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DIRECT EVALUATION OF THE IWASAWA AND TRIANGLE DECOMPOSITIONS FOR THE REAL FORMS OF LIE ALGEBRAS so(2n,C)

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1. Introduction

Various noncompact orthogonal groups(algebras)have arisen in the study of physical problems, such as De Sitter Groups O(3,2) and O(4,1) playing an important role in relativistic cosmology, the Lorents group O(3,1) which is of obvious crucial importance in relativistic physics, the O(2,1) group figuring in Regge pole theory and finally the conformal group O(4,2) playing an ever increasing role in particle physics, especially at very high energies.

The unitary representations of such groups are all infinite dimensional, and no general theory for them exists.

The Iwasawa /1/ and triangle /2/ decompositions of such algebras are the starting point for a construction of these representations. Also in our method /3/ of constructing skew-Hermitean boson realizations for an arbitrary real semisimple Lie algebra we use these decompositions. Explicit forms of this decompositions have been constructed by many authors for particular examples of these Lie algebras. The usual indirect method for evaluation of the Iwasawa decomposition of a noncompact semisimple Lie algebra involves search for solution to certain simultaneous eigenvalue type equations; for details see the book by Hermann /4/. Cornwell /5/ uses the fact that for any two Cartan subalgebras, there exist an inner automorphism which maps one into the other for the construction of these decomposition. This inner automorphism, however must be guessed for each particular case.

The present paper gives the Iwasawa and triangle decomposition for any real forms of Lie algebras so(2n,ℂ) in a much more simply way. For any real form of the algebra so(2n,ℂ) we construct the Cartan automorphism and explicit form of this automorphism then specifies the decomposition directly. Organization of the paper is the following. In Section 2 , we describe in general terms the construction of the decompositions in the case when the Cartan automorphism which gives real forms is defined explicitly. For the algebras so(2n,ℂ) we give explicit forms of the Cartan automorphism in Section 3. In the last section, these automorphisms will be used for construction of the Iwasawa and triangle decompositions. The modification for the Lie algebras so(2n+1,$\mathbb{C}$) is trivial.

2. The Iwasawa and triangle decompositions

In this section, we have given a brief survey of the Iwasawa and triangle decompositions. We followed Chapter 5 of Helgasson /7/ and Chapter 3, §3 of Zhelobenko and Stern /2/.

Let $\tilde{g}$ be a semisimple Lie algebra. We denote by $\tilde{h}$ a Cartan subalgebra $\tilde{g}$ and by Δ the root system of $\tilde{g}$ with respect $\tilde{h}$. Any semisimple Lie algebra has a Cartan-Weyl basis $\{H_1,\dots,H_n, E_\alpha\ ;\ \alpha\in\Delta\}$ for which:

$$\{H_1,\dots,H_n\} \quad \text{is a basis in } \tilde{h} \tag{1a}$$

$$\{h, E_\alpha\} = \alpha(h)E_\alpha \tag{1b}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta} \quad (N_{\alpha\beta} = -N_{-\alpha,-\beta}) \quad \text{for } \alpha+\beta\in\Delta \tag{1c}$$

$$B(E_\alpha, E_{-\alpha}) = 1 \tag{1d}$$

where B(.,.) is a Killing form on $\tilde{g}$.

Using this Cartan-Weyl basis of $\tilde{g}$ we can define an antilinear mapping on $\tilde{g}$ by

$$\psi(H_i) = -H_i \qquad \text{for } i=1,2,\dots,n \tag{2a}$$

$$\psi(E_\alpha) = -E_{-\alpha} \qquad \text{for all } \alpha\in\Delta. \tag{2b}$$

This mapping is the involutive antilinear automorphism on $\tilde{g}$ and the real form corresponding to

$$g_\psi = \{X\in\tilde{g};\ \psi(X) = X\} \tag{3}$$

is compact.

A linear automorphism is called a Cartan automorphism if

$$\theta^2 = 1 \tag{4a}$$

$$\theta\cdot\psi = \psi\cdot\theta \tag{4b}$$

$$\theta(\tilde{h}) = \tilde{h}\ . \tag{4c}$$

If θ is a Cartan automorphism on $\tilde{g}$, then θ defines the real forms of $\tilde{g}$ as a set for

$$g_\theta = \{X\in\tilde{g};\ \theta\cdot\psi(X) = X\}. \tag{5}$$

With the help of the Cartan automorphism θ, we can construct very simply a Cartan decomposition of g_θ

$$g_\theta = g^t_\theta \oplus g^p_\theta\ , \quad \text{where} \tag{6a}$$

$$g^t_\theta = \{Y\in g_\theta\ ;\ \theta(Y) = Y\} \tag{6b}$$

$$g^p_\theta = \{Y\in g_\theta\ ;\ \theta(Y) = -Y\}. \tag{6c}$$

The subalgebra g^t_θ is a maximal compact sublagebra in the algebra g_θ and the subspace g^p_θ is a complementary subspace to g^t_θ with respect to Killing form B(.,.).

Semilarly we put

$$h_\theta = \{X\in\tilde{h},\ \ \theta\psi(X) = X\} \tag{7a}$$

and then we have

$$h_\theta = h^t_\theta \oplus h^p_\theta \quad \text{where} \tag{7b}$$

$$h^t_\theta = \{Y\in h_\theta,\ \theta(Y) = Y\} \tag{7c}$$

$$h^p_\theta = \{Y\in h_\theta,\ \theta(Y) = -Y\}. \tag{7d}$$

The h^p_θ is a maximal commutative subalgebra in g^p_θ .

Now, let $\{H_1,\dots,H_q\}$ be a basis in h^p_θ and further let $\{iH_{q+1},\dots,iH_n\}$ be a basis in h^t_θ . Then $\{H_1,\dots,H_n\}$ is a basis in $\tilde{h}$. By Δ_+ we shall denote a system of positive roots with respect this basis.

For any $\alpha\in\Delta$ we define

$$\alpha^\theta(H) = \alpha(\theta(H)) \qquad \text{for any } H\in\tilde{h}. \tag{8}$$

After performing the following easy calculation

$$[H, \theta(E_\alpha)] = \theta[\theta(H), E_\alpha] = \alpha(\theta(H))\,\theta(E_\alpha) \tag{9}$$

we become that α^θ is also the element of Δ .

Using this fact we may define

$$P^+_\theta = \{\alpha\ ;\ \alpha\in\Delta_+\ , \alpha\neq\alpha^\theta\} \tag{10a}$$

$$\tilde{n}^+_\theta = \sum_{\alpha\in P^+_\theta} g^\alpha\ , \qquad n^+_\theta = \tilde{n}^+_\theta \cap g_\theta \tag{10b}$$

$$\tilde{n}_\theta^- = \sum_{\alpha \in P_\theta^+} g^{-\alpha}, \quad n_\theta^- = \tilde{n}_\theta^- \cap g_\theta \tag{10c}$$

where $g^\alpha = \mathbb{C}\{E_\alpha\}$

$$g_\theta^o = \{ Y \in g_\theta \, ; \, [Y, h_\theta^p] = 0 \} \tag{10d}$$

and we get the required Iwasawa and triangle decompositions of the real form g_θ at last

$$g_\theta = g_\theta^t \oplus h_\theta^p \oplus n_\theta^+ \quad \text{(Iwasawa decomposition)} \tag{11a}$$

$$g_\theta = n_\theta^+ \oplus g_\theta^o \oplus n_\theta^- \quad \text{(triangle decomposition).} \tag{11b}$$

3. The Cartan automorphisms of Lie algebras so(2n,$\mathbb{C}$)

The algebra so(2n,$\mathbb{C}$) is the n(2n-1) - dimensional complex Lie algebra with the standard basis $\{L_{ij};\ i,j= \pm 1, \pm 2, \ldots, \pm n\}$ the elements of which obey:

$$L_{ij} = -L_{-j,-i} \tag{12}$$

and the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{il} L_{kj} - \delta_{j,-l} L_{i,-k} + \delta_{i,-k} L_{-l,j} \,. \tag{13}$$

The standard Cartan subalgebra $\tilde{h}$ in $\tilde{g}$ = so(2n,$\mathbb{C}$) is generated by "diagonal" elements $\{L_{ii};\ i=1,2,\ldots,n;\}$ its dimension, i.e. rank of $\tilde{g}$ equals n. We will use the following Cartan-Weyl basis in $\tilde{g}$

$$H_i = L_{ii} \tag{14a}$$

$$E_{\lambda_i - \lambda_k} = \frac{1}{\sqrt{4n-4}} L_{ik}, \quad E_{-(\lambda_i - \lambda_k)} = \frac{1}{\sqrt{4n-4}} L_{ki}; \quad 0 < i < k \tag{14b}$$

$$E_{\lambda_i + \lambda_k} = \frac{1}{\sqrt{4n-4}} L_{i,-k}, \quad E_{-(\lambda_i + \lambda_k)} = \frac{1}{\sqrt{4n-4}} L_{-i,k}; \quad 0 < i,k \,. \tag{14c}$$

The relations (13) imply that (14b), (14c) are the root vectors corresponding to the root $\pm(\lambda_i - \lambda_k)$ and $\pm(\lambda_i + \lambda_k)$ because for $H(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i L_{ii}$ we get

$$[H(\lambda_1, \ldots, \lambda_n), L_{ik}] = (\lambda_i - \lambda_k) \; L_{ik} \quad \text{for } 0 < i < k, \text{ etc.} \tag{15}$$

For the root system we get

$$\Delta = \{ \pm(\lambda_i - \lambda_k), \pm(\lambda_i + \lambda_k);\ 0 < i,k < n, i \neq k \} \,. \tag{16}$$

Then the equality

$$-(\lambda_i \pm \lambda_k) = (\lambda_k \pm \lambda_i) \tag{17}$$

implies that in this case the mapping (2a-b) equals

$$\psi(L_{ik}) = -L_{ki} \,. \tag{18}$$

For any q=1,2,...,n we define linear mappings θ_q on $\tilde{g}$ in this way:

$$\theta_q(L_{st}) = -L_{ts} \,, \tag{19a}$$

$$\theta_q(L_{s,\alpha}) = -L_{-\alpha,s} \,, \qquad \theta_q(L_{\alpha,s}) = -L_{s,-\alpha} \,, \tag{19b}$$

$$\theta_q(L_{\alpha,\beta}) = L_{\alpha\beta} \,, \tag{19c}$$

where $s,t = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \ldots, \pm n$.

For n=2q even we define further

$$\theta'(L_{s,t}) = L_{s+q_s, t+q_t} \,, \tag{20a}$$

$$\theta'(L_{s,t+q_t}) = L_{s+q_s,t} \,, \qquad \theta'(L_{s+q_s,t}) = L_{s,t+q_t} \,, \tag{20b}$$

where $s,t = \pm 1, \pm 2, \ldots, \pm q$ and $q_s = q \cdot \mathrm{sgn}\, s$,

and consequently for n=2q+1 odd we define

$$\theta''(L_{st}) = L_{s+q_s, t+q_t} \,, \tag{21a}$$

$$\theta''(L_{s,t+q_t}) = L_{s+q_s,t} \,, \qquad \theta''(L_{s+q_s,t}) = L_{s,t+q_t} \,, \tag{21b}$$

$$\theta''(L_{s,\alpha}) = -L_{s+q_s,\alpha} \,, \qquad \theta''(L_{s+q_s,\alpha}) = L_{s,\alpha} \,, \tag{21c}$$

$$\theta''(L_{\alpha\alpha}) = L_{\alpha\alpha} \,, \tag{21d}$$

where $s,t = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha = \pm n$.

Theorem: The linear mappings θ', θ'', θ_q are Cartan automorphisms on $\tilde{g}$.

Proof: One can by direct calculation verify that the conditions (4a-c) are fulfilled.

4. Explicit forms of the decompositions

Using the method described in Section 2 and explicit forms of the automorphisms θ', θ'', θ_q on so(2n,C) we shall construct the Iwasawa and triangle decompositions. By the construction we will use following equalities

$$\psi(X+\psi(X)) = X+\psi(X) \quad (22a)$$

$$\psi(iX-i\psi(X)) = iX-i\psi(X) \quad \text{for any } X\in\tilde{g} \quad (22b)$$

$$\theta(Y+\theta(Y)) = Y+\theta(Y) \quad (22c)$$

$$\theta(Y-\theta(Y)) = -(Y-\theta(Y)) \quad \text{for any } Y\in g_\theta \quad (22d)$$

which are the direct consequence of the definitions (2a-b) and (4a-c). The calculations of the decompositions are simple and we carry only final results.

I. The case θ_q; q=1,2,...,n.

For a subalgebra $g^t_{\theta_q}$ and subspace $g^p_{\theta_q}$ we get

$$g^t_{\theta_q} = \mathbb{R}\{(L_{st}-L_{ts}),\ (L_{\alpha\beta}-L_{\beta\alpha}),\ i(L_{\alpha\beta}+L_{\beta\alpha}),\ (L_{s\alpha}+L_{s,-\alpha})-(L_{\alpha,s}+L_{-\alpha,s}),\ i(L_{s,\alpha}-L_{s,-\alpha})+i(L_{\alpha s}-L_{-\alpha,s})\} \quad (23a)$$

$$g^p_{\theta_q} = \mathbb{R}\{(L_{s\alpha}+L_{s,-\alpha})+(L_{\alpha s}+L_{-\alpha,s}),\ i(L_{s\alpha}-L_{s,-\alpha})-i(L_{\alpha s}-L_{-\alpha,s}),\ (L_{st}+L_{ts})\} \quad (23b)$$

where s,t= $\pm1,\ldots,q$ and $\alpha,\beta = \pm(q+1),\ldots,\pm n$.

We put

$$\{L_{11},\ldots,L_{qq}\} \quad (24)$$

a basis in a subalgebra $h^p_{\theta_q}$ and

$$\{iL_{q+1,q+1},\ldots,iL_{nn}\} \quad (25)$$

in a subalgebra $h^t_{\theta_q}$.

For a set of the roots $P^+_{\theta_q}$ we get

$$P^+_{\theta_q} = \{(\lambda_s-\lambda_t),(\lambda_s+\lambda_t);\ 0<s<t \text{ and } (\lambda_s-\lambda_\alpha),(\lambda_s+\lambda_\alpha);\ s>0, \text{ where } s,t=1,2,\ldots,q,\ \alpha=(q+1),\ldots,n.\} \quad (26)$$

and further

$$n^+_{\theta_q} = \mathbb{R}\{L_{st};\ 0<s<|t|,\ (L_{s\alpha}+L_{s,-\alpha}),\ i(L_{s\alpha}-L_{s,-\alpha});\ s>0\}, \quad (27a)$$

$$n^-_{\theta_q} = \mathbb{R}\{L_{st};\ 0>s>|t|\ (L_{s\alpha}+L_{s,-\alpha}),\ i(L_{s\alpha}-L_{s,-\alpha});\ s<0\}, \quad (27b)$$

$$g^o_{\theta_q} = \mathbb{R}\{L_{ss},\ (L_{\alpha\beta}-L_{\beta\alpha}),\ i(L_{\alpha\beta}+L_{\beta\alpha})\}. \quad (27c)$$

The case θ' n=2q.

For a subalgebra $g^t_{\theta'}$ and a subspace $g^p_{\theta'}$ we have

$$g^t_{\theta'} = \mathbb{R}\{(L_{st}-L_{t+q_t,s+q_s})-(L_{ts}-L_{s+q_s,t+q_t}),\ i(L_{st}+L_{t+q_t,s+q_s})+i(L_{ts}+L_{s+q_s,t+q_t}),\ (L_{s,t+q_t}-L_{t,s+q_s})+(L_{s+q_s,t}-L_{t+q_t,s}),\ i(L_{s,t+q_t}+L_{t,s+q_s})+i(L_{s+q_s,t}+L_{t+q_t,s})\} \quad (28a)$$

$$g^p_{\theta} = \mathbb{R}\{(L_{st}-L_{t+q_t,s+q_s})+(L_{ts}-L_{s+q_s,t+q_t}),\ i(L_{st}+L_{t+q_t,s+q_s})-i(L_{ts}+L_{s+q_s,t+q_t}),\ (L_{s,t+q_t}-L_{t,s+q_s})-(L_{s+q_s,t}-L_{t+q_t}),\ i(L_{s,t+q_t}+L_{t,s+q_s})-i(L_{s+q_s,t}+L_{t+q_t})\}. \quad (28b)$$

We put

$$\{(L_{11}-L_{q+1,q+1}),\ldots,(L_{qq}-L_{2q,2q})\} \quad (29a)$$

a basis in a subalgebra $h^p_{\theta'}$ and in a subalgebra $h^t_{\theta'}$

$$\{i(L_{11}+L_{q+1,q+1}),\ldots,i(L_{qq}+L_{2q,2q})\}. \quad (29b)$$

For a set of the roots $P^+_{\theta'}$ we get

$$P^+_{\theta'} = \{(\lambda_s-\lambda_t),(\lambda_{t+q_t}-\lambda_{s+q_s});\ 0<s<t,\ (\lambda_s+\lambda_t),(\lambda_{t+q_t}+\lambda_{s+q_s});\ s,t>0,\ (\lambda_s-\lambda_{t+q_t});\ s,t>0,\ (\lambda_s+\lambda_{t+q_t}),-(\lambda_t-\lambda_{s+q_s});\ 0<s<t\}, \quad (30)$$

and further

$$n^{+}_{\theta'} = R\{(L_{st}-L_{t+q_t,s+q_s}), i(L_{st}+L_{t+q_t,s+q_s});\ 0<s<t \vee s>0>t\ ,$$
$$(L_{s,t+q_t}-L_{t,s+q_s}), i(L_{s,t+q_t}+L_{t,s+q_s});\ s,t>0 \vee 0<s<-t \vee 0<t<-s\}, \qquad (31a)$$

$$g^{o}_{\theta'} = R\{(L_{ss}-L_{s+q_s,s+q_s}), i(L_{ss}+L_{s+q_s,s+q_s}),$$
$$(L_{s,-(s+q_s)}-L_{-s,s+q_s}), i(L_{s,-(s+q_s)}+L_{-s,s+q_s})\}, \qquad (31b)$$

$$n^{-}_{\theta'} = R\{(L_{ts}-L_{s+q_s,t+q_t}), i(L_{ts}+L_{s+q_s,t+q_t});\ 0<s<t \vee s>0>t\ ,$$
$$(L_{t,s+q_s}-L_{s,q_t+t}), i(L_{t,s+q_s}+L_{s,q_t+t});\ s,t>0 \vee 0<s<-t \vee 0<t<-s\}, \qquad (31c)$$

where $s,t = \pm 1, \pm 2, \ldots, \pm q$.

III. The case θ''.

For this case $n=2q+1$ and the formulae in this case are full analogical as in the case θ'. We introduce only the formulae for $n^{+}_{\theta''}$, $g^{o}_{\theta''}$ and $n^{-}_{\theta''}$.

$$n^{+}_{\theta''} = R\{(L_{st}-L_{t+q_t,s+q_s}), i(L_{st}+L_{t+q_t,s+q_s});\ 0<s<t \vee s>0>t\ ,$$
$$(L_{s,t+q_t}-L_{t,s+q_s}), i(L_{s,t+q_t}+L_{t,s+q_s});\ s,t>0 \vee 0<s<-t \vee 0<t<-s \qquad (32a)$$
$$,(L_{s,\alpha}+L_{\alpha,s+q_s}), i(L_{s,\alpha}-L_{\alpha,s+q_s}),\ s>0\}$$

$$g^{o}_{\theta''} = R\{(L_{ss}-L_{s+q_s,s+q_s}), i(L_{ss}+L_{s+q_s,s+q_s}),$$
$$(L_{s,-(s+q_s)}-L_{-s,s+q_s}), i(L_{s,-(s+q_s)}+L_{-s,s+q_s}), iL_{\alpha\alpha}\} \qquad (32b)$$

$$n^{-}_{\theta''} = R\{(L_{ts}-L_{s+q_s,t+q_t}), i(L_{ts}+L_{s+q_s,t+q_t}),\ 0<s<t \vee s>0>t\ ,$$
$$(L_{t,s+q_s}-L_{s,q_t+t}), i(L_{t,s+q_s}+L_{s,q_t+t});\ s,t>0 \vee 0>s>-t\ \ 0<t<-s \qquad (32c)$$
$$(L_{\alpha,s}+L_{s+q_s,\alpha}), i(L_{\alpha,s}-L_{s+q_s,\alpha});\ s>0\},$$

where $s,t = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha = \pm n$.

Dimension of the subalgebras g^{p}_{θ} and h^{p}_{θ} are characteristic values of the given real form g_{θ}. For the automorphisms θ', θ_q, θ'' we get from (23b), (24), (28b), (29a).

θ	dim g^{p}_{θ}	dim h^{p}_{θ}
θ'	$n(n-1)$	$\frac{n}{2}$
θ''	$n(n-1)$	$\frac{n-1}{2}$
θ_q	$q(2n-q)$	q

A comparison of these results with a list of the real forms so(2n,C) in book /2/ p.85 implies a following theorem.

Theorem: The algebra $g_{\theta'}$, $g_{\theta''}$ are isomorphic $so^{*}(2n)$ and the algebras g_{θ_q} are isomorphic $so(q,2n-q)$ for any $q=1,2,\ldots,n$.

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Конструкция разложений Ивасавы и треугольных разложений для вещественных форм алгебр Ли so(2n, C)

Построены разложения Ивасавы и треугольные разложения для всех вещественных форм алгебр Ли so(2n, C). Метод конструкции основан на вычислении в явном виде автоморфизмов Картана, при помощи которых определяются вещественные формы алгебры Ли so(2n, C).

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Direct Evaluation of the Iwasawa and Triangle Decompositions for the Real Forms of Lie Algebras so(2n,C)

The Iwasawa and triangle decompositions for any real form of Lie algebra so(2n, C) are given. Construction of this decomposition is based on the explicit calculation of the Cartan automorphism with the help of which the real forms of Lie algebras so(2n, C) are defined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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