

Proof. Let $H = \bigoplus_{a\in A} H_a$, where A has the cardinal n, and $m(H_a)<\infty$ for any $a \in A$. If n is a finite cardinal, then m has a separable carrier, see[5] and Lemma 2.6.

Let now $n \ge \aleph_0$.Then any $m_a := m|\mathcal{L}(H_a)$, $a \in A$, is a finite measure. Hence, $H_a = P_a \oplus N_a$, where P_a is a separable carrier of m_a. Put $P_A = \bigoplus_{a\in A} P_a$. It is evident that $\dim P_a = n$, and $m_A := m|\mathcal{L}(P_A)$ is an n-additive n-finite measure on $\mathcal{L}(P_A)$. Applying Lemma 3.1 we see that $P_A = P_0 \oplus N_0$, where P_0 is a carrier of m_A. To show that $P := P_0$ is a carrier of m see the proof of Theorem 2.7.

Q.E.D.

Theorem 3.3. Under the hypotheses of Theorem 3.2 any m-additive n-finite measure m on $\mathcal{L}(H)$ is totally additive. Moreover, there is a unique symmetric bilinear form t such that (2.1) holds.

Proof. The first part of the assertion is the same as that in Theorem 2.8. The second one may be proved using the methods developed in[10].

Q.E.D.

According to[10], we say that a signed measure m is f-finite if $\sup_{Q<P} |m(Q)| < \infty$ whenever $|m(P)|<\infty$. The following result may be proved analogically to Theorem 4.4 in[10].

Theorem 3.4. Let $\mathcal{L}(H)$ be the logic of a Hilbert space H whose dimension is a non-real measurable cardinal $I \ne 2$. Let n and m be two cardinals such that $n \le I$, $m \ge \aleph_0$.Then for any f-bounded m-additive n-additive signed measure m, there is a unique symmetric bilinear form t such that (2.2) holds. Moreover, if $|m(P)|<\infty$ and $P = \bigoplus_{a\in A} P_a$, where A is an arbitrary index set, then (2.0) holds.

4. APPLICATION

In this section we apply the result on the existence of supports of σ-finite measures to the problem of the existence of a joint distribution of observables in a σ-finite measure.

Let $(\Omega, \mathcal{S})$ be a measurable space, that is, Ω is a non-empty set, and the non-empty system $\mathcal{S}$ of subsets of Ω is closed with respect to the formation of the union of countably many elements from $\mathcal{S}$, and if $E \in \mathcal{S}$, then $\Omega - E \in \mathcal{S}$. A map $X : \mathcal{S} \to \mathcal{L}(H)$ is an $\mathcal{S}$-observable of $\mathcal{L}(H)$ if (i) $X(\Omega)=H$; (ii) $X(E) \perp X(F)$ whenever $E \cap F = \emptyset$; (iii) $X(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} X(E_i)$, $\{E_i\} \subset \mathcal{S}$. Observables correspond to measurable quantities in quantum mechanics[1]. An observable X is σ-finite with respect to a measure m if there is a sequence $\{E_i\} \subset \mathcal{S}$ such that $\bigcup_{i=1} E_i = \Omega$, $E_i \cap E_j = \emptyset$, $i \ne j$, and $m(X(E_i)) < \infty$, $i \ge 1$.

We say that a finite system of $\mathcal{S}_i$-observables, X_i, of $\mathcal{L}(H)$, $i = 1,\dots,$ where $(\Omega_i, \mathcal{S}_i)$ is a measurable space, has a joint distribution in a measure m if there is a measure μ on $\mathcal{S}_1 \times \dots \times \mathcal{S}_n$ such that

$$\mu(E_1 \times \dots \times E_n) = m(\bigwedge_{i=1}^{n} X_i(E_i)), \qquad (4.1)$$

for all $E_i \in \mathcal{S}_i$, $i=1,\dots,n$.

Theorem 4.1. Let $\mathcal{L}(H)$ be the quantum logic of a Hilbert space H whose dimension is a non-real measurable cardinal $\ne 2$. Let X_i be an $\mathcal{S}_i$-observable of $\mathcal{L}(H)$, $i = 1,\dots,n$, and let at least one observable is σ-finite with respect to m. Then $X_1, \dots, X_n$ have a joint distribution in a measure m iff

$$X_{i_1}(E_{i_1})\dots X_{i_n}(E_{i_n})P = X_1(E_1)\dots X_n(E_n)P \qquad (4.2)$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and all $E_i \in \mathcal{S}_i$, $i = 1,\dots,n$; here P is the carrier of m.

Proof. It is evident that m is σ-finite measure. Due to Theorem 2.7, m has a separable carrier, and, consequently, $m(P^\perp)=0$. The final result follows from [11, Lemma 3.9], where (4.2) is proved only under the assumption $m(P^\perp)<\infty$.

Q.E.D.

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Двуреченский А.

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О теореме Глизона для мер с бесконечными значениями

В качестве модели квантовой механики изучается квантовая логика всех замкнутых подпространств пространства гильберта. Доказывается, что каждая σ-конечная мера на этой квантовой логике гильбертова пространства, размерность которого - нереальное измеримое кардинальное число $\neq 2$, имеет сепарабельный носитель, а также, что она вполне аддитивна. Результаты применяются к проблеме совместного распределения наблюдаемых в мере.

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Dvurečenskij A.

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On Gleason's Theorem for Measures with Infinite Values

The quantum logic of all closed subspaces of a Hilbert space as a model of quantum mechanics is studied. It is proved that any σ-finite measure on this quantum logic of a Hilbert space whose dimension is a non-real measurable cardinal $\neq 2$ has a separable carrier, and it is totally additive. The results are applied to the problem of a joint distribution of observables in a measure.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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