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DIRECT EVALUATION OF THE IWASAWA  
AND TRIANGLE DECOMPOSITIONS  
FOR THE REAL FORMS  
OF LIE ALGEBRAS  $sp(n, \mathbb{C})$

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## 1. Introduction

A number of physical problems ranging from an attempted relativistic extension of internal symmetries in particle physics, through the spectrum-generating noninvariance (or dynamical) groups for "classical" quantum - mechanical systems, to the labelling of states in nuclear shell theory are more or less responsible for interest in and a deeper knowledge of the properties of unitary irreducible representations of noncompact Lie groups.

The Iwasawa <sup>1/</sup> decomposition plays the crucial role in the construction of infinite-dimensional unitary irreducible representations of these groups by the induced representations method. <sup>2/</sup>

Also for our method <sup>3/</sup> of constructing skew-Hermitian realizations for an arbitrary real semisimple Lie algebra the Iwasawa and triangle decompositions are a starting point.

Explicit forms of this decompositions have been constructed by many authors for particular examples of real semisimple Lie algebras. The usual indirect method for evaluation of the Iwasawa decomposition of a noncompact semisimple Lie algebra involves search for solutions to certain simultaneous eigenvalue type equations; for details see the book by Hermann <sup>4/</sup>.

The other method has been formulated by Cornwell <sup>5/</sup>. It is based on the canonical form of Lie algebra and on the fact that for any two Cartan subalgebras there exists an inner automorphism which maps one into the other. This inner automorphism, however must be guessed for each particular case.

The present paper gives the Iwasawa decomposition for real forms of Lie algebras  $sp(n, \mathbb{C})$  in a much more simple way. For any real form of the algebra  $sp(n, \mathbb{C})$  we construct the above-mentioned Cartan automorphism and explicit form of this automorphism then specifies the decompositions directly. Organisation of the paper is the following.

In Section 2, we describe in general terms the construction of the decompositions in the case when the Cartan automorphism which gives real forms is defined explicitly. For the algebras  $sp(n, \mathbb{C})$ , we give explicit forms of the Cartan automorphism in Section 3. In the last section, these automorphism will be used for construction of the Iwasawa and triangle decompositions.

## 2. The Iwasawa and Triangle Decompositions

In this section, we have given a brief survey of the Iwasawa and triangle decompositions. We followed Chapter 5 of Helgasson<sup>/6/</sup> and Chapter 3, §3 of Zhelobenko and Stern<sup>/7/</sup>.

Let  $\tilde{g}$  be a semisimple Lie algebra. We denote by  $\tilde{h}$  a Cartan subalgebra of  $\tilde{g}$  and by  $\Delta$  the root system of  $\tilde{g}$  with respect to  $\tilde{h}$ . Any semisimple Lie algebra has a Cartan-Weyl basis  $\{H_1, \dots, H_n, E_\alpha; \alpha \in \Delta\}$  for which:

$$\{H_1, \dots, H_n\} \text{ is a basis in } \tilde{h} \quad (1a)$$

$$[h, E_\alpha] = \alpha(h)E_\alpha \quad (1b)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad (N_{\alpha\beta} = -N_{-\alpha, -\beta}) \text{ for } \alpha + \beta \in \Delta \quad (1c)$$

$$B(E_\alpha, E_{-\alpha}) = 1, \quad (1d)$$

where  $B(\cdot, \cdot)$  is a Killing form on  $\tilde{g}$ .

Using this Cartan-Weyl basis of  $\tilde{g}$  we can define an antilinear mapping on  $\tilde{g}$  by

$$\psi(H_i) = -H_i \quad \text{for } i=1, 2, \dots, n \quad (2a)$$

$$\psi(E_\alpha) = -E_{-\alpha} \quad \text{for all } \alpha \in \Delta. \quad (2b)$$

This mapping is the involutive antilinear automorphism on  $\tilde{g}$  and the real form corresponding to  $\psi$

$$g_\psi = \{X \in \tilde{g}; \psi(X) = X\} \quad (3)$$

is compact.

A linear automorphism is called a Cartan automorphism if

$$\theta^2 = 1 \quad (4a)$$

$$\theta \cdot \psi = \psi \cdot \theta \quad (4b)$$

$$\theta(\tilde{h}) = \tilde{h}. \quad (4c)$$

If  $\theta$  is a Cartan automorphism on  $\tilde{g}$ , then  $\theta$  defines the real forms of  $\tilde{g}$  as a set for

$$g_\theta = \{X \in \tilde{g}; \theta \cdot \psi(X) = X\}. \quad (5)$$

With the help of the Cartan automorphism  $\theta$ , we can construct very simply a Cartan decomposition of  $g_\theta$

$$g_\theta = g_\theta^t \oplus g_\theta^p, \quad \text{where} \quad (6a)$$

$$g_\theta^t = \{Y \in g_\theta; \theta(Y) = Y\}, \quad (6b)$$

$$g_\theta^p = \{Y \in g_\theta; \theta(Y) = -Y\}. \quad (6c)$$

The subalgebra  $g_\theta^t$  is a maximal compact subalgebra in the algebra  $g_\theta$  and the subspace  $g_\theta^p$  is a complementary subspace to  $g_\theta^t$  with respect to Killing form  $B(\cdot, \cdot)$ .

Similarly we put

$$h_\theta = \{X \in \tilde{h}, \theta \cdot \psi(X) = X\} \quad (7a)$$

and then we have

$$h_\theta = h_\theta^t \oplus h_\theta^p \quad \text{where} \quad (7b)$$

$$h_\theta^t = \{Y \in h_\theta, \theta(Y) = Y\}, \quad (7c)$$

$$h_\theta^p = \{Y \in h_\theta, \theta(Y) = -Y\}. \quad (7d)$$

The  $h_\theta^p$  is a maximal commutative subalgebra in  $g_\theta^p$ .

Now, let  $\{H_1, \dots, H_q\}$  be a basis in  $h_\theta^p$  and further let  $\{iH_{q+1}, \dots, iH_n\}$  be a basis in  $h_\theta^t$ . Then  $\{H_1, \dots, H_n\}$  is a basis in  $\tilde{h}$ . By  $\Delta_+$  we shall denote a system of positive roots with respect to this basis.

For any  $\alpha \in \Delta$  we define

$$\alpha^\theta(H) = \alpha(\theta(H)) \quad \text{for any } H \in \tilde{h}. \quad (8)$$

After performing the following easy calculation

$$[H, \theta(E_\alpha)] = \theta[\theta(H), E_\alpha] = \alpha(\theta(H))\theta(E_\alpha) \quad (9)$$

we get that  $\alpha^\theta$  is also the element of  $\Delta$ .

Using this fact we may define

$$g_\theta^+ = \{\alpha; \alpha \in \Delta_+, \alpha \neq \alpha^\theta\}, \quad (10a)$$

$$\tilde{n}_\theta^+ = \sum_{P_\theta^+} g^\alpha, \quad n_\theta^+ = \tilde{n}_\theta^+ \cap g_\theta, \quad (10b)$$

$$\tilde{n}_\theta^- = \sum_{P_\theta^-} g^{-\alpha}, \quad n_\theta^- = \tilde{n}_\theta^- \cap g_\theta, \quad (10c)$$

where  $g^\alpha = \mathbb{C}\{E_\alpha\}$

$$g_\theta^0 = \{Y \in g_\theta; [Y, h_\theta^p] = 0\} \quad (10d)$$

and we get the required Iwasawa and triangle decompositions of the real form  $g_\theta$  at last

$$g_\theta = g_\theta^t \oplus h_\theta^p \oplus n_\theta^+ \quad (\text{Iwasawa decomposition}) \quad (11a)$$

$$g_\theta = n_\theta^+ \oplus g_\theta^0 \oplus n_\theta^- \quad (\text{triangle decomposition}). \quad (11b)$$

### 3. The Cartan Automorphism of Lie Algebras $sp(n, \mathbb{C})$

The algebra  $sp(n, \mathbb{C})$  is the  $n(2n+1)$  - dimensional complex Lie algebra with the standard basis  $\{L_{ij}; i, j = \pm 1, \pm 2, \dots, \pm n\}$  the elements of which obey

$$L_{ij} = -\epsilon_i \epsilon_j L_{-j, -i}, \quad \text{where } \epsilon_i = \text{sgn } i \quad (12)$$

and the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{il} L_{kj} + \epsilon_i \epsilon_j \delta_{j-1} L_{k, -i} - \epsilon_i \epsilon_j \delta_{i, -k} L_{-j, k}. \quad (13)$$

The standard Cartan subalgebra  $\tilde{h}$  in  $\tilde{g} = sp(n, \mathbb{C})$  is generated by the "diagonal" elements  $\{L_{ii}; i = 1, 2, \dots, n\}$  its dimension, i.e. rank of  $\tilde{g}$  equals  $n$ . We choose the following Cartan-Weyl basis

$$H_i = L_{ii}, \quad (14a)$$

$$E_{\lambda_1 - \lambda_k} = \frac{1}{\sqrt{4n+4}} L_{1k}, \quad E_{-\lambda_1 + \lambda_k} = \frac{1}{\sqrt{4n+4}} L_{k1}; \quad 0 < i < k, \quad (14b)$$

$$E_{\lambda_1 + \lambda_k} = \frac{1}{\sqrt{4n+4}} L_{1, -k}, \quad E_{-\lambda_1 - \lambda_k} = \frac{1}{\sqrt{4n+4}} L_{-k, 1}; \quad 0 < i < k, \quad (14c)$$

$$E_{2\lambda_1} = \frac{1}{\sqrt{2n+2}} L_{1, -1}, \quad E_{-2\lambda_1} = \frac{1}{\sqrt{2n+2}} L_{-1, 1} \quad 0 < i \quad (14d)$$

The relations (13) imply that (14b), (14c) and (14d) are the root vectors corresponding to the root  $\lambda_1 - \lambda_k, \lambda_1 + \lambda_k$  and  $2\lambda_1$  because

for  $H(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i L_{ii}$  we get

$$[H(\lambda_1, \dots, \lambda_n), L_{ik}] = (\lambda_i - \lambda_k) L_{ik}, \quad \text{etc.} \quad (15)$$

For the root system we get

$$\Delta = \{\pm(\lambda_i - \lambda_k), \pm(\lambda_i + \lambda_k), \pm 2\lambda_i; 0 < i, k < n, i \neq k\}. \quad (16)$$

Then the equality

$$-(\lambda_i \pm \lambda_k) = (\lambda_k \pm \lambda_i) \quad (17)$$

implies that in this case the mapping (2a-b) equals

$$\psi(L_{ik}) = -L_{ki}. \quad (18)$$

We define a linear mapping  $\theta'$  on  $\tilde{g}$  in the basis  $\{L_{ij}; i, j = \pm 1, \dots, \pm n\}$

$$\theta'(L_{ij}) = -L_{ji} \quad (19)$$

and further we define for any  $q=1, 2, \dots, [\frac{n}{2}]$  linear mappings  $\theta_q$  on  $\tilde{g}$  in this way:

$$\theta_q(L_{st}) = L_{s+q_s, t+q_t}, \quad \theta_q(L_{s+q_s, t+q_t}) = L_{st}, \quad (20a)$$

$$\theta_q(L_{s, t+q_t}) = L_{s+q_s, t}, \quad \theta_q(L_{s+q_s, t}) = L_{s, t+q_t}, \quad (20b)$$

$$\theta_q(L_{s, \alpha}) = -L_{s+q_s, \alpha}, \quad \theta_q(L_{s+q_s, \alpha}) = -L_{s, \alpha}, \quad (20c)$$

$$\theta_q(L_{\alpha, s}) = -L_{\alpha, s+q_s}, \quad \theta_q(L_{\alpha, s+q_s}) = L_{\alpha, s}, \quad (20d)$$

$$\theta_q(L_{\alpha, \beta}) = L_{\beta, \alpha}. \quad (20e)$$

where  $s, t = \pm 1, \pm 2, \dots, \pm q$ ,  $\alpha, \beta = \pm(2q+1), \dots, \pm n$  and  $q_s = q \text{ sgn } s$ .

Theorem: The linear mappings  $\theta', \theta_q$  are Cartan automorphisms on  $\tilde{g}$ .

Proof: One can by direct calculation verify that the conditions (4a-c) are fulfilled.

#### 4. Explicit Forms of the Decompositions

Using the method described in Section 2 we shall construct the explicit forms of the Iwasawa and triangle decompositions for the real forms of the Lie algebra  $sp(n, \mathbb{C})$  which are defined with the help of the automorphisms  $\theta, \theta_q$ . By the construction we will use following equalities

$$\psi(X + \psi(X)) = X + \psi(X) \quad (21a)$$

$$\psi(1X - 1\psi(X)) = 1X - 1\psi(X) \quad \text{for any } X \in \tilde{g} \quad (21b)$$

$$\theta(Y + \theta(Y)) = Y + \theta(Y) \quad (21c)$$

$$\theta(Y - \theta(Y)) = -(Y - \theta(Y)) \quad \text{for any } Y \in g_\theta \quad (21d)$$

which are the direct consequence of the definitions (2a-b) and (4a-c). The calculations of the decompositions are simple and we carry only final results.

#### I. The case $\theta'$

For a subalgebra  $g_{\theta'}^t$  and a subspace  $g_{\theta'}^p$ , we have

$$g_{\theta'}^t = \mathbb{R}\{L_{1j} - L_{j1}; i, j = \pm 1, \pm 2, \dots, \pm n\}, \quad (22a)$$

$$g_{\theta'}^p = \mathbb{R}\{L_{1j} + L_{j1}; i, j = \pm 1, \pm 2, \dots, \pm n\}. \quad (22b)$$

In this case a subalgebra  $h_{\theta'}^t = 0$  and the set  $\{L_{11}, L_{22}, \dots, L_{nn}\}$  form a basis in  $h_{\theta'}^p$ . For  $k_{\theta'}$  with respect to this basis we get

$$P_{\theta'}^+ = \{\lambda_1 - \lambda_k, \lambda_1 + \lambda_k, 2\lambda_1; i < k, i, k = 1, 2, \dots, n\} \quad (23)$$

and this implies consequently that

$$n_{\theta'}^+ = \mathbb{R}\{L_{1j}, L_{1,-j}, L_{1,-1}; i < j, i, j = 1, 2, \dots, n\}, \quad (24a)$$

$$n_{\theta'}^- = \mathbb{R}\{L_{1j}, L_{1,-j}, L_{1,-1}; i > j, i, j = 1, 2, \dots, n\} \quad (24b)$$

and

$$g_{\theta'}^0 = h_{\theta'}^p. \quad (24c)$$

#### II. The case $\theta_q, q=1, 2, \dots, [\frac{n}{2}]$

For a subalgebra  $g_{\theta_q}^t$  and subspace  $g_{\theta_q}^p$  we get

$$g_{\theta_q}^t = \mathbb{R}\{(L_{st} - L_{t+s, s+q_s}) - (L_{ts} - L_{s+q_s, t+q_t}), \\ i(L_{st} + L_{t+q_t, s+q_s}) + i(L_{ts} + L_{s+q_s, t+q_t}), \\ (L_{s, t+q_t} - L_{t, s+q_s}) + (L_{s+q_s, t} - L_{t+q_t, s}), \\ i(L_{s, t+q_t} + L_{t, s+q_s}) + i(L_{s+q_s, t} + L_{t+q_t, s}), \\ (L_{s, \alpha} + L_{\alpha, s+q_s}) - (L_{\alpha, s} + L_{s+q_s, \alpha}), \\ i(L_{s, \alpha} - L_{\alpha, s+q_s}) - i(L_{\alpha, s} - L_{s+q_s, \alpha}), \\ (L_{\alpha, \beta} - L_{\beta, \alpha}), i(L_{\alpha, \beta} + L_{\beta, \alpha})\}, \quad (25a)$$

$$g_{\theta_q}^p = \mathbb{R}\{(L_{st} - L_{t+q_t, s+q_s}) + (L_{ts} - L_{s+q_s, t+q_t}), \\ i(L_{st} + L_{t+q_t, s+q_s}) - i(L_{ts} + L_{s+q_s, t+q_t}), \\ (L_{s, t+q_t} - L_{t, s+q_s}) - (L_{s+q_s, t} - L_{t+q_t, s}), \\ i(L_{s, t+q_t} + L_{t, s+q_s}) - i(L_{s+q_s, t} + L_{t+q_t, s}), \\ (L_{s, \alpha} + L_{\alpha, s+q_s}) + (L_{\alpha, s} + L_{s+q_s, \alpha}), \\ i(L_{s, \alpha} - L_{\alpha, s+q_s}) + i(L_{\alpha, s} - L_{s+q_s, \alpha})\}. \quad (25b)$$

We put

$$\{(L_{11} - L_{q+1, q+1}), \dots, (L_{qq} - L_{2q, 2q})\} \quad (26a)$$

a basis in a subalgebra  $h_{\theta_q}^p$  and

$$\{i(L_{11} + L_{q+1, q+1}), \dots, i(L_{qq} + L_{2q, 2q}), iL_{2q+1, 2q+1}, \dots, iL_{nn}\} \quad (26b)$$

in a subalgebra  $h_{\theta_q}^t$ .

For a set of the roots  $P_{\theta_q}^+$  we get

$$P_{\theta_q}^+ = \{(\lambda_s - \lambda_t), (\lambda_{t+q_t} - \lambda_{s+q_s}); 0 < s < t,$$

$$\begin{aligned}
& (\lambda_s + \lambda_t), (\lambda_{t+q_t} + \lambda_{s+q_s}) ; s > 0, \\
& (\lambda_s - \lambda_{q_t+t}), (\lambda_s + \lambda_{q_t+t}), \\
& (\lambda_s - \lambda_\alpha), \lambda_\alpha - \lambda_{s+q_s} ; s > 0 \text{ where} \\
& s, t = 1, 2, \dots, q \text{ and } \alpha = \pm 2q+1, \dots, \pm n \} ,
\end{aligned} \tag{27}$$

and further

$$\begin{aligned}
n_{\theta_q}^+ = \mathbb{R} \{ & (L_{st} - L_{t+q_t, s+q_s}), i(L_{st} + L_{t+q_t, s+q_s}) ; 0 < s < t \vee s > 0 > t, \\
& (L_{s, t+q_t} - L_{t, s+q_s}), i(L_{s, t+q_t} + L_{t, s+q_s}) ; s, t > 0 \vee 0 < s < -t \vee 0 < t < -s, \\
& (L_{s, \alpha} + L_{\alpha, s+q_s}), i(L_{s, \alpha} - L_{\alpha, s+q_s}) ; s > 0 \} ,
\end{aligned} \tag{28a}$$

$$\begin{aligned}
g_{\theta_q}^0 = \mathbb{R} \{ & (L_{ss} - L_{s+q_s, s+q_s}), i(L_{ss} + L_{s+q_s, s+q_s}), L_{\alpha\beta} - L_{\beta\alpha}, i(L_{\alpha\beta} + L_{\beta\alpha}), \\
& (L_{s+q_s, -s} - L_{-s, s+q_s}), i(L_{s+q_s, -s} + L_{-s, s+q_s}) \} ,
\end{aligned} \tag{28b}$$

$$\begin{aligned}
n_{\theta_q}^- = \mathbb{R} \{ & (L_{st} - L_{t+q_t, s+q_s}), i(L_{st} + L_{t+q_t, s+q_s}) ; 0 > s > t \vee s > 0 > t \\
& (L_{t+q_t, s} - L_{s+q_s, t}), i(L_{t+q_t, s} + L_{s+q_s, t}) ; s, t > 0 \vee 0 < s < -t \vee 0 < t < -s \\
& (L_{s, \alpha} + L_{\alpha, s+q_s}), i(L_{s, \alpha} - L_{\alpha, s+q_s}) ; s < 0 \}
\end{aligned} \tag{28c}$$

where  $s, t = \pm 1, \pm 2, \dots, \pm q$  and  $\alpha, \beta = \pm(2q+1), \dots, \pm n$ .

Dimensions of the subalgebras  $g_{\theta}^p$  and  $h_{\theta}^p$  are characteristic values of the given real form  $g_{\theta}$ . For the automorphisms  $\theta, \theta_q$  we get from (22b), (25b) and (26a)

$\theta$	$\dim g_{\theta}^p$	$\dim h_{\theta}^p$
$\theta'$	$n(2n+1)$	$n$
$\theta_q$	$4(n-q)q$	$q$

A comparison of these results with a list of the real forms  $sp(n, \mathbb{C})$  in book /7/ p.85 implies a following theorem.

**Theorem:** The algebra  $g_{\theta'}$  is isomorphic  $sp(n, \mathbb{R})$  and further the algebras  $g_{\theta_q}$  are isomorphic  $sp(n-q, q)$  for any  $q=1, 2, \dots, [\frac{n}{2}]$ .

### Conclusion

In the present paper we give the explicit form of the Iwasawa and triangle decompositions for the real forms of Lie algebras  $sp(n, \mathbb{C})$ . Also for the real forms of complex Lie algebras  $gl(n+1, \mathbb{C})$  these decompositions have been obtained (see our paper 8). The results for algebras  $B_n$  and  $D_n$  will be presented in a forthcoming paper.

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